SLAC-PUB 7326
SNUTP-96/105
SU-ITP 96-44
hep-th/9610157

CLASSICAL AND QUANTUM ASPECTS OF BPS BLACK HOLES IN $N = 2, D = 4$ HETEROTIC STRING COMPACTIFICATIONS[1]

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ABSTRACT

We study classical and quantum aspects of $D = 4, N = 2$ BPS black holes for T_2 compactification of $D = 6, N = 1$ heterotic string vacua. We extend *dynamical relaxation phenomena* of moduli fields to background consisting of a BPS soliton or a black hole and provide a simpler but more general derivation of the Ferrara-Kallosh's extremized black hole mass and entropy. We study quantum effects to the the BPS black hole mass spectra and to their dynamical relaxation. We show that, despite non-renormalizability of string effective supergravity, quantum effect modifies BPS mass spectra only through coupling constant and moduli field renormalizations. Based on target-space duality, we establish a perturbative non-renormalization theorem and obtain exact BPS black hole mass and entropy in terms of renormalized string loop-counting parameter and renormalized moduli fields. We show that similar conclusion holds, in the large T_2 limit, for leading non-perturbative correction. We finally discuss implications to type-I and type-IIA Calabi-Yau black holes.

Submitted to Nuclear Physics B

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[1]Work supported in part by DOE DE-AC03-76SF00515, NSF Grant PHY-9219345, NSF-KOSEF Bilateral Grant, KRF International Collaboration Grant and Non-Directed Research Grant 81500-1341, KOSEF Purpose-Oriented Research Grant 94-1400-04-01-3 and SRC Program, the Ministry of Education BSRI Grant 96-2418, and the Seoam Foundation Fellowship.

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1 INTRODUCTION

In recent exciting development [1, 2, 3, 4], string theory has provided with a microscopic first–principle from which long–standing puzzle of black hole thermodynamics [5] can be understood. The development was made possible, on one hand, from better understanding of non-perturbative string theory including strong-weak coupling duality [6], various string-string dualities [7, 8, 9, 10, 11, 12] and D-brane soliton sector carrying Ramond-Ramond charges [13], and, on another, from deeper understanding of BPS states in string theory and their stability throughout weak to strong coupling regime.

While a definite relation between the statistical mechanics of microscopic stringy states and the macroscopic laws of the black hole thermodynamics has been established for various specific black holes, it has not yet provided a *universal* derivation of entropy–area relation for *all* classes of black holes. In particular, all specific examples explored so far have had large $N \geq 4$ supersymmetries in four dimensions. Together with the fact that the scalar fields at black hole horizon were fixed [14, 15] so that the horizon area were independent of the scalar fields at infinity, large $N \geq 4$ local supersymmetries were stringent enough to determine the macroscopic black hole entropy uniquely (up to constants). In view of this, extension of the previous studies to the less stringent yet controllable situations is posed as an interesting problem and might offer further insights to the black–hole physics. In this respect, black holes arising in $N = 2$ supersymmetric theories are unique in that there exist controllable BPS states yet smaller supersymmetries renders underlying dynamics richer enough.

Recently, initiated by the pioneering work of Ferrara, Kallosh and Strominger [16], macroscopic aspects of $N = 2$ BPS black holes have been studied extensively [17]–[21]. In these works, the special geometry [22] that governs interactions of $N = 2$ supergravity with vector and hyper multiplets has played an important role. On the microscopic side, examples of D-brane configurations in $D = 4, N = 2$ compactifications have been found [23] and microscopic state counting has shown complete agreement with macroscopic entropy formula of corresponding black holes.

Particularly interesting subset of $N = 2$ BPS black holes are are the ones having constant moduli everywhere outside black hole horizon [17, 18]. These, so-called double extreme black holes [19], are distinguished from other BPS black holes in that they have the lightest possible mass. With such special properties, one might expect that the double-extreme black holes play a special role among all $N = 2$ BPS black holes and open up new understanding uncovered so far. In this respect, better understanding of the double-extreme black holes is desirable. The first motivation and contents of the present work is to address various aspects of them.

In all previous microscopic and macroscopic studies, however, only classical aspects of $N = 2$ BPS black holes were considered. What distinguishes $N = 2$ supersymmetry from $N \geq 4$ ones is that nontrivial quantum effects are present generically at perturbative and non-perturbative levels [24]. In establishing entropy–area relation for black holes in $N \geq 4$ supersymmetric theories, stability of BPS states against strong coupling extrapolation has served an integral part of underlying physics. Hence, it is of interest to what extent the nontrivial quantum effects are reflected in the $N = 2$ black holes and their physical properties. The second motivation and contents of the present work is study quantum effects for BPS states in rigid and local $N = 2$ theories.

In this paper, we study the above aspects for four–dimensional $N = 2$ heterotic string compactifications. For definiteness, we focus on rank-3, so-called STU model, theories that arises from compactification on T_2 of a $D = 6, N = 1$ heterotic string theory, which have been obtained from $D = 10$ by compactifying on K_3 with instanton numbers $(12, 12)$[3]. This theory is known to be dual either to the type IIA compactification on the Calabi-Yau threefold $\mathbf{P}_{1,1,2,8,12}(24)$ [9] or to the T_2 compactification of the $D = 6$ type-I orientifold on K_3 orbifold $T_4/\mathbf{Z}_2$ with one tensor multiplet and completely Higgs gauge group [28]. Utilizing each duality map, we may learn otherwise inaccesible properties of type IIA and type I black holes from heterotic STU black holes as well.

This paper is organized as follows. In Section 2, we first consider rigid $N = 2$ theory and interpret BPS mass minimization as dynamical relaxation of scalar fields. We extend this to local $N = 2$ theory and obtain what we call Kähler-BPS condition. This provides a simpler and more general derivation of the result by Ferrara and Kallosh [18]. In Section 3, we consider classical aspect of Kähler-BPS black holes for the heterotic STU model. In Section 4, we study quantum effects to Kähler-BPS configurations. After recalling the rigid $N = 2$ supersymmetry non-renormalization theorems to BPS masses, we study quantum effects to BPS black holes of the heterotic STU model. We show that target-space duality symmetry provides a strong constraint to quantum corrections. We establish a perturbative non-renormalization theorem based on the symmetry and show that the BPS black holes continues to saturate the BPS bound in terms of renormalized string loop-counting parameter and moduli fields. In the large T_2 limit, we also derive leading non-perturbative corrections. In Section 5, we conclude with brief discussions on utilizing string-string duality maps to type–I and type–IIA string theory black holes.

[3]This compactification is identical to other ones with different instanton numbers $(10, 14)$ or $(11, 13)$ at least perturbatively. Non-perturbative effects, however, may reveal differences among them[25, 26] .

2 DYNAMICAL RELAXATION OF $N = 2$ BPS MASS SPECTRA

2.1 BPS Mass and Dynamical Relaxatioin in Rigid $N = 2$ Theory

2.1.1 RIGID $N = 2$ SPECIAL GEOMETRY

Before we dwell into the technically more involved $N = 2$ supergravity theory, we first study dynamical relaxation of the extremal BPS mass spectra for rigid $N = 2$ supersymmetric Yang-Mills theory. The Lagrangian of the $N = 2$ supersymmetric gauge theory is constructured out of $N = 2$ vector multiplets. The $N = 2$ vector multiplets X^A are constrained chiral superfields and contain a complex scalar, $SO(2)$ doublet Majorana-Weyl gauginos and an abelian gauge field. We denote them as $(X^A, \Omega^A_a, \Omega^{Aa}, A^A_\mu)$, $A = 1, \cdots, n_V; a = 1, 2$ for both left-handed Ω_a and right-handed Ω^a's. Associated with them, one introduces a holomorphic prepotential $F(X)$ of the vector multiplets. Coupling of the scalar fields with the vector field strengths is then specified by a holomorphic tensor:

$$\overline{\mathcal{N}}_{AB} \equiv \frac{\theta_{AB}}{2\pi} + i\frac{4\pi}{g^2_{AB}} = \partial_A \partial_B F. \tag{2.1}$$

To describe scalar self-interaction, we first combine X^A and $F_A \equiv \partial F / \partial X^A$ together, and construct a symplectic vector V:

$$V = \begin{pmatrix} X^A \\ \\ F_A \end{pmatrix}. \tag{2.2}$$

Denoting symplectic inner product in terms of matrix multiplication

$$\langle V | W \rangle \equiv V^t \cdot \omega \cdot W \ ; \qquad \omega = \begin{pmatrix} 0 & +\mathbf{I} \\ \\ -\mathbf{I} & 0 \end{pmatrix}, \tag{2.3}$$

Kähler potential defined by

$$K(Z, \overline{Z}) = i \langle V | \overline{V} \rangle = i \Big(X^A \overline{F}_A(\overline{X}) - \overline{X}^A F_A(X) \Big). \tag{2.4}$$

specifies a n_V-dimensional Kähler manifold. By adopting so-called 'rigid special coordinates' $Z^A = X^A$, we find the Kähler metric

$$K_{A\overline{B}} = \partial_A \overline{\partial}_B K = -2 (\mathrm{Im} \mathcal{N})_{AB}. \tag{2.5}$$

The fact that couplings of scalar self-interactions and scalar-vector interactions are the same is nothing but a manifestation of the underlying $N = 2$ supersymmetry. By the same reason, the scalar-gaugino interaction couplings should also be governed by $\mathcal{N}_{AB}$. Indeed, expanding the Lagrangian defined by chiral F-term of the prepotential into component fields, one obtains:

$$L = -\frac{1}{4\pi}\mathrm{Im}\Big(\frac{1}{2}\overline{\mathcal{N}}_{AB}\mathcal{F}^{-A}_{\mu\nu}\mathcal{F}^{-B\mu\nu} + 2\overline{\mathcal{N}}_{AB}\mathcal{D}_\mu X^A\mathcal{D}^\mu\overline{X}^B + \overline{\mathcal{N}}_{AB}\overline{\Omega}^{Aa}\not{\mathcal{D}}\Omega^B_a\Big) + \cdots. \quad (2.6)$$

Here, the ellipses denote non-minimal coupling terms including magnetic moment interactions and scalar potential.

Heuristically, one can draw an analogy to an electrodynamics in a macroscopic media [29]. The coupling matrix $\mathcal{N}_{AB}$ is then interpreted as a generalized electric permittivity and magnetic permeability tensors:

$$\begin{aligned}\epsilon_{AB} &= [1+\chi_E]_{AB} = [\,\mathrm{Im}\mathcal{N}\,]_{AB} \\ \mu^{AB} &= [1+\chi_M]^{AB} = [(\mathrm{Im}\mathcal{N})^{-1}]^{AB}\end{aligned} \quad (2.7)$$

The novelty of the analog macroscopic media is that the speed of light remains unity as can be confirmed from the fact $\epsilon_{AB}\mu^{BC} = [\mathrm{Im}\mathcal{N}]_{AB}[(\mathrm{Im}\mathcal{N})^{-1}]^{BC} = \delta^C_A$. This is a necessary condition to maintain the manifest Lorentz covariance of the theory[4] The analogy with macroscopic media turns out quite useful later for interpreting dynamical relaxation of the BPS mass as a result of (anti)-screening of microscopic electric and magnetic charges by the macroscopic media. Motivated by this analogy, one then introduce a symplectic vector of the anti-self-dual field strengths:

$$\mathcal{Z}^- = (\mathcal{F}^{-A}, \mathcal{G}^-_A). \quad (2.9)$$

Symplectic vector of self-dual field strengths $\mathcal{Z}^+$ is defined by a complex conjugate relation of Eq.(2.9). The field $\mathcal{F}^A$ corresponds to the generalized electric and magnetic induction fields, **E** and **B**. Similarly, the field $\mathcal{G}_A$ corresponds to the generalized electric displacement and magnetic fields, **D** and **H**. These two sets of field-strength sections are related each other

$$\mathcal{G}^-_A \equiv \overline{\mathcal{N}}_{AB}\mathcal{F}^{-B}. \quad (2.10)$$

This is a direct counterpart of the so-called 'constitutive relation' [29] in the electrodynamics of a macroscopic media, viz., a functioinal relation of $\mathbf{D}, \mathbf{H}$ in terms of $\mathbf{E}, \mathbf{B}$.

[4]We note that effective supergravity theory of the type-I open string provides an another interesting analog macroscopic media with a unit speed of light, for which the electric permittivity and the magnetic permeability is self-field dependent:

$$[\epsilon]^\mu_\nu = [\frac{1}{\mu}]^\mu_\nu = \Big[1 - \frac{1}{4\pi\alpha'}F^{\alpha\beta}F_{\alpha\beta}\Big]^{-1/2}\delta^\mu_\nu. \quad (2.8)$$

In the presence of electric and magnetic four-currents $\mathcal{J} = (J_e^A, J_{Am})$, $8n_V$-component Maxwell's equation is expressed compactly as:

$$\mathrm{d} \wedge \mathrm{Re}\mathcal{Z}^- = \wedge \mathcal{J}. \tag{2.11}$$

Integrating this equation over the space, we obtain a symplectic vector of microscopic electric and magnetic charges:

$$\begin{aligned} \mathcal{Q} &\equiv (P^A, Q_A), \\ \oint_{S_2} \mathcal{F}^A &= P^A \ ; \qquad \oint_{S_2} \mathcal{G}_A = Q_A. \end{aligned} \tag{2.12}$$

Classically, the charges are continuous, real-valued in units of appropriate electric and magnetic coupling constants. Quantum mechanically, however, the charges should obey the Dirac-Schwinger-Zwanziger quantization condition, $\langle \mathcal{Q} | \mathcal{Q}' \rangle = P^\Lambda Q'_\Lambda - Q_\Lambda P'^\Lambda$ is an integer multiple of the Dirac unit $(2\pi\hbar)$. This in turn implies that the symplectic charge vector $\mathcal{Q}$ is covariant only under $Sp(2n_V; \mathbf{Z}) \in Sp(2n_V; \mathbf{R})$.

2.1.2 $N = 2$ CENTRAL CHARGE AND BPS SPECTRA

One can derive the central charge directly from the supersymmetry algebra. The supersymmetry current $S_{\mu a}$ and the supercharge of Eq.(2.6) are given by:

$$\begin{aligned} S_\mu^a &= -(\mathrm{Im}\mathcal{N})_{AB} \Big[\gamma_\nu \gamma_\mu \Omega_a^A \mathcal{D}^\nu \overline{X}^B - \epsilon_{ab} \gamma^\alpha \gamma^\beta \gamma_\mu \Omega^{Ab} \mathcal{F}_{\alpha\beta}^{-B} \Big]; \\ \mathbf{Q}_a &\equiv \int d^3\mathbf{x} \, S_a^0. \end{aligned} \tag{2.13}$$

From Eq.(2.6), one also derives anti-commutation relations for the gaugino fields:

$$\{ \Omega_a^A(t, \mathbf{x}), \Omega^{\dagger Bb}(t, \mathbf{y}) \}_{\mathrm{ET}} = \{ \Omega^{Aa}(t, \mathbf{x}), \Omega_b^{\dagger B}(t, \mathbf{y}) \}_{\mathrm{ET}} = \Big[\frac{i}{\mathrm{Im}\mathcal{N}} \Big]^{AB} \delta_b^a \delta(\mathbf{x} - \mathbf{y}). \tag{2.14}$$

One then evaluates the supercharge anti-commutators:

$$\begin{aligned} \{ \mathbf{Q}_a, \overline{\mathbf{Q}}^b \} &= +\delta_a^b \gamma^\mu P_\mu, \\ \{ \mathbf{Q}_a, \overline{\mathbf{Q}}_b \} &= -\epsilon_{ab} \int d^3\mathbf{x} \Big[\vec{\mathcal{G}}_A \cdot \vec{\mathcal{D}} X^A - \vec{\mathcal{F}}^A \cdot \vec{\mathcal{D}} \overline{F}_A \Big]. \end{aligned} \tag{2.15}$$

The central charge is defined from the second anti-commutator after integrating by parts and using the Maxwell's equation Eq.(2.12):

$$\begin{aligned} \mathbf{Z} &\equiv X^A \oint \mathcal{G}_A - F_A \oint \mathcal{F}^A = X^A Q_A - F_A P^A \\ &= \langle V | \mathcal{Q} \rangle. \end{aligned} \tag{2.16}$$

Note the topological nature of the central charge as it is defined as the surface integral at spatial infinity. Diagonalizing the supersymmetry algebra, one finds the BPS inequality for the mass spectra:

$$\mathbf{M}^2 \geq \mathbf{M}^2_{\rm BPS} \equiv |\mathbf{Z}|^2 = |\langle V|\mathcal{Q}\rangle|^2. \tag{2.17}$$

2.1.3 DYNAMICAL RELAXATION OF BPS MASS

We now introduce a notion of *dynamical relaxation of BPS mass and free energy*. Consider a single, isolated BPS state carrying electric and magnetic charges specified by the symplectic vector $\mathcal{Q}$. The BPS mass $\mathbf{M}_{\rm BPS}$ defines a mass gap separating the BPS state from vacuum state, and is a function of the gauge coupling constants. In supersymmetric gauge theory, the gauge coupling matrix $\mathcal{N}_{AB}$ is not a constant but a function of the coordinates Z^A on Kähler manifold. These coordinates are dynamical fields in $N = 2$ supersymmetric gauge theory, hence, can relax dynamically and minimize the mass gap $\mathbf{M}_{\rm BPS}$. Since the BPS state is characterized by electric and magnetic charges it carries, the relaxation configuration of the special coordinates Z^A should be determined entirely in terms of the symplectic charge vector $\mathcal{Q}$. Therefore, up to $Sp(2n_V;\mathbf{R})$ symplectic transformations, one can associate a one-to-one mapping from the n_V-dimensional Kähler manifold parametrized by the scalar fields $X^A = Z^A$ to the space of electric and magnetic charges of a given BPS state. Such a mapping is a harmonic one and the BPS mass can be taken as the positive-definite free energy associated with the harmonic mapping. Obviously, this notion can be extended to situations of multiple BPS states.

To exemplify the notion of dynamical minimization of BPS mass gap, consider a situation for the gauge group of rank one. The BPS mass spectra may be written as

$$\mathbf{M}^2_{BPS} = \mathtt{M}^2_{\rm W}\, [Q^2 + (\frac{4\pi}{g^2}\, P)^2]. \tag{2.18}$$

Here, $\mathtt{M}_{\rm W}$ denotes mass of heavy charged gauge boson and is related to vacuum expectation value v of Higgs fields as $\mathtt{M}_{\rm W} = gv$. Using the Cauchy-Schwarz inequality [30],

$$\mathbf{M}^2_{\rm BPS} \geq 2\mathtt{M}^2_{\rm W}\ \alpha_g \sqrt{P^2Q^2} = \frac{4\pi}{g^2}\, [8\pi v^2]\, P^2, \tag{2.19}$$

and the equality is saturated at $\alpha_g \equiv \frac{g^2}{4\pi} = \sqrt{P^2/Q^2}$.

It is straightforward to generalize the example to rank-N ($N > 1$) gauge group. For simplicity, we consider the minimal coupling $\overline{\mathcal{N}}_{AB} = [(\theta/2\pi) + i(4\pi/g^2)]\delta_{AB}$, but the foregoing result can be generalized to non-minimal case straightforwardly . The gauge group is spontaneously broken to $[U(1)]^N$. Taking into account of the Witten effect [31] and introducing a notation $\langle A, B\rangle$ for the quadratic form in the charge space, the BPS mass spectra is given by

$$\mathbf{M}^2_{\rm BPS} = \mathtt{M}^2_{\rm W}\, [\langle (Q - \frac{\theta}{2\pi}P), (Q - \frac{\theta}{2\pi}P)\rangle + (\frac{4\pi}{g^2})^2 \langle P, P\rangle]. \tag{2.20}$$

By introducing a new basis of electric and magnetic charges:

$$\mathbf{p} \equiv P; \qquad \mathbf{q} \equiv \frac{g^2}{4\pi}(Q - \frac{\theta}{2\pi}P), \tag{2.21}$$

one has

$$\mathbf{M}^2_{\rm BPS} = \mathtt{M}^2_{\rm W} \left(\frac{4\pi}{g^2}\right)^2 \Big[\langle \mathbf{p}, \mathbf{p}\rangle + \langle \mathbf{q}, \mathbf{q}\rangle\Big]. \tag{2.22}$$

Again, one finds that there exists a special configuration of the coupling constants at which BPS mass gap is minimized:

$$\begin{aligned} \mathbf{M}^2_{\rm BPS} &\geq \frac{\mathtt{M}^2_{\rm W}}{\alpha_g^2} \sqrt{\langle \mathbf{p}, \mathbf{p}\rangle \, \langle \mathbf{q}, \mathbf{q}\rangle} \\ &\geq \frac{\mathtt{M}^2_{\rm W}}{\alpha_g^2} \, (\mathbf{p} \cdot \mathbf{q}) \\ &\geq 0. \end{aligned} \tag{2.23}$$

Here, we have used the Cauchy-Schwarz inequality and the Hölder's inequalities [30] at the first and the second steps respectively. Each inequality then provides with separate conditions to the coupling constant $g^2/4\pi$ and to the vacuum angle $\theta/2\pi$ for which the BPS mass is minimized. First, the extremal vacuum angle is determined by saturating the Hölder's inequality, viz., the second line in Eq.(2.23):

$$\begin{aligned} (\mathbf{p} \cdot \mathbf{q}) &= \alpha_g \, \langle P, (Q - \frac{\theta}{2\pi}P)\rangle = 0; \\ \rightarrow \qquad \frac{\theta}{2\pi} &= \frac{(P \cdot Q)}{\langle P, P\rangle}. \end{aligned} \tag{2.24}$$

Next, the extremal coupliing constant is determined by saturating the Cauchy-Schwarz inequality, viz., the first line in Eq.(2.23):

$$\begin{aligned} \langle P, P\rangle &= \langle \mathbf{p}, \mathbf{p}\rangle = \langle \mathbf{q}, \mathbf{q}\rangle = \Big(\frac{g^2}{4\pi}\Big)^2 \, \langle (Q - \frac{\theta}{4\pi}P), (Q - \frac{\theta}{4\pi}P)\rangle; \\ \rightarrow \qquad \frac{4\pi}{g^2} &= \frac{1}{\langle P, P\rangle} \sqrt{\langle Q, Q\rangle \langle P, P\rangle - (P \cdot Q)^2}. \end{aligned} \tag{2.25}$$

To obtain the last expression, we have inserted the extremal vacuum angle Eq.(2.24). Alternatively, one may first tune the the vacuum angle to $\theta = 0$ by a Peccei-Quinn transformation and determine the extremal gauge coupling constant by Cauchy-Schwarz inequality. It is given by:

$$[\frac{4\pi}{g^2}]_{\theta=0} = \sqrt{\frac{\langle Q, Q\rangle}{\langle P, P\rangle}}. \tag{2.26}$$

One then undo the Peccei-Quinn rotation of the vacuum angle θ. Recalling that the vacuum angle shift also introduces an electric charge by the Witten effect [31], one finds that the BPS mass minimized under the condition $\theta = 0$ can be lowered further. Saturation of the new BPS mass then yields exactly the same result as the one based on the Hölder's inequality in Eq.(2.23). Hence, the vacuum angle relaxes to the extremal value Eq.(2.24) and, in turn, the coupling constant further to the extremal value Eq.(2.25). In either methods, one finally obtain the extremal BPS mass gap or free energy:

$$\mathbf{M}^2_{\rm BPS} = \frac{\mathtt{M}^2_{\rm W}}{\alpha_g}\sqrt{\langle Q, Q\rangle\langle P, P\rangle - (P\cdot Q)^2} = \frac{4\pi}{g^2}\,[8\pi v^2]\,\langle P, P\rangle. \tag{2.27}$$

The main idea of the BPS mass minimization is that the the gauge coupling constants parametrized by n_V complex scalar fields on the Kähler manifold are actually not constants but can relax. Since the BPS mass-squared is a positive-definite quadratic form of $2n_V$ electric and magnetic charges, it can be taken as a free energy that determines the relaxation configuration. In the analog electrodynamics of a macroscopic media, one allows the electric permittivity and the magnetic permeability to relax dynamically so that the screening of electric and magnetic monopole charges becomes as perfect as possible. Because of the Lorentz covariance relation $\epsilon\cdot\mu = 1$, screening of electric and screening of magnetic charges compete each other. The above extremal configuration is where the competition is balanced. While it is rather artificial in non-supersymmetric theories, the notion of dynamical relaxation is quite natural in supersymmetric field theories, supergravity theories and superstring theory. In fact, the idea has been used repeatedly for minimizing vacuum energy and determine physical parameters dynamically for various situations [32]. The only novelty in the present situation is that the background under consideration is not a flat spacetime but a BPS soliton or, as we will extend later, a black hole carrying nonvanishing electric and magnetic charges.

2.2 Local Special Geometry and $N = 2$ Supergravity

2.2.1 LOCAL $N = 2$ SPECIAL GEOMETRY

Consider the space of the n_V complex scalar fields Z^A associated with vector multiplets in the $N = 2$ supergravity. Locally this space form a Kähler-Hodge manifold, endowed with a Kähler potential $K(Z, \overline{Z})$ and a Kähler metric $K_{A\overline{B}} \equiv \partial_A\overline{\partial}_B K$. The local $N = 2$

supergravity algebra constrains that the Riemann curvature tensor of the Kähler-Hodge manifold should obey so-called 'special-geometry' relation:

$$R_{A\overline{B}C\overline{D}} = K_{A\overline{B}}K_{C\overline{D}} + K_{A\overline{D}}K_{C\overline{B}} - Y_{ACE}\overline{Y}_{\overline{B}\overline{D}\overline{F}}K^{E\overline{F}}. \tag{2.28}$$

To define the $N = 2$ supergravity couplings, we start by defining symplectic sections of the Hodge bundle:

$$V \equiv (L^\Lambda, M_\Lambda); \qquad \Lambda = 0, 1, \cdots, n_V. \tag{2.29}$$

They are covariantly holomorphic

$$D_{\overline{A}}V \equiv [\overline{\partial}_{\overline{A}} - \frac{1}{2}K_{\overline{A}}]V = 0. \tag{2.30}$$

Projection of L^Λ's to Z^A's is achieved by a gauge fixing. Demanding that the scalar and the graviton kinetic terms decouple, we find the choice:

$$\langle V|\overline{V}\rangle = (L^\Lambda\overline{M}_\Lambda - \overline{L}^\Lambda M_\Lambda) = i. \tag{2.31}$$

In addition to the section V, one can construct n_V new symplectic sections U_A out of V:

$$U_A = D_A V \equiv (\partial_A + \frac{1}{2}\partial_A K)V. \tag{2.32}$$

Then the special-geometry constraint Eq.(2.28) is solved by the above (n_V+1) symplectic sections if they satisfy (n_V+1) relations:

$$V = \begin{pmatrix} 0 & \mathcal{N}^{-1} \\ \mathcal{N} & 0 \end{pmatrix} \cdot V; \qquad U_A = \begin{pmatrix} 0 & \overline{\mathcal{N}}^{-1} \\ \overline{\mathcal{N}} & 0 \end{pmatrix} U_A. \tag{2.33}$$

Here, a symmetric matrix $\mathcal{N}$ is solved by combining the $V, \overline{U}_A$ sections together into $(n_V+1)\times(n_V+1)$ matrix $(V, \overline{U}_{\overline{A}})^T$ and inverting the (n_V+1) relations:

$$\begin{pmatrix} V \\ \overline{U}_{\overline{A}} \end{pmatrix} = \begin{pmatrix} 0 & +\mathcal{N}^{-1} \\ \mathcal{N} & 0 \end{pmatrix}\begin{pmatrix} V \\ \overline{U}_{\overline{A}} \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 0 & \mathcal{N}^{-1} \\ \mathcal{N} & 0 \end{pmatrix} = \begin{pmatrix} V \\ \overline{U}_{\overline{A}} \end{pmatrix} \cdot \begin{pmatrix} V \\ \overline{U}_{\overline{A}} \end{pmatrix}^{-1}. \tag{2.34}$$

Note that the gauge fixing condition Eq.(2.31) becomes

$$(\mathcal{N}_{\Lambda\Sigma} - \overline{\mathcal{N}}_{\Lambda\Sigma})L^\Lambda\overline{L}^\Sigma = i. \tag{2.35}$$

It is straightforward to solve $\mathcal{N}$ in terms of the symplectic sections and holomorphic matrix $\mathcal{F}$:

$$\mathcal{N}_{\Lambda\Sigma} = \overline{\mathcal{F}}_{\Lambda\Sigma} + 2i\frac{(\mathrm{Im}\mathcal{F})_{\Lambda\Gamma}L^\Gamma(\mathrm{Im}\mathcal{F})_{\Sigma\Delta}L^\Delta}{L^\Gamma(\mathrm{Im}\mathcal{F})_{\Gamma\Delta}L^\Delta}; \quad \mathcal{F}_{\Lambda\Sigma} \equiv \partial_\Lambda F_\Sigma(X). \tag{2.36}$$

One further finds that

$$\begin{aligned}
\langle V|U_A\rangle &= \langle V|\overline{U}_{\overline{A}}\rangle = 0, \\
K_{A\overline{B}} &= -i\langle U_A|U_{\overline{B}}\rangle, \\
Y_{ABC} &= \langle D_A U_B|U_C\rangle.
\end{aligned} \tag{2.37}$$

It is possible to fix an overall scale of the symplectic section as

$$V = \mathrm{M}_{\mathrm{Pl}} e^{K/2}\Omega; \qquad \Omega \equiv \begin{pmatrix} X^\Lambda \\ \\ F_\Lambda \end{pmatrix}. \tag{2.38}$$

It follows that Ω are holomorphic sections over a line bundle. All the above relations are then straightforwardly rewritten in terms of the holomorphic sections. In terms of Ω, the Kähler potential is given by

$$K = -\log[i\langle\Omega|\overline{\Omega}\rangle]. \tag{2.39}$$

Under the Kähler transformation $K \rightarrow K + \Lambda + \overline{\Lambda}$, $\Omega \rightarrow e^{-\Lambda}\Omega$. Therefore X^Λ provides a homogeneous local coordinate system on the Kähler manifold. One possible choice of the coordinate system is so-called 'special coordinates':

$$Z^A = \frac{X^A}{X^0}. \tag{2.40}$$

The electric and the magnetic charges provide with the source to the black hole mass. To manifest the $Sp(2n_V+2)$ symplectic structure, it is convenient to introduce anti-self-dual field strengths:

$$\mathcal{Z}^- \equiv (\mathcal{F}^{-\Lambda}, \mathcal{G}^-_\Lambda); \qquad \mathcal{G}^-_\Lambda = \overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Sigma}. \tag{2.41}$$

The equations of motion and the Bianchi identities are then compactly expressed as

$$\mathrm{d}\wedge(\mathrm{Re}\mathcal{Z}^-) = \wedge\mathcal{J}. \tag{2.42}$$

As in the rigid theory, the electric and the magnetic charges combine to a symplectic vector:

$$\mathcal{Q} = (P^\Lambda, Q_\Lambda); \qquad \oint_{S_2} \mathrm{Re}\mathcal{F}^{-\Lambda} = P^\Lambda, \qquad \oint_{S_2} \mathrm{Re}\mathcal{G}^-_\Lambda = Q_\Lambda. \tag{2.43}$$

Again, quantum mechanically, the electric and the magnetic charges are required to obey the Dirac-Schwinger-Zwanziger quantization conditions. Therefore, the charge symplectic vector $\mathcal{Q}$ is covariant only under the $Sp(2n_V+2;\mathbf{Z})$ transformations.

2.2.2 CENTRAL CHARGE AND BPS MASS SPECTRA

For a manifest supersymmetric multiplet formulation, it turns out convenient to reorganize the gauge field strengths into a new (n_V+1) linearly independent combinations:

$$\begin{aligned} T^- &\equiv \langle V|\mathcal{Z}^-\rangle = (M_\Lambda \mathcal{F}^{-\Lambda} - L^\Lambda \mathcal{G}^-_\Lambda), \\ F^{-A} &\equiv G^{A\overline{B}}\langle \overline{U}_{\overline{B}}|\mathcal{Z}^-\rangle = G^{A\overline{B}}(\overline{D}_B \overline{M}_\Lambda \mathcal{F}^{-\Lambda} - \overline{D}_B \overline{L}^\Lambda \mathcal{G}^-_\Lambda). \end{aligned} \tag{2.44}$$

That there are no other linearly independent field strength combinations is easy to understand from the two identities:

$$\langle \overline{V}|\mathcal{Z}^-\rangle = 0 = \langle U_A|\mathcal{Z}^-\rangle. \tag{2.45}$$

The T^- and F^-_A, $(A = 1, \cdots, n_V)$ are the gravi-photon of the supergravity multiplet and the gauge fields of n_V vector multiplets respectively.

Associated to the new (n_V+1) linearly independent combinations of the gauge field strengths are complex-valued, (n_V+1)-component central charge vector:

$$\mathbf{Z} \equiv -\frac{1}{2}\oint_{S_2} T^- = \langle V|\mathcal{Q}\rangle = (L^\Lambda Q_\Lambda - M_\Lambda P^\Lambda), \tag{2.46}$$

$$\mathbf{Z}_A \equiv -\frac{1}{2}\oint_{S_2} G_{A\overline{B}} F^{+\overline{B}} = (Q_\Lambda D_A L^\Lambda - P^\Lambda D_A M_\Lambda) = \langle U_A|\mathcal{Q}\rangle = D_A \mathbf{Z}. \tag{2.47}$$

One notes that, under the Kähler transformation $K \to K + \Lambda + \overline{\Lambda}$, the central charge transforms as holomorphic sections: $\mathbf{Z} \to e^{-\Lambda}\mathbf{Z}$, $\mathbf{Z}_A \to e^{-\Lambda}\mathbf{Z}_A$. These central charge vectors satisfy quadratic sum rules

$$\begin{aligned} |\mathbf{Z}|^2 + |\mathbf{Z}_A|^2 &= -\frac{1}{2}\mathcal{Q}^T \cdot \mathbf{M}(\mathcal{N}) \cdot \mathcal{Q}, \\ |\mathbf{Z}|^2 - |\mathbf{Z}_A|^2 &= -\frac{1}{2}\mathcal{Q}^T \cdot \mathbf{M}(\mathcal{F}) \cdot \mathcal{Q}. \end{aligned} \tag{2.48}$$

where

$$\mathbf{M}(\mathcal{N}) = \mathcal{R}^T(\mathrm{Re}\mathcal{N}) \cdot \mathcal{D}(\mathrm{Im}\mathcal{N}) \cdot \mathcal{R}(\mathrm{Re}\mathcal{N}) \tag{2.49}$$

and

$$\mathcal{R}(\mathrm{Re}\mathcal{N}) = \begin{pmatrix} \mathbf{I} & 0 \\ -\mathrm{Re}\mathcal{N} & \mathbf{I} \end{pmatrix}; \qquad \mathcal{D}(\mathrm{Im}\mathcal{N}) = \begin{pmatrix} \mathrm{Im}\mathcal{N} & 0 \\ 0 & (\mathrm{Im}\mathcal{N})^{-1} \end{pmatrix}. \tag{2.50}$$

The $\mathcal{R}(\mathrm{Re}\mathcal{N})$ matrix defines a symplectic transformation associated with the Witten effect [31]:

$$\mathcal{R}^T \begin{pmatrix} 0 & +\mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix} \mathcal{R} = \begin{pmatrix} 0 & +\mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix};$$

$$\mathcal{R}\begin{pmatrix} 0 & +\mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix}\mathcal{R} \;=\; \begin{pmatrix} 0 & +\mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix}. \tag{2.51}$$

Similarly, the matrices $\mathbf{M}(\mathcal{F})$, $\mathcal{R}(\mathrm{Re}\mathcal{F})$ and $\mathcal{D}(\mathrm{Im}\mathcal{F})$ are defined by replacing $\mathcal{N}$ in Eqs.(2.49, 2.50) into $\mathcal{F}$ respectively. In terms of factorized matrices, the quadratic sum rules Eq.(2.48) are given by:

$$\begin{aligned} |\mathbf{Z}|^2 + |\mathbf{Z}_A|^2 &= -\frac{1}{2}(Q - \overline{\mathcal{N}}\cdot P)_\Lambda \Big[\frac{1}{\mathrm{Im}\mathcal{N}}\Big]^{\Lambda\Sigma}(Q - \mathcal{N}\cdot P)_\Sigma; \\ |\mathbf{Z}|^2 - |\mathbf{Z}_A|^2 &= -\frac{1}{2}(Q - \overline{\mathcal{F}}\cdot P)_\Lambda \Big[\frac{1}{\mathrm{Im}\mathcal{F}}\Big]^{\Lambda\Sigma}(Q - \mathcal{F}\cdot P)_\Sigma. \end{aligned} \tag{2.52}$$

The two formula shows clearly that $\mathrm{Im}\mathcal{N}$ defines a negative-definite metric, while $\mathrm{Im}\mathcal{F}$ defines a metric of signature $(1, n_V)$ in the quadratic central charge sum rules.

Since the central charge and its derivatives are projections of the electric and the magnetic charges with respect to the symplectic sections, they are in general functions of moduli fields and complex-valued. The utility of the new linear combinations of the gauge field strengths and associated the central charges becomes transparent once one solves the condition for nontrivial BPS black holes to exist. We now turn to these conditions.

2.3 $N = 2$ Supersymmetric Black Holes

Consider $N = 2$ supergravity theory coupled to n_V vector multiplets. Explicit construction of the $N = 2$ supersymmetric black hole utilizing the special geometry was initiated by Ferrara, Kallosh and Strominger [16].

One obtains the metric, the $(n_V + 1)$ gauge fields including the gravi-photon field and the n_V scalar fields Z^A configurations by solving the supersymmetry Killing spinor conditions to the gravitino and the gaugino supersymmetry transformation rules:

$$\begin{aligned} \delta\Psi_{\mu a} &= \mathcal{D}_\mu \epsilon_a + T^-_{\mu\nu}\gamma^\nu\, \epsilon_{ab}\epsilon^b, \\ \delta\Omega^{Aa} &= i(\nabla\!\!\!\!/ Z^A)\epsilon^a + F^{-A}_{\mu\nu}\gamma^\mu\gamma^\nu\, \epsilon_b. \end{aligned} \tag{2.53}$$

The classical, supersymmetric black hole configuration is obtained by demanding an existence of covariantly constant spinors, $\delta_\epsilon \psi^a_\mu = \delta_\epsilon \Omega^\Lambda_a = 0$ with an ansatz

$$\begin{aligned} ds^2 &= e^{+2U}dt^2 - e^{-2U}d\vec{x}^2\ ; \qquad e^{-U} = (1 + \frac{M}{r}), \\ \mathcal{F}^\Lambda &= \frac{\mathbf{q}^\Lambda}{r^2}\,[e^{2U}dt\wedge dr] + \frac{\mathbf{p}^\Lambda}{r^2}\,[r^2 d\Omega_2]. \end{aligned} \tag{2.54}$$

Here, $\mathbf{p}$ and $\mathbf{q}$ are arbitrary constant denoting the magnetic and electric charges measured from F^Λ field at spatial infinity. Recall that, in an analogy with the electrodynamics of a macroscopic media, $\mathcal{F}^\Lambda$ field corresponds to the generalized electric and magnetic induction fields, $\mathbf{E}$ and $\mathbf{B}$. Therefore, one expects that the electric charge $\mathbf{q}$ is not the microscopic charge but the total charge including the screening and the Witten effect [31]. Given the constitutive relations Eq.(2.42), one should then find a relation to the fundamental, microscopic charges $\mathcal{Q} = (P, Q)$. By a straightforward calculation, one finds that

$$\begin{aligned} Q_\Lambda &= (\mathrm{Re}\mathcal{N})_{\Lambda\Sigma}\mathbf{p}^\Sigma - (\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}\mathbf{q}^\Sigma; \\ P^\Lambda &= \mathbf{p}^\Lambda. \end{aligned} \tag{2.55}$$

The afore-mentioned screening and Witten effects are manifest from the charge relations.

A particularly interesting class of the $N = 2$ black hole configurations is the ones with a frozen special coordinate fields, Z^A = constant [17, 18]. The gaugino supersymmetry transformation rules in Eq.(2.53) then imply that the gauge fields associated with the vector multiplets should vanish everywhere:

$$F^{-A} = 0 \qquad \leftrightarrow \qquad F^{+\overline{A}} = 0. \tag{2.56}$$

Inferring Eq.(2.47) for the above gauge field configuration , one finds

$$\mathbf{Z}_A = D_A \mathbf{Z} = 0, \tag{2.57}$$

and concludes that the black holes with frozen special coordinate fields exhibit a special feature that central charge $\mathbf{Z}$ is covariantly constant with respect to the special coordinates. Being n_V independent equations, Eq.(2.57) in turn determines uniquely the configuration of the n_V special coordinates Z^A.

The fact that the constant moduli ansatz leads to the lowest BPS mass of the black hole may be understood as follows. The total energy $\mathcal{E}$ of the black hole configuration may be expressed schematically as:

$$\mathcal{E}_{\mathrm{BH}} = \int_{M_3} \Big[R^{(3)} + ||\nabla Z^A||^2 + \frac{1}{2}\Big(\mathbf{E} \cdot \mathbf{D}(Z) + \mathbf{B} \cdot \mathbf{H}(Z) \Big) \Big]. \tag{2.58}$$

That this expression is positive-definite is guaranteed by the Witten's positive energy theorem [33], applied to the background of a black hole with nonvanishing electric and magnetic charges [34]. The second term represents a harmonic map to the Kähler manifold whose coordinates are represented by the complex scalar fields Z^A. Since the weight $||\nabla Z^A||^2$ is manifestly positive definite, the lowest but nonzero BPS energy is achieved by a *constant* harmonic map: $\nabla Z^A = 0$, viz. map the entire space of the black hole exterior (outside the horizon) to a single point in the Kähler manifold. Because of $N = 2$ supersymmetry, this in turn requires that the the gauge field strengths F^A paired with Z^A fields vanish as well. In what follows, we will call the constant harmonic map as *Kähler-BPS limit*, since it follows from the minimization of the Kähler sigma model contribution to the BPS black hole energy.

2.4 Dynamical Relaxation of the BPS Black Hole Spectra

2.4.1 KÄHLER-BPS SCALAR FIELDS

One now solves the Kähler-BPS condition and determines explicitly the constant scalar fields Z^A as a function of charges:

$$\mathbf{Z}_A \equiv D_A \mathbf{Z} = 0 \qquad \leftrightarrow \qquad \langle U_A | \mathcal{Q} \rangle = 0. \tag{2.59}$$

In this case the two quadratic chrage sum rules Eq.(2.52) reduce down to the square of the central charge $\mathbf{Z}$ itself:

$$\begin{aligned} |\mathbf{Z}|^2 &= -\frac{1}{2} \mathcal{Q}^T \cdot \mathbf{M}(\mathcal{N}) \cdot \mathcal{Q} \\ &= -\frac{1}{2} \mathcal{Q}^T \cdot \mathbf{M}(\mathcal{F}) \cdot \mathcal{Q}. \end{aligned} \tag{2.60}$$

While $\mathcal{N}$ and $\mathcal{F}$ matrices are different in general, for the sum rules of the minimized central charge, quadratic forms formed out of either matrices are the same. In subsequent calculations, we will use exclusively the quadratic form using the holomorphic matrix $\mathcal{F}$ mainly for calculational convenience. However, all the formulas we derive in this paper are straightforwardly generalizable to the representation using the coupling matrix $\mathcal{N}$ by replacing wherever $\mathcal{F}$ appears into $\mathcal{N}$.

The minimization condition Eq.(2.59) determines vacuum expectation value of the moduli fields as a function of the electric and the magnetic charges. The condition can be inverted as follows. One first recalls the symplectic orthogonality relation Eq.(2.37) of the symplectic covariant vector U_A:

$$\langle U_A | V \rangle = 0 = \langle U_A | \overline{V} \rangle. \tag{2.61}$$

Then the minimization condition Eq.(2.59) is solved by a *linear map* between the symplectic section V and the electric and the magnetic charge $\mathcal{Q}$:

$$\mathcal{Q} = c_1 V + c_2 \overline{V} \tag{2.62}$$

where c_1, c_2 are complex-valued parameters to be determined. Consistency condition imposes the symplectic inner product of this solution with $V, \overline{V}$ respectively:

$$
\begin{aligned}
\mathbf{Z} &\equiv \langle V|\mathcal{Q}\rangle \quad \rightarrow \quad c_2 = -i\mathbf{Z}, \\
\overline{\mathbf{Z}} &\equiv \langle \overline{V}|\mathcal{Q}\rangle \quad \rightarrow \quad c_1 = +i\overline{\mathbf{Z}}.
\end{aligned} \tag{2.63}
$$

Hence, we finally get the Kähler-BPS condition:

$$
\begin{aligned}
\mathcal{Q} &= i(\overline{\mathbf{Z}}V - \mathbf{Z}\overline{V}), \\
P^\Lambda &= 2\mathrm{Im}(\overline{\mathbf{Z}}L^\Lambda), \\
Q_\Lambda &= 2\mathrm{Im}(\overline{\mathbf{Z}}M_\Lambda).
\end{aligned} \tag{2.64}
$$

Note that the right hand side of Eq.(2.64) is invariant under the Kähler transformation. This is a necessary condition since the electric and the magnetic charges carry no Kähler weights.

In order to obtain an explicit form of the constant scalar fields, one needs to solve the Kähler BPS condition Eq.(2.64). Using the Kähler-BPS saturated quadratic sum rules of the central charges Eq.(2.60), one finds the solution as:

$$
V = -\frac{1}{2\overline{\mathbf{Z}}}\Big[\begin{pmatrix} 0 & +\mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix} \cdot \mathbf{M}(\mathcal{F}) + i \begin{pmatrix} +\mathbf{I} & 0 \\ 0 & +\mathbf{I} \end{pmatrix} \Big] \cdot \mathcal{Q}. \tag{2.65}
$$

On the right-hand side, the first term denotes a particular solution that satisfies $\langle V|\mathcal{Q}\rangle = \mathbf{Z}$, while the second term is a homogeneous solution that satisfies the Kähler-BPS condition Eq.(2.64). It is straightforward to check that the solution Eq.(2.65) satisfies the symplectic constraint $\langle V|\overline{V}\rangle = i$. Expanding Eq.(2.65) in $2n_V$ components,

$$
\begin{aligned}
-2\overline{\mathbf{Z}}L^\Lambda &= \Big[iP - (\mathrm{Im}\mathcal{N})^{-1}(\mathrm{Re}\mathcal{N})\,P + (\mathrm{Im}\mathcal{N})^{-1}\,Q\Big]^\Lambda \\
-2\overline{\mathbf{Z}}M_\Lambda &= \Big[iQ - ((\mathrm{Im})\mathcal{N}) + (\mathrm{Re}\mathcal{N})(\mathrm{Im}\mathcal{N})^{-1}(\mathrm{Re}\mathcal{N}))\,P + (\mathrm{Re}\mathcal{N})(\mathrm{Im}\mathcal{N})^{-1}\,Q\Big]_\Lambda
\end{aligned} \tag{2.66}
$$

one finds an agreement with earlier result by Ferrara and Kallosh[18].

A comment is in order about the relation between the BPS black hole free energy and the topological free energy [35] which appears naturally in threshold corrections to string effective supergravity. In terms of holomorphic sections Ω, the central charge is given:

$$
\mathbf{Z}(\mathcal{Q}) = e^{K/2}\langle \mathcal{Q}|\Omega\rangle \equiv e^{K/2}\mathcal{M}(\mathcal{Q}); \qquad \Omega = (X^\Lambda, F_\Lambda). \tag{2.67}
$$

Here, $\mathcal{M}$ is the so-called holomorphic mass. We have emphasized that the expression is for a fixed charge vector $\mathcal{Q}$ by denoting the dependence on it explicitly. The topological free-energy is given by

$$e^{-F_{\text{Top.}}} = \text{Det}_{\{\mathcal{Q}\}} e^K \mathcal{M}(\mathcal{Q}) \cdot \mathcal{M}^\dagger(\mathcal{Q}), \tag{2.68}$$

where the determinant is over the fermionic mass matrix. Thus the topological free energy sums up contributions of all *virtual* BPS states, hence, is an infinite sum over the logarithm of the free-energy associated with a single BPS black hole background. In general, however, minima of the topological free energy is distinct from that for the free energy of a single BPS black. Minima of the BPS black hole free energy is given by the Kähler-BPS condition

$$D_A \mathbf{Z} = 0 \quad \rightarrow \quad e^{K/2} \langle \mathcal{Q} | (\partial_A + K_A) \Omega \rangle = 0. \tag{2.69}$$

viz. condition for a Kähler covariantly constant holomorphic mass:

$$\begin{aligned} \mathcal{D}_A \mathcal{M}(\mathcal{Q}) &\equiv (\partial_A + K_A) \mathcal{M}(\mathcal{Q}) = 0, \\ \rightarrow \qquad \partial_A \log \mathcal{M}(\mathcal{Q}) &= -\partial_A K(\mathcal{Q}). \end{aligned} \tag{2.70}$$

On the other hand, the minima of topological free energy is determined by:

$$\partial_A F_{\text{Top.}} = \sum_{\{\mathcal{Q}\}} [\partial_A \log \mathcal{M}(\mathcal{Q}) + \partial_A K] = 0. \tag{2.71}$$

In Eq.(2.71), while the summand equals to the condition Eq.(2.70), it does not necessarily require for each term in the summand to vanish. It is evident that minima of the topological free energy is generically different from that determined by the Kähler-BPS condition.

2.4.2 KÄHLER-BPS CONDITION IN THE SHIFTED BASIS

It is possible to simplify the solution given in Eq.(2.65, 2.66) further by making a symplectic transformation that amounts to the Witten effect [31] and associated shift of the electric charge. One first recalls that the quadratic sum rule of the central charge becomes manifestly a positive-definite quadratic form once the Kähler-BPS condition is satisfied:

$$\begin{aligned} |\mathbf{Z}|^2 &= -\frac{1}{2} \mathcal{Q}^T \cdot \mathbf{M}(\mathcal{F}) \cdot \mathcal{Q}, \\ \mathbf{M}(\mathcal{F}) &= \mathcal{R}^T(\text{Re}\mathcal{F}) \cdot \mathcal{D}(\text{Im}\mathcal{F}) \cdot \mathcal{R}(\text{Re}\mathcal{F}). \end{aligned} \tag{2.72}$$

Define the following new symplectic sections and symplectic charges[5]:

$$\mathbf{V} \equiv \begin{pmatrix} \mathbf{L}^\Lambda \\ \\ \mathbf{M}_\Lambda \end{pmatrix} = \mathcal{R}(\mathrm{Re}\mathcal{F}) \cdot V = \begin{pmatrix} \mathbf{I} & 0 \\ \\ -\mathrm{Re}\mathcal{F} & \mathbf{I} \end{pmatrix} \cdot \begin{pmatrix} L^\Lambda \\ \\ M_\Lambda \end{pmatrix}, \tag{2.73}$$

$$\mathbf{Q} \equiv \begin{pmatrix} \mathbf{p}^\Lambda \\ \\ (\mathrm{Im}\mathcal{F})_{\Lambda\Sigma} \cdot \mathbf{q}^\Sigma \end{pmatrix} = \mathcal{R}(\mathrm{Re}\mathcal{F}) \cdot \mathcal{Q} = \begin{pmatrix} \mathbf{I} & 0 \\ \\ -\mathrm{Re}\mathcal{F} & \mathbf{I} \end{pmatrix} \begin{pmatrix} P^\Lambda \\ \\ Q_\Lambda \end{pmatrix}. \tag{2.74}$$

A few comments are in order for the electric and the magnetic charges in the new shifted basis. First, while the fundamental electric and magnetic charges of the symplectic charge vector $\mathcal{Q} = (P^\Lambda, Q_\Lambda)$ are integer-valued, hence, independent of the special coordinates Z^A, the shifted electric charge $(\mathrm{Im}\mathcal{F}) \cdot \mathbf{q}$ after the symplectic transformation depends on the special coordinates, hence, takes an arbitrary value. This is the manifestation of induced charge effect both due to the screening from $\mathrm{Im}\mathcal{N}$ and due to the Witten effect[6]. The shifted magnetic charge $\mathbf{p}^\Lambda$, however, remains unchanged $\mathbf{p}^\Lambda = P^\Lambda$. Second, the shifted charges $\mathbf{Q} = (\mathbf{p}^\Lambda, \mathbf{q}^\Lambda)$ in the new basis are precisely the ones that appear in the gauge field strength of the BPS black hole solution Eq(2.54), viz. charges that are measured outside the horizon.

One also notes that the constitutive relations for the new symplectic sections is given by[7]

$$\begin{aligned} \mathbf{M}_\Lambda &= [-(\mathrm{Re}\mathcal{F}) \cdot L + M]_\Lambda \\ &= [-(\mathrm{Re}\mathcal{F}) \cdot L + \mathcal{F} \cdot L]_\Lambda \\ &= i\,(\mathrm{Im}\mathcal{F})_{\Lambda\Sigma} \mathbf{L}^\Sigma. \end{aligned} \tag{2.76}$$

For Kähler-BPS states, the quadratic form of the central charge reads in the new shifted basis as:

$$\begin{aligned} |\mathbf{Z}|^2 &= -\frac{1}{2} \mathbf{Q}^T \cdot \mathcal{D}(\mathrm{Im}\mathcal{F}) \cdot \mathbf{Q} \\ &= -\frac{1}{2} \Big[\mathbf{p} \cdot \mathrm{Im}\mathcal{F} \cdot \mathbf{p} + \mathbf{q} \cdot \mathrm{Im}\mathcal{F} \cdot \mathbf{q} \Big] \end{aligned} \tag{2.77}$$

[5]As mentioned at the end of Section2.2, one can define another symplectic transformed vectors and charges in which the holomorphic matrix $\mathcal{F}$ in the following equations is replaced by the coupling matrix $\mathcal{N}$.

[6]This difference is also reflected on the Dirac-Schwinger-Zwanziger quantization condition of $\mathbf{p}, \mathbf{q}$ charges:

$$\mathrm{Im}\mathcal{F}_{\Lambda\Sigma}(\mathbf{p}^\Lambda \mathbf{q}'^\Sigma - \mathbf{p}'^\Lambda \mathbf{q}^\Sigma) = (2\pi\hbar) n. \tag{2.75}$$

[7]In deriving this relations we have used Eq.(2.30) and the fact that $\mathcal{N} \cdot L = \mathcal{F} \cdot L$, which can be checked directly from Eq.(2.33).

Since the Kähler-BPS solution Eq.(2.65) is symplectically covariant, one may simply replace the original sections and charges into the new shifted ones so that the form of equation is unchanged[8] :

$$\mathbf{V} = -\frac{1}{2\overline{\mathbf{Z}}}\Big[\begin{pmatrix} 0 & +\mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix} \cdot \mathcal{D}(\mathrm{Im}\mathcal{F}) + i \begin{pmatrix} +\mathbf{I} & 0 \\ 0 & +\mathbf{I} \end{pmatrix}\Big] \cdot \mathbf{Q}. \tag{2.78}$$

It is easy to check that the $(2n_V+2)\times(2n_V+2)$ matrix inside the bracket on the right hand side of Eq.(2.78) has a rank (n_V+1) only. This is as it should be since the L^Λ and the M_Λ are related each other for a given $(2n_V+2)$ electric and magnetic charges. Therefore, it is enough to solve only half components of Eq.(2.78). Typically since the L^Λ sections are not modified by the shift symplectic transformation, it is more convenient to solve them. In terms of the shifted holomorphic sections, this is easily seen:

$$\begin{aligned} \mathbf{L}^\Lambda &= L^\Lambda = \mathrm{M_{Pl}} e^{K/2} X^\Lambda, \\ \mathbf{M}_\Lambda &= i(\mathrm{Im}\mathcal{F})_{\Lambda\Sigma}\mathbf{L}^\Sigma = i(\mathrm{Im}\mathcal{F})_{\Lambda\Sigma} L^\Sigma. \end{aligned} \tag{2.79}$$

To keep the Einstein-Hilbert term in the $N=2$ supergravity Lagrangian, it is necessary to choose the $X^0 = 1$ gauge. This gauge choice then determines the Kähler-BPS central charge in terms of the Kähler potential once the electric and magnetic charges are specified:

$$-2\mathrm{M_{Pl}}\overline{\mathbf{Z}} e^{K/2} = [i\mathbf{p} + \mathbf{q}]^0. \tag{2.80}$$

Therefore, the Kähler-BPS black hole mass and the macroscopic entropy is given:

$$\mathbf{M}^2_{\mathrm{BPS}} = \mathrm{M}^2_{\mathrm{Pl}}\Big(\frac{\mathrm{S_{BH}}}{\pi}\Big) = \mathrm{M}^2_{\mathrm{Pl}}|\mathbf{Z}|^2 = \frac{1}{4}e^{-K}[(\mathbf{p}^0)^2 + (\mathbf{q}^0)^2]. \tag{2.81}$$

Using the gauge fixing relation Eq.(2.80), the n_V special coordinates that saturate the Kähler-BPS bound can be expressed as a homogeneous rational function of the charges:

$$Z^A = \Big(\frac{\mathbf{L}^A}{\mathbf{L}^0}\Big) = \Big(\frac{X^A}{X^0}\Big) = \frac{[i\mathbf{p}+\mathbf{q}]^A}{[i\mathbf{p}+\mathbf{q}]^0}. \tag{2.82}$$

On the other hand,

$$\Big(\frac{\mathbf{M}_\Lambda}{\mathbf{L}^0}\Big) = -(\mathrm{Re}\mathcal{F})_{\Lambda\Sigma} Z^\Sigma + \Big(\frac{F_\Lambda}{X^0}\Big) = i(\mathrm{Im}\mathcal{F})_{\Lambda\Sigma}\frac{[i\mathbf{p}+\mathbf{q}]^\Sigma}{[i\mathbf{p}+\mathbf{q}]^0}. \tag{2.83}$$

The first equality clearly shows a generalization of the Witten effect. By the exactly same argument given in section 2.1.2 for the rigid $N=2$ case, one may first shift the generalized

[8]One can derive this new, shifted Kähler-BPS solution directly from Eq. (2.65) using Eq.(2.51).

vacuum angle Re$\mathcal{F}$ to zero by the symplectic transformation and minimize the Im$\mathcal{F}$. Subsequently the Re$\mathcal{F}$ may be re-introduced by un-doing the symplectic transformation. As shown in the rigid $N=2$ case, this procedure is essentially equivalent to maintaining the symplectic section M_Λ or, equivalently, the holomorphic section F_Λ unshifted in the new basis defined by the symplectic transformation Eq.(2.73). This method will be adopted when we solve the Kähler-BPS conditions and determine the special coordinates explicitly for the heterotic string theory.

2.5 Example: N=4 Supergravity

To demonstrate the utility of the shifted symplectic basis, consider the $N=4$ supergravity described as the $N=2$ supergravity coupled to two vector fields [19]. The prepotential of vector fields is given by $F=-iX^0X^1$. Calculating the holomorphic matrix $\mathcal{F}_{\Lambda\Sigma}\equiv\partial_\Lambda\partial_\Sigma F(X)$:

$$\mathcal{F}=\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}; \qquad \mathcal{F}^{-1}=\begin{pmatrix} 0 & +i \\ +i & 0 \end{pmatrix}. \tag{2.84}$$

One finds immediately the special coordinate for Kähler-BPS states:

$$Z\equiv\Big(\frac{X^1}{X^0}\Big)=\frac{[\mathbf{p}^1+i\mathbf{q}^1]}{[\mathbf{p}^0+i\mathbf{q}^0]}=\frac{[iP^1-Q_0]}{[iP^0-Q_1]}. \tag{2.85}$$

The Kähler-BPS black hole mass and the entropy is given by

$$\begin{aligned}\mathbf{M}^2_{\rm BH}=\mathrm{M}^2_{\rm Pl}(\frac{\mathrm{S}_{\rm BH}}{\pi}) &= \mathrm{M}^2_{\rm Pl}|\mathbf{Z}|^2=-\frac{1}{2}\mathbf{Q}^T\cdot\begin{pmatrix} \mathrm{Im}\mathcal{F} & 0 \\ 0 & (\mathrm{Im}\mathcal{F})^{-1}\end{pmatrix}\cdot\mathbf{Q} \\ &= [\mathbf{p}^0\mathbf{p}^1+\mathbf{q}^0\mathbf{q}^1]^2 \\ &= [P^0P^1+Q_0Q_1]^2. \end{aligned} \tag{2.86}$$

Applying the following symplectic transformation,

$$\begin{aligned}\hat{\Omega}:\ (X^0,X^1,F_0,F_1) &\quad\rightarrow\quad (\hat{X}^0,\hat{X}^1,\hat{F}_0,\hat{F}_1) \\ (P^0,P^1,Q_0,Q_1) &\quad\rightarrow\quad (\hat{P}^0,\hat{P}^1,\hat{Q}_0,\hat{Q}_1),\end{aligned} \tag{2.87}$$

where

$$\hat{\Omega} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \\ 0 & +1 & 0 & 0 \end{pmatrix}. \tag{2.88}$$

one obtains the $SU(4)$ formulation of the $N=4$ supergravity. Since the matrix $\mathcal{D}(\mathrm{Im}\mathcal{F})$ transforms as

$$\hat{\Omega}: \ \mathcal{D} \quad \rightarrow \quad \hat{\Omega}\cdot\mathcal{D}\cdot\hat{\Omega}^T = \begin{pmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{pmatrix}, \tag{2.89}$$

one finds the Kähler-BPS black hole mass and entropy in the $SU(4)$ basis as:

$$\mathbf{M}_{\mathrm{BH}}^2 = \mathrm{M}_{\mathrm{Pl}}^2(\frac{\mathrm{S}_{BH}}{\pi}) = \mathrm{M}_{\mathrm{Pl}}^2|\mathbf{Z}|^2 = [\hat{\mathbf{p}}^0\hat{\mathbf{q}}^1 - \hat{\mathbf{p}}^1\hat{\mathbf{q}}^0]^2 = [\hat{P}^1\hat{Q}_0 - \hat{Q}_0\hat{P}^1]^2. \tag{2.90}$$

3 HETEROTIC EXTREME BLACK HOLES: CLASSICAL ASPECTS

3.1 Rank-3 Heterotic Compactification: The STU Model

The simplest yet nontrivial $D=4, N=2$ heterotic string vacua is obtained from from further T_2 compactification of the $D=6, N=1$ heterotic string compactified on K_3 with instanton number $(12,12)$ [9]. At generic point in the moduli space, gauge symmetry is completely Higgsed and one is left with supergravity multiplet and one tensor multiplet. Further compactification on T_2 then gives rise to rank-3 gauge groups[10] in four dimensions, of which one is associated with the heterotic dilaton vector multiplet and the other two with the T, U moduli vector multiplets. It is now well-established [9] [27] that this, so-called STU-model is dual to the $D=4, N=2$ type IIA string compactification on a Calabi-Yau three-fold defined by a weighted-projective space $\mathbf{P}_{1,1,2,8,12}(24)$. It is also known [28] to be dual to T_2 compactification of $D=6$ type-I orientifold model on K_3

[9]For extensive study of heterotic K_3 compactification, see [38].

[10]This is the rank of the gauge group not including the gravi-photon.

orbifold $T_4/\mathbf{Z}_2$ with one tensor multiplet and completely Higgs gauge group. On the type IIA side, the classical prepotential in the large Kähler volume limit is given by

$$\begin{aligned} F_I I &\equiv d_{ABC} X^A X^B X^C / X^0 \\ &= (X^0)^2 STU \end{aligned} \tag{3.1}$$

where $d_{ABC} \equiv \int J_A \wedge J_B \wedge J_C$ (J_A's are generators of Kähler cone) denotes the classical intersection numbers and the heterotic special coordinates are used in the last expression. The prepotential displays an explicit STU triality associated with the special Kähler manifold $\mathcal{M}_{SK} = [SU(1,1)/U(1)]_S \times [SO(2,2)/SO(2)\times SO(2)]_{T,U}$. By type IIA-heterotic string-string duality map, one obtains the heterotic STU model, for which the heterotic dilaton S is picked up as a special polarization in the Kähler moduli space. On the heterotic side, it is known that [37] one needs to make a symplectic transformation corresponding to a strong-weak coupling exchange in order to yield a uniform weak coupling behavior as Im$S \to \infty$. This symplectic transformation is defined as:

$$X^1 \to F_1^{(0)}; \qquad F_1^{(0)} \to -X^1. \tag{3.2}$$

In this case, one finds that the holomorphic section is given by

$$\Omega = \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix} = \begin{pmatrix} X^\Lambda \\ S\eta_{\Lambda\Sigma} X^\Sigma \end{pmatrix}, \tag{3.3}$$

where

$$\eta_{\Lambda\Sigma} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \qquad \eta^2 = \mathbf{I}. \tag{3.4}$$

In special coordinates,

$$\begin{aligned} X^\Lambda &= (1, -TU, T, U); \\ F_\Lambda^{(0)} &= (-STU, S, SU, ST). \end{aligned} \tag{3.5}$$

One now finds that the new sections are not mutually independent but are constrained as:

$$\langle X, X\rangle = 0 \qquad \text{where} \qquad \langle A, B\rangle \equiv A^\Lambda \eta_{\Lambda\Sigma} B^\Sigma. \tag{3.6}$$

It is important to realize that this constraint has to be obeyed not only at classical level but also at quantum level. This is because the constraint has stemmed from the geometrical aspect that one of the original sections, X^1, that are needed for parametrizing freely the special Kähler manifold $\mathcal{M}_{SK}$ is lost[11] by the symplectic transformation Eq.(3.2) and from the fact that the heterotic dilaton S, which defines the string-loop counting parameter resides only to the F_Λ sections. One also finds that $\langle F^{(0)}, F^{(0)}\rangle = 0$. However, this relations is not expected to be valid beyond the classical approximation, since the sections F_Λ depend on the heterotic dilaton S already at the classical level, hence, subject to quantum corrections.

The Kähler potential of the heterotic STU model is easily found:

$$\begin{aligned} K &= -\log[i\langle\Omega|\overline{\Omega}\rangle] \\ &= -\log[2\mathrm{Im}S] - \log[\langle X, \overline{X}\rangle] \\ &= -\log[2\,\mathrm{Im}S] - \log[4\,\mathrm{Im}T\,\mathrm{Im}U]. \end{aligned} \tag{3.7}$$

One finds the following Kähler metric at classical level:

$$\begin{aligned} K_{S\overline{S}} &= \frac{1}{4(\mathrm{Im}S)^2}, \\ K_{\Lambda\Sigma} &= -\frac{1}{\langle X, \overline{X}\rangle}\Big[\eta_{\Lambda\Sigma} - \frac{\overline{X}_\Lambda X_\Sigma + \overline{X}_\Sigma X_\Lambda}{2\langle X, \overline{X}\rangle}\Big]; \\ \rightarrow \quad K_{T\overline{T}} &= \frac{1}{4(\mathrm{Im}T)^2}, \qquad K_{U\overline{U}} = \frac{1}{4(\mathrm{Im}U)^2}. \end{aligned} \tag{3.8}$$

One also finds the holomorphic matrix as:

$$\begin{aligned} \mathcal{F}^{(0)}_{\Lambda\Sigma} \equiv \partial_\Lambda F^{(0)}_\Sigma &= S\eta_{\Lambda\Sigma}; & \Big[\frac{1}{\mathcal{F}^{(0)}}\Big]^{\Lambda\Sigma} &= \frac{1}{S}\eta^{\Lambda\Sigma}, \\ (\mathrm{Im}\mathcal{F}^{(0)})_{\Lambda\Sigma} &= (\mathrm{Im}S)\eta_{\Lambda\Sigma}; & \Big[\frac{1}{\mathrm{Im}\mathcal{F}^{(0)}}\Big]^{\Lambda\Sigma} &= (\mathrm{Im}S)^{-1}\eta^{\Lambda\Sigma}. \end{aligned} \tag{3.9}$$

From this one finds the metric $\mathcal{D}(\mathrm{Im}\mathcal{F})$ that defines the quadratic form of the BPS mass spectra Eq.(2.77):

$$\mathcal{D}(\mathrm{Im}\mathcal{F}) = \begin{pmatrix} (\mathrm{Im}S)\,\eta & 0 \\ 0 & (\mathrm{Im}S)^{-1}\,\eta \end{pmatrix}. \tag{3.10}$$

One notes that the (classical) heterotic dilaton $\mathrm{Im}S$ sets a universal coupling parameter to all gauge fields, hence, the BPS mass spectra and the central charge take precisely the same form as those of the rigid $N = 2$ supersymmetric gauge theories studied in section

[11] Because of this $SO(2,2)$ is realized nonlinearly in the new basis.

2.1.3. One thus expects that the Kähler-BPS conditions of the heterotic string theory follow a similar pattern as those in the rigid $N = 2$ theories, Eqs.(2.24, 2.25). In the next section, we will find that this turns out the case.

The T, U special coordinates are the moduli fields that parametrize inequivalent ground state of spontaneously broken rank-2 gauge group. It is known that the $U(1)\times U(1)$ gauge group associated with T, U fields are enhanced to $SU(2) \times U(1)$ at $T = U \neq 1$ complex curve, to $SU(2) \times SU(2)$ at $T = U = 1$ point and to $SU(3)$ at $T = 1/U = \exp(i\pi/6)$ point. Thus, the values of T, U fields away from these enhanced gauge symmetry points and curve can be interpreted as the string counterpart of the vacuum expectation value of Higgs fields in the rigid $N = 2$ gauge theory. In fact, it is known that [36], much as the Higgs expectation value v in the latter theory sets the mass scale of the elementary excitations in the theory such as the heavy charged gauge boson mass $\mathtt{M}_W = gv$, the vacuum expecation values of T, U fields sets the mass scale of the elementary string excitations such as the Kaluza-Klein and the winding string states that arise upon compactification on T_2. Likewise, the mass scale of non-perturbative soliton excitations such as magnetic monopoles and dyons in the rigid $N = 2$ theory and black holes in the local $N = 2$ theory is also determined by the same vacuum expectation values of the Higgs field and of the T, U fields respectively.

3.2 Dynamical Relaxation of Heterotic Black Hole Mass and Entropy

One now solve the classical Kähler-BPS condition:

$$\mathbf{V} = -\frac{1}{2\overline{\mathbf{Z}}}\begin{pmatrix} i\mathbf{I} & (\mathrm{Im}S)^{-1}\cdot\eta \\ -\mathrm{Im}S\cdot\eta & i\mathbf{I}\end{pmatrix}\cdot\mathbf{Q}. \tag{3.11}$$

Expanding the components in terms of holomorphic sections, one gets

$$\begin{aligned} -2\overline{\mathbf{Z}}\mathrm{M}_{\mathrm{Pl}}e^{K/2}X^\Lambda &= [i\mathbf{p}+\mathbf{q}]^\Lambda, \\ -2\overline{\mathbf{Z}}\mathrm{M}_{\mathrm{Pl}}e^{K/2}S\,(\eta\cdot X)_\Lambda &= i\mathrm{Im}S\,[\eta\cdot(i\mathbf{p}+\mathbf{q})]_\Lambda. \end{aligned} \tag{3.12}$$

As explained in section 2.4.2 on a general ground, one finds that the second set of equations in Eq.(3.12) is identical to those of the first set. One now solves the first set of equations to determine the Kähler-BPS configurations of S, T, U fields as well as the BPS black hole mass and entropy.

Since the S-field sets the (classical) heterotic string coupling parameters, one first determine Kähler-BPS configuration of this field. It is determined by the requirement that the Kähler-BPS configurations of the scalars respect the geometric constraint Eq.(3.6):

$$\langle X, X\rangle = 0 \qquad \rightarrow \qquad \langle (i\mathbf{p}+\mathbf{q}), (i\mathbf{p}+\mathbf{q})\rangle = 0. \tag{3.13}$$

Decomposing this constraint into real and imaginary parts, one obtains conditions

$$\langle \mathbf{p}, \mathbf{p}\rangle = \langle \mathbf{q}, \mathbf{q}\rangle, \tag{3.14}$$

$$(\mathbf{p}\cdot\mathbf{q}) = 0. \tag{3.15}$$

Recalling the definition of shifted charged vector $\mathbf{Q} = (\mathbf{p}, \mathbf{q})$ given in Eq.(2.74) and the structure of the classical holomorphic matrix Eq.(3.9), one notes that these two constraints are precisely of the same conditions Eqs.(2.24, 2.25) needed for the minimized the BPS mass spectra in the rigid $N = 2$ gauge theories.

Following exactly the same steps as in the rigid case, one first solves the Eq.(3.15) and determine ReS-field configuration in terms of the microscopic charges:

$$\begin{aligned} (\mathbf{p}\cdot\mathbf{q}) &= (\frac{1}{\mathrm{Im}S})\,[-(\mathrm{Re}S)\langle P, P\rangle + (P\cdot Q)] = 0; \\ \rightarrow \qquad \mathrm{Re}S &= \frac{(P\cdot Q)}{\langle P, P\rangle}. \end{aligned} \tag{3.16}$$

Using this result, one also solves the Eq.(3.14) and obtain the ImS-field configuration as:

$$\begin{aligned} \langle P, P\rangle = \langle \mathbf{p}, \mathbf{p}\rangle &= \langle \mathbf{q}, \mathbf{q}\rangle = \frac{1}{(\mathrm{Im}S)^2}\langle (Q - \frac{(P\cdot Q)}{\langle P, P\rangle}P), (Q - \frac{(P\cdot Q)}{\langle P, P\rangle}P)\rangle; \\ \rightarrow \qquad \mathrm{Im}S &= \sqrt{\frac{\langle Q, Q\rangle}{\langle P, P\rangle} - \frac{(P\cdot Q)}{\langle P, P\rangle}}. \end{aligned} \tag{3.17}$$

Altogether, we have the Kähler-BPS configuration of the S-field:

$$S \equiv [\frac{\theta}{2\pi} + i\frac{4\pi}{g^2}]_{\mathrm{Cl}} = \frac{(P\cdot Q)}{\langle P, P\rangle} + i\sqrt{\frac{\langle Q, Q\rangle}{\langle P, P\rangle} - \frac{(P\cdot Q)}{\langle P, P\rangle}}. \tag{3.18}$$

Kähler-BPS configuration of the moduli fields T, U are obtained as a homogeneous rational functions of charges:

$$T = \Big(\frac{X^2}{X^0}\Big) = \frac{[i\mathbf{p}+\mathbf{q}]^2}{[i\mathbf{p}+\mathbf{q}]^0}, \tag{3.19}$$

$$U = \Big(\frac{X^3}{X^0}\Big) = \frac{[i\mathbf{p}+\mathbf{q}]^3}{[i\mathbf{p}+\mathbf{q}]^0}. \tag{3.20}$$

In addition, one also finds

$$-TU = \Big(\frac{X^1}{X^0}\Big) = \frac{[i\mathbf{p}+\mathbf{q}]^1}{[i\mathbf{p}+\mathbf{q}]^0}. \tag{3.21}$$

That this equals to the product of the two expressions of Eqs.(3.19, 3.20) is guaranteed by the constraint: $\langle X, X\rangle = 0$. One can verify this by direct multiplication of Eqs.(3.19, 3.20) and comparison with Eq.(3.21) using the charge relations Eqs.(3.14, 3.15).

A consistency check of the above Kähler-BPS configuration is provided by deriving the BPS black hole mass and entropy explicitly. Taking an inner product of the Kähler-BPS configuration Eq.(3.12) with a complex-conjugate of itself and using the Eq.(3.15), one obtains

$$4\mathrm{M}_{\mathrm{Pl}}^2 e^K \langle X, \overline{X}\rangle |\mathbf{Z}|^2 = [\langle \mathbf{p}, \mathbf{p}\rangle + \langle \mathbf{q}, \mathbf{q}\rangle]. \tag{3.22}$$

Using the form of the Kähler potential given in Eq.(3.7) one finds the Kähler-BPS black hole mass and the entropy:

$$\begin{aligned} \mathbf{M}_{\mathrm{BPS}}^2 = \mathrm{M}_{\mathrm{Pl}}^2 \Big(\frac{S_{\mathrm{BH}}}{\pi}\Big) = \mathrm{M}_{\mathrm{Pl}}^2 |\mathbf{Z}|^2 &= \frac{1}{2}(\mathrm{Im}\, S)[\langle \mathbf{p}, \mathbf{p}\rangle + \langle \mathbf{q}, \mathbf{q}\rangle] \ge (\mathrm{Im} S)\sqrt{\langle \mathbf{p}, \mathbf{p}\rangle \langle \mathbf{q}, \mathbf{q}\rangle} \\ &= (\mathrm{Im} S)\, \langle \mathbf{p}, \mathbf{p}\rangle = (\mathrm{Im} S)\, \langle P, P\rangle \\ &= \sqrt{\langle P, P\rangle \langle Q, Q\rangle - (P \cdot Q)^2}. \end{aligned} \tag{3.23}$$

At the end of the first line, we have indicated the Cauchy-Schwarz inequality valid generally for the quantity of the preceding expression. To obtain the second line, we have used the condition Eq.(3.14) to the middle expression of the first line. Noting that this equals the value of the last expression in the first line, one concludes that the geometric constraints Eqs.(3.14,3.15) are nothing but the condition for saturating the Cauchy-Schwarz inequality. Note that the condition for saturating the Hölder's inequality Eq.(3.15) has been used already in obtaining Eq.(3.22).

4 HETEROTIC EXTREME BLACK HOLES: QUANTUM ANALYSIS

4.1 Quantum BPS Mass and Dynamical relaxation in Rigid $N = 2$ Theory

4.1.1 RIGID $N = 2$ NON-RENORMALIZATION THEOREMS

We first recall the known results for the non-renormalization theorems in rigid $N = 2$ supersymmetric gauge theories [40] [12] The one-loop radiative correction is most straightforwardly calculated in the background field method by evaluating determinants of Gaussian fluctuations around a fixed background of bosons, fermions and Faddeev-Popov ghosts. In terms of $N = 1$ supermultiplets, denoting the wave function renormalizations of the

[12] The results were subsequently extended to coupling to supergravity [40] with consistent regularization and renormalization prescriptions [41].

vector field, the scalar fields in the adjoint representation and chiral matter fields in the complex-conjugate pair representations as $\mathtt{Z}_V, \mathtt{Z}_A, \mathtt{Z}_Q, \mathtt{Z}_{\overline{Q}}$, the gauge coupling renormalization as $\mathtt{Z}_g$ and the mass of adjoint scalar and complex-conjugate pair scalars as $\mathtt{M}_W$ and $\mathtt{M}_Q$, it was found [40] that they are related each other as:

$$
\begin{aligned}
\mathtt{Z}_g \cdot [\mathtt{Z}_V]^{\frac{1}{2}} &= 1, \\
\mathtt{Z}_g \cdot \mathtt{Z}_Q^{\frac{1}{2}} \cdot \mathtt{Z}_{\overline{Q}}^{\frac{1}{2}} \mathtt{Z}_A &= 1, \\
\mathtt{Z}_Q^{\frac{1}{2}} \mathtt{Z}_{\overline{Q}}^{\frac{1}{2}} [\mathtt{M}_Q]_{\mathrm{Bare}} &= [\mathtt{M}_Q]_{\mathrm{Ren}}.
\end{aligned} \tag{4.1}
$$

For $N = 2$ supersymmetric gauge theories, the requisite $N = 2$ vector and hyper multiplet structures impose two conditions $\mathtt{Z}_V = \mathtt{Z}_A$ and $\mathtt{Z}_Q = \mathtt{Z}_{\overline{Q}}$ respectively. Therefore, one finds the following two $N = 2$ nonrenormalization theorems:

$$
\mathtt{Z}_g \cdot \mathtt{Z}_V^{1/2} = 1 \qquad \rightarrow \qquad [\mathtt{M}_W]_{\mathrm{Ren}} = [\mathtt{M}_W]_{\mathrm{Bare}}, \tag{4.2}
$$

$$
\mathtt{Z}_Q \cdot \mathtt{Z}_{\overline{Q}} = 1 \qquad \rightarrow \qquad [\mathtt{M}_Q]_{\mathrm{Ren}} = [\mathtt{M}_Q]_{\mathrm{Bare}}, \tag{4.3}
$$

protecting the spectra of the massive vector and hyper multiplets from renormalization. While the non-renormalization theorem was derived for the logarithmic corrections, it should hold also for finite renormalizations including threshold corrections due to heavy particles.

Similar non-renormalization theorem applies to the $N = 2$ BPS monopoles and dyons. One might naively expect that the BPS masses are not renormalized at all since the BPS spectra is determined by the central charge that has a topological origin, see Eq.(2.16). A heuristic argument was that the Gaussian fluctuation spectra of bosons and fermions around any supersymmetric configuration are equal each other by supersymmetry, hence, a sum over the fluctuation energy cancels out between the bosonic and the fermionic contributions [42, 43]. However, for $N = 2$ supersymmetric theories, this turned out not to be the case [44]. A subtle but important point was that the Gaussian fluctuations around the BPS soliton configuration contain not only discrete, bound states but also a continuum scattering states. Thus, schematically, quantum correction to the BPS soliton mass is given by:

$$
[\Delta \mathbf{M}]_{1-\mathrm{loop}} = \sum\!\!\!\!\!\!\int \hbar \Omega_{\mathrm{boson}} - \frac{1}{2} \sum\!\!\!\!\!\!\int \hbar \Omega_{\mathrm{fermion}} - \frac{1}{2} \sum\!\!\!\!\!\!\int \hbar \Omega_{\mathrm{ghost}}. \tag{4.4}
$$

For the continuum contributions, while bosonic and fermionic spectra are always paired as dictated by the supersymmetry, the density of states turns out not equal for the bosonic and the fermionic parts [13] Explicit one-loop calculations for $N = 2$ BPS monopole have shown that the BPS mass receives a logarithmic radiative correction

$$[\mathbf{M}_{\rm BPS}]_{1-{\rm loop}} = [\mathbf{M}_{\rm BPS}]_{\rm bare}\Big[1 - (\frac{g^2}{4\pi})\frac{\hbar}{\pi}\,\log(\frac{2\Lambda}{M_W})^2\Big] \tag{4.5}$$

where $[\mathbf{M}_{\rm BPS}]_{\rm bare} = 4\pi v/g$ and $\mathtt{M}_W = gv$ denotes the heavy charged gauge boson mass. At same order, one also finds one-loop radiative correction to the gauge coupling constant

$$\Big(\frac{4\pi}{g^2}\Big)_{1-{\rm loop}} = \Big(\frac{4\pi}{g^2}\Big)_{\rm bare}\Big[1 - (\frac{g^2}{4\pi})\,\frac{\hbar}{\pi}\,\log(\frac{2\Lambda}{M_W})^2\Big]. \tag{4.6}$$

Comparing Eq.(4.5) with Eq.(4.6) one finds that radiative correction to the BPS monopole mass is entirely due the radiative correction to the gauge coupling constant that the BPS monopole mass depends on. One now use the $N = 2$ non-renormalization theorem Eq.(4.2) and re-express the quantum BPS monopole mass as:

$$\begin{aligned} [\mathbf{M}_{\rm BPS}]_{\rm ren} &= \Big(\frac{4\pi v}{g}\Big)_{\rm ren} \\ &= \frac{[g\,v]_{\rm ren}}{\alpha_{\rm g,ren}} = \frac{[g\,v]_{\rm bare}}{\alpha_{\rm g,ren}} \\ &= \frac{\mathtt{M}_W}{\alpha_{\rm g,ren}}. \end{aligned} \tag{4.7}$$

The above analysis indicates that, while the $N = 2$ BPS spectra receives nontrivial radiative corrections, the BPS mass spectra takes exactly the same form once all the quantities are expressed in terms of renormalized physical quantities. In particular, if there are no massless charged states in the elementary spectra, then the renormalization effect will be dominated by threshold corrections. It is in this case that the notion of BPS states as those of balancing long-distance static force between them retains a well-defined meaning at quantum level. All the short-distance details are summarized into the renormalization effects to physical parameters and the renormalization effects are exponentially suppressed at a distance larger than the typical Compton wavelength of the heavy charged states.

It is then a natural question to what extent the above results for the quantum BPS spectra for rigid supersymmetric case extends to hold once they are embedded into supergravity theory.

We now turn to this question by embedding the gauge theories into heterotic string theory.

[13]The BPS monopole mass of $N = 4$ supersymmetric gauge theory is not renormalized since the difference of density of states between bosons and fermions turns out to vanish identically for the $N = 4$ multiplet spectra.

4.2 Perturbative Quantum Effect: Heterotic STU Model

Consider perturbative string-loop corrections to the Kähler-BPS black hole configuration, mass and entropy formula of the heterotic STU model studied in section 3.2. In this section, we study the perturbative string-loop effects to the classical Kähler-BPS black hole by solving the Kähler-BPS conditions in which the holomorphic matrix $\mathcal{F}$ is derived from quantum-corrected holomorphic sections [45, 46, 47, 48, 49]. While this procedure sounds right, one needs to have a careful thought for its consistency. In general perturbative renormalization around a black hole background shows quite different structure from that around a flat spacetime. In calculating perturbative corrections to physical quantities such as BPS black hole mass and entropy, one thus needs first to evaluate quantum corrections to the holomorphic sections around the black hole background and then to use it for solving the Kähler-BPS conditions. When deriving perturbative corrections to the holomorphic sections, Refs. [45]–[48] have assumed a flat spacetime, vanishing gauge field strengths and constant scalar fields[14]. The Kähler-BPS black hole configuration is rather special in this aspect. It is distinguished from other BPS black holes by the fact that the scalar fields are constant everywhere outside the horizon. Therefore, so long as one restricts to a macroscopic Kähler-BPS black hole, whose horizon is large enough that spatial curvature and gauge field strengths at black hole exterior are small enough, one may use the quantum-corrected holomorphic sections of Ref. [47] when solving the Kahler-BPS conditions. In this section, we will adopt this strategy and *a posteriori* justify this procedure by showing that the scalar fields remain constant everywhere, viz. the Kähler-BPS bound is saturated at quantum level.

4.2.1 PERTURBATIVE CORRECTIONS

Perturbative corrections to the heterotic prepotential has been calculated by a direct threshold calculation and by mapping the classical prepotential of type IIA string compactifications on Calabi-Yau three-folds [50] to the heterotic side using the type IIA-heterotic string duality [9]. Under the string duality, the type IIA worldsheeet instanton corrections to the type IIA prepotential is mapped to the spacetime instanton corrections that depend on heterotic S-field and to the worldsheet instanton corrections. The $N=2$ supersymmetry non-renormalization theorem guarantees that the quantum effects on the heterotic side comes from one-loop and from non-perturbative effects. In this section, we study the perturbative one-loop correction exclusively. The non-perturbative corrections

[14] With a notable exception of Ref. [49], where non-vanishing but constant background fields were considered.

will be discussed in the next section. The non-renormalization theorem dictates that the perturbative corrections to the prepotential and to the holomorphic section depends only on T, U moduli fields. In the large $\mathrm{Im}T, \mathrm{Im}U$ limit, the heterotic STU model prepotential contains cubic polynomials in T, U, constant term and exponentially suppressed $\mathcal{O}(e^{2\pi i T}, e^{2\pi i U})$ worldsheet instanton correction terms. The direct calculations [47, 45, 46] have shown that the target-space duality symmetry $SL(2, \mathbf{Z})_T \times SL(2, \mathbf{Z})_U \times \mathbf{Z}_2^{T\leftrightarrow U}$ acting on the moduli T and U constrains strongly these one-loop corrections. Only third derivatives of $F^{(P)}$ with respect to T and U fields transform as well-defined modular forms under the modular group $SL(2, \mathbf{Z})_T \times SL(2, \mathbf{Z})_U$. Integrating back to obtain $F^{(P)}$ then poses a quadratic ambiguity that amounts to redefinition ambiguity of the dilaton $S \to S + \alpha T + \beta U$, where α, β are arbitrary parameters. Comparing with the tree-level prepotential, one finds that this redefinition gives rise to an ambiguity to the perturbative part of the prepotential $F^{(P)} \to F^{(P)} + \alpha T^2 U + \beta T U^2$. By adopting a convention[15] that the invariant dilaton is the same as the special coordinate dilaton, viz. $\alpha = \beta = 0$, one finds that the perturbative correction to the prepotential is given by

$$
\begin{aligned}
F^{(P)}(T, U) = \quad & (X^0)^2 (T^2 U + \frac{1}{3} U^3)\, \theta(\mathrm{Im}T - \mathrm{Im}U) \\
+ \quad & (X^0)^2 (U^2 T + \frac{1}{3} T^3)\, \theta(\mathrm{Im}U - \mathrm{Im}T) \\
+ \quad & 240 \cdot \frac{\zeta(3)}{(2\pi)^3} + \mathcal{O}(e^{2i\pi T}, e^{2i\pi U}).
\end{aligned} \tag{4.8}
$$

The first and the second lines are the S-field redefinition ambiguity-free, cubic polynomial part of the prepotential. It, however, depends on the Weyl chamber divided at $\mathrm{Im}T = \mathrm{Im}U$ that is symmetric under $\mathbf{Z}_2^{T\leftrightarrow U}$ exchange. The third line denotes constant part, which depends on the famous $\zeta(3)$ and the Euler number $\chi(CY) = -480$ of the corresponding Calabi-Yau space and contributions of exponentially suppressed worldsheet instanton effects. It is important to note that, at quantum level, the heterotic S-field is a special coordinates but is not invariant under the target–space duality. As we will see, however, so long as the nonperturbative quantum corrections are ignored, the Kähler-BPS configuration of the S field remains target-space duality invariant. In fact, because of this, we have chosen the quadratic ambiguity of the prepotential in the simplest manner, viz. the perturbative S-field is the same as the classical one. On the other hand, it is *not* the S or S_{inv} that organizes the perturbation expansions, but the

[15] Alternative possible choice is $\alpha = 0, \beta = -1$ [51] This choice ensures the $S \leftrightarrow T$ exchange symmetry [52] present in the theory. We are motivated to choose the above convention as the Kähler-BPS configuration of the heterotic S-field turns out to be invariant under the target-space duality perturbatively. Different choices, however, should be all physically equivalent.

one including threshold corrections. It is possible to define a renormalized dilaton, which is also invariant under the target–space duality [45, 47]. In what follows, since we are mainly in the weak coupling limit, we will start with the special coordinate dilaton, S.

Holomorphic sections including the perturbative corrections are given by

$$\begin{aligned}
X^\Lambda &= (X^0, X^1, X^2, X^3) = (1 - TU, T, U), \\
F_\Lambda &= F_\Lambda^{(0)} + F_\Lambda^{(P)}, \\
F^{(P)} &\equiv (-\frac{1}{3}U^3 - T^2U,\ 0\ , 2TU\ , T^2 + U^2).
\end{aligned} \tag{4.9}$$

Quantum correction to the holomorphic coupling matrix is easily calculated. The explicit change of it is given by

$$\mathcal{F}^{(P)}_{\Lambda\Sigma} \equiv \partial_\Lambda F^{(P)}_\Sigma = \begin{pmatrix} \frac{2}{3}U^3 + 2T^2U & 0 & -2TU & -T^2 - U^2 \\ 0 & 0 & 0 & 0 \\ -2TU & 0 & +2U & +2T \\ -T^2 - U^2 & 0 & +2T & +2U \end{pmatrix}. \tag{4.10}$$

One first notes that the entries in the second rows and columns are zero to any finite orders in perturbation theory. Later, in identifying the perturbative quantum corrections to the black hole BPS mass and the entropy, this observation will play an important role. One can also verify that the holomorphic matrix $\mathcal{F}^{(P)}$ satisfies the homogeneous degree-two condition: $\partial_{(\Delta}\mathcal{F}^{(P)}_{\Lambda\Sigma)}X^\Lambda = 0$.

The quantum corrected Kähler potential is easily derived:

$$\begin{aligned}
K &= -\log\left[X^\Lambda \overline{F}_\Lambda - \overline{X}^\Lambda F_\Lambda\right] \\
&\equiv [K_S + \hat{K}(T,U)]_{\rm Pert} \\
K_S^{\rm Pert} &= -\log[2({\rm Im}S + V_{GS}(T,U))] \\
\hat{K}^{\rm Pert} &= -\log[i\langle X, \overline{X}\rangle] = \log[2{\rm Im}T \cdot 2{\rm Im}U].
\end{aligned} \tag{4.11}$$

One first notes that the Kähler potential of the T, U moduli fields is not modified at all by perturbative quantum effects. This stems from the fact that the holomorphic sections X^Λ does not contain the heterotic dilaton S. This in turn ensures that the geometric nonlinear constraint

$$\langle X, X\rangle_{\rm Pert} = 0. \tag{4.12}$$

remains valid to all orders in perturbation theory. All the one-loop quantum correction then goes to the so-called Green-Schwarz term V_{GS} in the Kähler potential of the heterotic S-field. This correction comes from universal threshold corrections for the heterotic rank-3 model:

$$
\begin{aligned}
V_{GS}(T,U) &= i\frac{[2-\mathrm{Im}T\partial_{\mathrm{Im}T}-\mathrm{Im}U\partial_{\mathrm{Im}U}](F^{(P)}+\overline{F}^{(P)})}{4\mathrm{Im}T\mathrm{Im}U} \\
&= -\mathrm{Im}T-\frac{\frac{1}{3}(\mathrm{Im}U)^2}{\mathrm{ImT}}+240\frac{\zeta(3)}{(2\pi)^3}\frac{1}{(\mathrm{Im}T\mathrm{Im}U)}.
\end{aligned}
\tag{4.13}
$$

While the classical heterotic string coupling is governed by the special coordinate S, at perturbative level, the heterotic string-loop expansion parameter is a combination of dilaton and and Green-Schwarz term [45]:

$$
\Big[\frac{4\pi}{g^2_{\mathrm{het}}}\Big]_{\mathrm{Pert.}} = \mathrm{Im}S + V^{\mathrm{GS}} \equiv \mathrm{Im}S_{\mathrm{inv}} + V^{\mathrm{GS}}_{\mathrm{inv}}. \tag{4.14}
$$

4.2.2 DYNAMICAL RELAXATION FOR QUANTUM BPS BLACK HOLE

One now solves the Kähler-BPS conditions including the perturbative quantum corrections

$$
\mathrm{M}_{\mathrm{Pl}}e^{K/2}\begin{pmatrix} X^\Lambda \\ \\ F_\Lambda \end{pmatrix} = -\frac{1}{2\overline{\mathbf{Z}}}\begin{pmatrix} i\mathbf{I} & (\mathrm{Im}\mathcal{F})^{-1} \\ \\ -\mathrm{Im}\mathcal{F} & i\mathbf{I} \end{pmatrix}\cdot\mathbf{Q}. \tag{4.15}
$$

It turns out, as in the classical case, that the geometric nonlinear constraint Eq.(4.12) plays an important role. Inserting Eq.(4.15) into Eq.(4.12), one finds:

$$
\langle \mathbf{p},\mathbf{p}\rangle_{\mathrm{Pert}} = \langle \mathbf{q},\mathbf{q}\rangle_{\mathrm{Pert}} \tag{4.16}
$$

$$
(\mathbf{p}\cdot\mathbf{q})_{\mathrm{Pert}} = 0. \tag{4.17}
$$

Note that, even though structure of the constraint looks the same as in the classical case, each components of the charge vector components are expected to receive quantum corrections. Recall that $\mathbf{P}=P$ denotes the microscopic, integer-valued magnetic charges, hence, do not change by quantum effects. The $\mathbf{q}$ charges, however, may be modified, since it depends on the holomorphic matrix and scalar fields. For the STU model, there are four components of $\mathbf{q}$. Imposing the two constraints Eqs.(4.16, 4.17) leaves two free components of $\mathbf{q}$ that can adjust as quantum effects are included. The argument clearly indicates nontrivial quantum effects to the Kähler-BPS configuration.

Quantum effects to the black hole mass and entropy arise in two possible ways. One is through *explicit* change of the Kähler potential K_S^{Pert} by the Green-Schwarz term.

Another is through *implicit* functional change of the scalar fields that depend on the charges $\mathbf{q}$ for Kähler-BPS configurations. Note that we have shown that components of $\mathbf{q}$ charges are not protected at all from the quantum corrections.

4.2.3 PERTURBATIVE NON-RENORMALIZATION THEOREM OF BLACK-HOLE MASS & ENTROPY

We now establish perturbative non-renormalization of the Kähler-BPS black hole mass and entropy. Before doing so, we first show that there is no implicit functional shift of the heterotic dilaton S-field from the classical configuration. From Eq.(4.15), one finds quantum corrections to the ImS:

$$\Delta(\mathrm{Im}S) = \Delta\Big[(\mathrm{Im}\mathcal{F})_{1\Sigma}\,\frac{[i\mathbf{p}+\mathbf{q}]^{\Sigma}}{[i\mathbf{p}+\mathbf{q}]^{0}}\Big]. \tag{4.18}$$

On the right hand side, the quantum corrections arise both from an explicit change of the first factor by Im$\mathcal{F}^{(P)}$ and from an implicit change of the $\mathbf{q}$ charge components. One first notes that the first row and column of the perturbative holomorphic matrix $\mathcal{F}^{(P)}$ in Eq.(4.10) vanish identically. While it was shown for large T, U limit, this holds true at finite T, U as well[16] and merely reflects the fact that the perturbative corrections are independent of the heterotic dilaton S. The observation leads to a conclusion that there is no explicit quantum correction from the first factor in Eq.(4.18). Taking the classical Kähler-BPS configuration for the first factor, one then also finds that the charge ratios in the second factor cancel out, hence, no implicit changes. A symplectic transformation that shifts the vacuum angle leads to the conclusion that ReS is not renormalized either. This completes the proof that the Kähler-BPS configuration of the heterotic dilaton S-field is not renormalized to all orders in perturbation theory.

We now establish the afore-mentioned non-renormalization theorem. Following exactly the same steps as in the classical analysis in Eq.(3.22), consider the inner product of the perturbative sections X^Λ that satisfy Eq.(4.15) with its complex-conjugate. This yields:

$$4\mathrm{M}_{\mathrm{Pl}}^2\,[e^{K}\,|\mathbf{Z}|^2\,\langle X,\overline{X}\rangle\,]_{\mathrm{Pert}} = [\langle\mathbf{p},\mathbf{p}\rangle + \langle\mathbf{q},\mathbf{q}\rangle]_{\mathrm{Pert}}, \tag{4.19}$$

hence,

$$\mathrm{M}_{\mathrm{Pl}}^2|\mathbf{Z}|^2_{\mathrm{Pert}} = \frac{1}{4}\Big([e^{-K_S}]\,[e^{\hat{K}}\langle X,\overline{X}\rangle]^{-1}[\langle\mathbf{p},\mathbf{p}\rangle + \langle\mathbf{q},\mathbf{q}\rangle]\Big)_{\mathrm{Pert}} \tag{4.20}$$

One now analyze quantum corrections to the right hand side of Eq.(4.20). Since the form of the Kähler potential $\hat{K}$ and the holomorphic sections X^Λ are not changed perturba-

[16] Note that the constant and the infinite worldsheet instanton expansion terms also depends only on T, U fields.

tively, the second factor is not renormalized and remains unity. The third factor is not renormalized either since

$$\Delta(\langle \mathbf{p},\mathbf{p}\rangle + \langle \mathbf{q},\mathbf{q}\rangle)_{\rm Pert} = 2\Delta\langle \mathbf{p},\mathbf{p}\rangle_{\rm Pert} = 2\Delta\langle P, P\rangle = 0. \tag{4.21}$$

In the first equality, the perturbative nonlinear constraint Eq.(4.16) is used. The second equality follows from the fact that $\mathbf{p}$ equals to the microscopic, integer-valued magnetic charge P. This charge cannot jump by quantum effects. Since ImS is not renormalized as shown above, the only perturbative correction to the Eq.(4.20) comes from the explicit dependence through the Green-Schwarz term in e^{-K_S}. One thus concludes that:

$$|\mathbf{Z}|^2_{\rm Pert} = \exp[-(K_S^{\rm Pert.} - K_S^{\rm Cl})]\cdot|\mathbf{Z}|^2_{\rm Cl} = \frac{({\rm Im}S + {\rm V}^{\rm GS})_{\rm Pert}}{({\rm Im}S)_{\rm Cl}}\cdot|\mathbf{Z}|^2_{\rm Cl}. \tag{4.22}$$

The perturbative BPS black hole mass and the entropy formula is then obtained from Eq.(4.22) as [17] :

$$\begin{aligned} [\mathbf{M}^2_{\rm BH}]_{\rm Pert} = \Big(\frac{S_{\rm BH}}{\pi}\Big)_{\rm Pert} = {\rm M}^2_{\rm Pl}|\mathbf{Z}|^2_{\rm Pert} &= \Big({\rm Im}S + V^{\rm GS}({\rm Im}T, {\rm Im}U)\Big)\ \langle P, P\rangle \\ &= \Big(\frac{4\pi}{g^2_{\rm het}}\Big)_{\rm Pert}\ \langle P, P\rangle \end{aligned} \tag{4.24}$$

where Eq.(4.14) was used. Note that, nowhere in deriving the non-renormalization theorem and Eq.(4.24), specific details of the perturbative correction to the prepotential were assumed except that the correction should be independent of S. This is a property for any $N = 2$ heterotic string compactifications should satisfy, hence, the perturbative black hole mass and entropy formula Eq.(4.24) is expected to be valid for any $N = 2$ heterotic string compactifications[18] In particular, for the heterotic STU model, the formula Eq.(4.24) is valid not only for large T, U limit but also for finite T, U points so long as one stays away from the enhanced gauge symmetry points. Recall that the Green-Schwarz term originates from the universal threshold corrections of heavy Kaluza-Klein

[17]Recently the authors of [21] have made a conjecture for the perturbatively corrected black hole mass/entropy formula. Our result disagrees with theirs. That the formula we have obtained should be the correct one can be understood simply in the following way. It is known that the combination $({\rm Im}S + V_{Gs})$ equals to the combination of so-called invariant dilaton and the invariant Green-Schwarz term

$${\rm Im}S + {\rm V}^{\rm GS} = {\rm Im}S_{\rm inv} + {\rm V}^{\rm GS}_{\rm inv}. \tag{4.23}$$

This combination defines heterotic string-loop expansion parameter with manifest target-space duality invariance. Therefore it is natural to expect for this combination to appear in physical quantities such as BPS black hole mass and entropy once evaluated in a consistent perturbative expansion.

[18]This should also hold, in particular, for heterotic compactifications which do not have known type-IIA side duals.

and winding states, that are charged under the $U(1)\times U(1)$ gauge group associated with the T, U special coordinates. Since there are no massless charged fields coupled to these gauge fields , the quantum correction is entirely summarized by the threshold corrections only.

It should be emphasized that it was the target-space duality symmetry that guaranteed for the non-renormalization theorem to hold. Both the fact that the Kähler potential $\hat{K}$ was not modified by perturbative effect and the fact that the nonlinear constraint $\langle X, X\rangle = 0$, from which the two important constraint Eqs.(4.16, 4.17) were derived, were the two ingredients in deducing the non-renormalization theorem from Eq.(4.20).

Actually, there exist finer structures to the non-renormalization theorem. In deriving the perturbative central charge Eq.(4.20), we have used the special aspect of T_2 compactification of the heterotic string. In $X^0 = 1$ gauge, in general, the central charge was obtained from the gauge fixing itself, see Eq.(2.81). The same procedure with perturbative corrections for the heterotic STU model yields:

$$\begin{aligned} \mathrm{M}_{\mathrm{Pl}}^2 |\mathbf{Z}|^2_{\mathrm{Pert}} &= \frac{1}{4} e^{-K}\Big[(\mathbf{p}^0)^2 + (\mathbf{q}^0)^2\Big] \\ &= \frac{1}{2}(\mathrm{Im}S + V^{GS}(T,U))\cdot \Big[e^{-\hat{K}(T,U)}\Big((\mathbf{p}^0)^2 + (\mathbf{q}^0)^2\Big)\Big]. \end{aligned} \tag{4.25}$$

Comparing this expression with Eq.(4.20), one finds a relation:

$$[(\mathbf{p})^2 + (\mathbf{q})^2] = e^{-\hat{K}}[(\mathbf{p}^0)^2 + (\mathbf{q}^0)^2]. \tag{4.26}$$

In Eq.(4.21), it was argued that the left hand side is not renormalized. Hence, on the right hand side of Eq.(4.26), renormalization of the two terms should cancel each other. The T, U moduli fields, in general, receives nontrivial renormalizations, as is evident from the $\mathcal{F}$ dependence of Eq.(4.15):

$$\begin{aligned} T &= \Big(\frac{X^2}{X^0}\Big)_{\mathrm{Pert}} = \Big(\frac{[i\mathbf{p}+\mathbf{q}]^2}{[i\mathbf{p}+\mathbf{q}]^0}\Big)_{\mathrm{Pert}}, \\ U &= \Big(\frac{X^3}{X^0}\Big)_{\mathrm{Pert}} = \Big(\frac{[i\mathbf{p}+\mathbf{q}]^3}{[i\mathbf{p}+\mathbf{q}]^0}\Big)_{\mathrm{Pert}}. \end{aligned} \tag{4.27}$$

Such renormalizations of the T, U moduli fields are string counterparts of those of the Higgs vacuum expectation values in the rigid $N = 2$ supersymmetric gauge theories. Despite the fact that T, U moduli fields are renormalized, we now show that the non-renormalization theorem is stronger enough that the two factors on the right hand side of Eq.(4.26) are not renormalized separately. Again, it turns out the target-space duality symmetry is to ensure their non-renormalizations.

4.2.4 PERTURBATIVE NON-RENORMALIZATION OF $[(\mathbf{p}^0)^2 + (\mathbf{q}^0)^2]$

Recall that the macroscopic charges are related to the microscopic charges as:

$$\begin{aligned}
\mathbf{p}^\Lambda &= P^\Lambda \\
\mathbf{q}^\Lambda &= [(\mathrm{Im}\mathcal{F})^{-1}]^{\Lambda\Sigma}\,[-\mathrm{Re}\mathcal{F}\cdot P + Q]_\Sigma.
\end{aligned} \tag{4.28}$$

Since P^Λ's are integer-valued, one finds:

$$\Delta\Big([(\mathbf{p}^0)^2 + (\mathbf{q}^0)^2]\Big) = \Delta(\mathbf{q}^0)^2. \tag{4.29}$$

The quantum corrections on the right hand side come in through explicit modification of the holomorphic matrix $\mathcal{F}$ and implicit modification of the moduli fields. Using the matrix identity

$$\frac{1}{\mathrm{Im}\mathcal{F}} = \frac{1}{\mathrm{Im}\mathcal{F}^{(0)} + \mathrm{Im}\mathcal{F}^{(P)}} \equiv \sum_{n=0}^{\infty} \frac{(-)^n}{\mathrm{Im}\mathcal{F}^{(0)}}\Big(\mathrm{Im}\mathcal{F}^{(P)}\frac{1}{\mathrm{Im}\mathcal{F}^{(0)}}\Big)^n, \tag{4.30}$$

the structure of the classical holomorphic matrix:

$$\Big[\frac{1}{\mathrm{Im}\mathcal{F}^{(0)}}\Big]^{\Lambda\Sigma} = \Big(\frac{1}{\mathrm{Im}S}\Big)\eta^{\Lambda\Sigma}, \tag{4.31}$$

and the fact the first row and column of the quantum correction part of the holomorphic matrix $\mathcal{F}^{(P)}$ are zero, one finds that all higher-order ($n \geq 1$) terms of $\Lambda = 0$ component in Eq.(4.30) vanish identically. Since $(\mathrm{Re}\mathcal{F}^{(P)})_{1\Sigma} = 0$ as well, one concludes that

$$\Delta(\mathbf{q}^0) = \Delta\Big((\frac{1}{\mathrm{Im}\mathcal{F}})^{0\Sigma}\cdot[-(\mathrm{Re}\mathcal{F})\cdot P + Q]_\Sigma\Big) = 0 \tag{4.32}$$

In the previous subsection, the $\mathrm{Im}S$, which defines the $n = 0$ classical term in $\mathbf{q}^0$, does not receive quantum corrections. One thus concludes that $((\mathbf{p}^0)^2 + (\mathbf{q}^0)^2)$ is not renormalized to all orders in perturbation theory.

4.2.5 PERTURBATIVE NON-RENORMALIZATION OF $\exp(-\hat{K})$

We now examine $e^{-\hat{K}} = 4\mathrm{Im}T\,\mathrm{Im}U$. From Eq.(4.27), one obtains

$$\begin{aligned}
\mathrm{Im}T &= \frac{[\mathbf{p}^2\mathbf{q}^0 - \mathbf{p}^0\mathbf{q}^2]}{[(\mathbf{p}^0)^2 + (\mathbf{q}^0)^2]}, \\
\mathrm{Im}U &= \frac{[\mathbf{p}^3\mathbf{q}^0 - \mathbf{p}^0\mathbf{q}^3]}{[(\mathbf{p}^0)^2 + (\mathbf{q}^0)^2]}.
\end{aligned} \tag{4.33}$$

The previous non-renormalization theorem guarantees that the denominators are not renormalized. It now remains whether the numerators are renormalized. From the non-vanishing entries of $\mathcal{F}^{(P)}$, it should be clear that individual numerators receive quantum corrections. This is not surprising since $\mathrm{Im}T, \mathrm{Im}U$ are the string counterpart of the vacuum expectation values of the scalar fields in the rigid theory. Surprising enough, we now show that the renormalization effects cancel each other so that the product $\mathrm{Im}T \cdot \mathrm{Im}U$, hence, the Kähler potential $\hat{K}$ is not renormalized at all. This is another manifestation of the target-space duality symmetry.

Consider the quantum correction to the product of the two numerators of Eq.(4.33):

$$\Delta\Big(\mathbf{p}^2\mathbf{p}^3(\mathbf{q}^0)^2 - \mathbf{p}^0\mathbf{p}^2\mathbf{q}^0\mathbf{q}^3 - \mathbf{p}^0\mathbf{p}^3\mathbf{q}^0\mathbf{q}^2 + (\mathbf{p}^0)^2\mathbf{q}^2\mathbf{q}^3\Big). \tag{4.34}$$

Since $\mathbf{p}^\Lambda = P^\Lambda$ are moduli-independent, integer-valued and $\mathbf{q}^0$ charges are not renormalized perturbatively, we focus only on $\mathbf{q}^{1,2,3}$ charges. The constraint $\langle X, X\rangle_{\rm Pert} = 0$ puts two conditions:

$$\begin{aligned} \mathbf{q}^0\Delta\mathbf{q}^1 + \Delta(\mathbf{q}^2\mathbf{q}^3) &= 0, \\ \mathbf{p}^0\Delta\mathbf{q}^1 + \mathbf{p}^2\Delta\mathbf{q}^3 + \mathbf{p}^3\Delta\mathbf{q}^2 &= 0. \end{aligned} \tag{4.35}$$

Using the two constraints, it is straightforward to check that Eq.(4.34) vanishes identically. This establishes the perturbative non-renormalization theorem of the Kähler potential $\hat{K}$, a product of two Higgs expectation values, as protected by the target-space duality[19].

4.3 Nonperturbative Quantum Corrections

So far, we have ignored possible nonperturbative effects such as corrections due to space-time gauge or gravitational instantons. Such instanton effects give rise to holomorphic but exponentially suppressed corrections to the holomorphic sections. At finite $\mathrm{Im}S$, such non-perturbative effects are intractibly complicated. In this section, we limit ourselves to the weak coupling limit,

$$\mathrm{Im}S \gg \mathrm{Im}T \gg \mathrm{Im}U \gg 1 \tag{4.36}$$

and calculates the leading–order non–perturbative corrections.

The classical Peccei-Quinn symmetry constrains the nonperturbative correction to the prepotential to be of a form:

$$\mathcal{F} = (X^0)^2[STU + F^{(P)}(T,U) + F^{(NP)}(e^{2\pi i S}, T, U)]. \tag{4.37}$$

[19]As we have shown above, the individual T and U fields are subject to renormalization. It is important to note that a specific form of the radiative correction to these fields are meaningless unless one specifies how the cubic polynomial ambiguity in the one-loop corrected prepotential is fixed. In the rigid $N = 2$ supersymmetric theories, similarly, the Higgs expectation value by itself is not a meaningful quantity unless renormalization prescription of the gauge coupling constant is specified.

After the strong-weak coupling symplectic transformation, one finds that the holomorphic sections are further corrected:

$$
\begin{aligned}
X^{\Lambda,(NP)} &= (0, -\partial_S F^{(NP)}, 0, 0) \\
F_\Lambda^{(NP)} &= ((2 - S\partial_S - T\partial_T - U\partial_U)F^{(\mathrm{NP})}, 0, \partial_T F^{(\mathrm{NP})}, \partial_U F^{(\mathrm{NP})}),
\end{aligned} \tag{4.38}
$$

where the superscript NP denotes nonperturbative correction parts to the sections. In the weak coupling limit Eq.(4.36), one finds the non-perturbative Kähler potential as:

$$
\begin{aligned}
K_{\mathrm{Nonp}} &\rightarrow K_S + [\hat{K}]_{\mathrm{Pert}} + \Delta\hat{K}_{\mathrm{NP}} \\
\Delta\hat{K}_{\mathrm{NP}} &= \frac{\mathrm{Im}S}{\mathrm{Im}T\mathrm{Im}U} \cdot \mathrm{Im}(\overline{TU}\partial_S F^{(\mathrm{NP})}) = \mathcal{O}\Big(\mathrm{Im}S\, e^{-2\pi\mathrm{Im}S}\Big)
\end{aligned} \tag{4.39}
$$

where $\Delta\hat{K}^{(\mathrm{NP})}$ denotes the leading-order non-perturbative corrections in the limit Eq.(4.36). Here, we have used the fact that $\partial_{S,T,U}F^{(NP)} = \mathcal{O}(F^{(NP)})$ and Eq.(4.36). Note that the non-perturbative correction $\Delta\hat{K}^{(\mathrm{NP})}$ is proportional to ImS, the classical heterotic coupling constant.

One observes two important changes compared to the pattern of the perturbative corrections. First, one finds that the holomorphic matrix $\mathcal{F}$ receives further nonperturbative corrections. The first row and column entries, which was vanishing perturbatively, are now nonzero:

$$
\mathcal{F}^{(NP)} = \begin{pmatrix}
(\mathrm{Pert}) & \partial_S\mathcal{D}F^{(\mathrm{NP})} & (\mathrm{Pert}) & (\mathrm{Pert}) \\
\partial_S\mathcal{D}F^{(\mathrm{NP})} & 0 & \partial_S\partial_T F^{(NP)} & \partial_S\partial_U F^{(NP)} \\
(\mathrm{Pert}) & \partial_T\partial_S F^{(\mathrm{NP})} & (\mathrm{Pert}) & (\mathrm{Pert}) \\
(\mathrm{Pert}) & \partial_U\partial_S F^{(\mathrm{NP})} & (\mathrm{Pert}) & (\mathrm{Pert})
\end{pmatrix} \tag{4.40}
$$

where $\mathcal{D} \equiv (2 - S\partial_S - T\partial_T - U\partial_U)$ and (Pert) denotes the entries that were non-vanishing already at the perturbative level. Recalling the proof of perturbative non-renormalization theorems, it is clear that the non-vanishing first row and column entries render the S-field is renormalized non-perturbatively. As in the rigid supersymmetric field theories, the correction may be interpreted as non-perturbative renormalization of the heterotic string loop-counting parameter. Second, it is not only F_Λ's but also X^Λ's that receive corrections. In particular, the nonlinear constraint $\langle X, X\rangle = 2\partial_S F \neq 0$. This indicates that the T, U moduli fields should be further renormalized in addition to the perturbative renormalization.

In the weak coupling limit Eq.(4.36) the non-perturbative violation of the geometric constraint $\langle X, X\rangle = 0$ is exponentially small. Therefore, making an expansion in powers of $\mathrm{Im}S$ and $\exp(-2\pi\mathrm{Im}S)$, the leading non-perturbative corrections to the BPS central charge is given by

$$\begin{aligned} M_{\rm Pl}^2|\mathbf{Z}|^2_{\rm NP} &= \frac{1}{4}\Big(e^{-K}\,[(\mathbf{p}^0)^2+(\mathbf{q}^0)^2]\Big)_{\rm Nonp} \\ &= \frac{1}{2}\Big((\mathrm{Im}S+V^{\rm GS})_{\rm Nonp}\,e^{-\Delta\hat{K}^{\rm NP}}\Big)\cdot\Big(\mathrm{Im}T\mathrm{Im}U[(\mathbf{p}^0)^2+(\mathbf{q}^0)^2]\Big)_{\rm Nonp} \end{aligned} \tag{4.41}$$

up to $\mathcal{O}(e^{-2\pi\mathrm{Im}S})$. In the last line, we have re-organized the leading order non-perturbative corrections such that the second bracket corresponds to the combination that were subject to the perturbative non-renormalization theorem. Even in this limit, however, it is not clear if the perturbative non-renormalization theorems established for each factors inside the second bracket remain uuchanged. In fact, as will be shown below, each non-renormalization theorems are violated non-perturbatively. However, we find the second bracket as a whole is subject to a new non-renormalization theorem.

4.3.1 NON-PERTURBATIVE RENORMALIZATION OF HETEROTIC COUPLING CONSTANT

We first analyze how the heterotic string-loop counting parameter is renormalized non-perturbatively. The minimization condition equation

$$\mathrm{Im}S = (\mathrm{Im}\mathcal{F})_{1\Sigma}\frac{[i\mathbf{p}+\mathbf{q}]^\Sigma}{[i\mathbf{p}+\mathbf{q}]^0} \tag{4.42}$$

suggests that the nonperturbative corrections may arise through either the overall holomorphic matrix or through the $\mathbf{q}^0$ charge vectors. The definition of $\mathbf{q}^0$ in Eq.(4.28) shows clearly that it is renormalized non-perturbatively. Recall that the perturbative non-renormalization of $\mathbf{q}^0$ was solely based on the fact that the first row and column entries of the $\mathcal{F}^{(P)}$ matrix was zero identically. For the S-field, however, the effect of nonperturbative renormalization of $\mathbf{q}^0$ is cancelled out in the ratio, and the nonperturbative correction comes entirely from the $(\mathrm{Im}\mathcal{F})_{1\Sigma}$ parts. Expanding in terms of components and using the minimal configuration of T, U special coordinates, one finds leading non-perturbative correction as

$$\begin{aligned} \Delta[\mathrm{Im}S] &= \Big(\partial_S\mathcal{D}F^{(NP)}+T\partial_T F^{(NP)}+U\partial_U F^{(NP)}\Big) \\ &= \mathrm{Im}\Big(\partial_S(2-S\partial_S)F^{(NP)}\Big)\equiv V^{(\rm NP)}=\mathcal{O}(\mathrm{Im}S\,e^{-2\pi\mathrm{Im}S}). \end{aligned} \tag{4.43}$$

This correction is proportional to $\mathrm{Im}S$, hence, is the same order as the correction to the Kähler potential Eq.(4.39). We are thus motivated to define a non-perturbative heterotic string coupling parameter as:

$$\Big(\frac{1}{g^2_{\rm het}}\Big)_{\rm Nonp}\equiv[(\mathrm{Im}S+\mathrm{V}^{\rm GS})_{\rm Cl}+V^{(\rm NP)}]\cdot e^{-\Delta\hat{K}_{\rm NP}}. \tag{4.44}$$

The leading order non-perturbative corrections give rise to non-perturbative renormalization of the heterotic coupling parameter.

4.3.2 NON-PERTURBATIVE NON-RENORMALIZATION OF $\mathrm{Im}T\mathrm{Im}U[(\mathbf{p}^0)^2+(\mathbf{q})^2]$

That the charge-squared sum and the Kähler potential $\hat{K}_{\rm Pert} = \mathrm{Im}T\mathrm{Im}U$ are non-perturbatively renormalized is again from the fact that the first row and column entries of the holomorphic matrix $\mathcal{F}^{(NP)}$ are non-perturbatively corrected. This in turn implies, as we have shown above, that $\mathbf{q}^0$ charge is renormalized and that the T, U fields receives further corrections. Nevertheless, we now establish a non-perturbative non-renormalization theorem that

$$\mathtt{M}^2 \equiv \mathrm{Im}T\mathrm{Im}U\,[(\mathbf{p}^0)^2 + (\mathbf{q}^0)^2] \tag{4.45}$$

is not renormalized non-perturbatively. In establishing the theorem, one again finds the constraint $\langle X, X\rangle = 0$, valid up to the order $\mathcal{O}(e^{-2\pi \mathrm{Im}S})$, plays a crucial role[20]. Noting that $\mathbf{q}^0$ is non-perturbatively renormalized, the real and imaginary part of this constraint now reads:

$$\begin{aligned} \mathbf{p}^0\Delta\mathbf{q}^1 + \mathbf{p}^1\Delta\mathbf{q}^0 + \mathbf{p}^2\Delta\mathbf{q}^3 + \mathbf{p}^3\Delta\mathbf{q}^2 &= 0, \\ \Delta(\mathbf{q}^0\mathbf{q}^1 + \mathbf{q}^2\mathbf{q}^3) = \Delta(\mathbf{p}^0\mathbf{p}^1 + \mathbf{p}^2\mathbf{p}^3) &= 0. \end{aligned} \tag{4.46}$$

Consider expanding the nonperturbative correction

$$\begin{aligned} \Delta\mathtt{M}^2 &= \Delta\Big((\mathrm{Im}T)(\mathrm{Im}U)[(\mathbf{p}^0)^2 + (\mathbf{q}^0)^2]\Big) \\ &= [(\mathbf{p}^0)^2 + (\mathbf{q}^0)^2]^{-2} \\ &\times \Big[\,(\mathbf{p}^0)^2\Big(\ \{\mathbf{p}^2\mathbf{p}^3\Delta(\mathbf{q}^0)^2 + (\mathbf{p}^0)^2\Delta(\mathbf{q}^2\mathbf{q}^3)\} \\ &\qquad\qquad -\{\mathbf{p}^0\mathbf{p}^2\Delta(\mathbf{q}^0\mathbf{q}^3) - \mathbf{p}^0\mathbf{p}^3\Delta(\mathbf{q}^0\mathbf{q}^2)\}\ \Big) \\ &\quad +\mathbf{p}^0\mathbf{p}^2\Big(\mathbf{q}^0\mathbf{q}^3\Delta(\mathbf{q}^0)^2 - (\mathbf{q}^0)^2\Delta(\mathbf{q}^0\mathbf{q}^3)\Big) \\ &\quad +\mathbf{p}^0\mathbf{p}^3\Big(\mathbf{q}^0\mathbf{q}^2\Delta(\mathbf{q}^0)^2 - (\mathbf{q}^0)^2\Delta(\mathbf{q}^0\mathbf{q}^2)\Big) \\ &\quad -(\mathbf{p}^0)^2\Big(\mathbf{q}^2\mathbf{q}^3\Delta(\mathbf{q}^0)^2 - (\mathbf{q}^0)^2\Delta(\mathbf{q}^2\mathbf{q}^3)\Big)\,\Big]. \end{aligned} \tag{4.47}$$

Using the first relation of Eq.(4.46), one can simplify this to

$$\Delta\mathtt{M}^2 = [(\mathbf{p}^0)^2 + (\mathbf{q}^0)^2]^{-2}$$

[20]Recall that the leading-order corrections to the non-perturbative heterotic coupling parameter was $\mathcal{O}(\mathrm{Im}S\, e^{-2\pi\mathrm{Im}S})$, hence, larger than the violation of the geometric constraint.

$$
\times \quad \Big((\mathbf{p}^0)^2 \Big[(\mathbf{p}^0\mathbf{p}^1 + \mathbf{p}^2\mathbf{p}^3)\Delta(\mathbf{q}^0)^2 + (\mathbf{p}^0)^2\Delta(\mathbf{q}^0\mathbf{q}^1 + \mathbf{q}^2\mathbf{q}^3) \Big]
$$
$$
+ (\mathbf{p}^0)^2 \Big[(\mathbf{q}^0)^2\Delta(\mathbf{q}^0\mathbf{q}^1 + \mathbf{q}^2\mathbf{q}^3) - (\mathbf{q}^0\mathbf{q}^1 + \mathbf{q}^2\mathbf{q}^3)\Delta(\mathbf{q}^0)^2 \Big] \Big). \tag{4.48}
$$

Using the second relation of Eq.(4.46), one finds that this expression vanishes identically.

4.3.3 NON-PERTURBATIVE BPS BLACK HOLE MASS AND ENTROPY

Based on the results of the preceding two subsections, we now have non-perturbative result of the BPS central charge in the weak coupling limit Eq.(4.36):

$$
\begin{aligned}
|\mathbf{Z}|^2_{\rm Nonp} &= \exp[-(K_S^{\rm Nonp} - K_S^{\rm Cl})] \cdot |\mathbf{Z}|^2_{\rm Cl} \\
&= \frac{[({\rm Im}S + {\rm V}^{\rm GS})_{\rm Cl} + V^{\rm (NP)}] \cdot e^{-\Delta \hat{K}_{\rm NP}}}{({\rm Im}S)_{\rm Cl}} |\mathbf{Z}|^2_{\rm Cl}.
\end{aligned} \tag{4.49}
$$

Therefore, using the definition of heterotic string coupling parameter Eq.(4.44), the BPS black hole mass and the entropy formula with the leading order non-perturbative correction is obtained as:

$$
\begin{aligned}
\Big[\mathbf{M}^2_{\rm BH}\Big]_{\rm Nonp} = M^2_{\rm Pl}|\mathbf{Z}|^2_{\rm Nonp} &= [({\rm Im}S + {\rm V}^{\rm GS})_{\rm Cl} + V^{\rm (NP)}]\, e^{-\Delta \hat{K}_{\rm NP}} \cdot \langle P, P\rangle \\
&\equiv \Big[\frac{4\pi}{g^2_{\rm het}}\Big]_{\rm Nonp} \cdot \langle P, P\rangle.
\end{aligned} \tag{4.50}
$$

5 DISCUSSION

In this paper, we have studied classical and quantum configurations of the Kähler-BPS black holes in $D = 4, N = 2$ heterotic string compactification. We have first given an interpretation of the Kähler-BPS limit as a dynamical relaxation phenomena of the scalar fields. We have emphasized that the interpretation offers a unified description of the Kähler-BPS limit for both rigid and local $N = 2$ supersymmetric cases. We have analyzed quantum corrections to the BPS black hole mass and macroscopic entropy. Much as in the rigid supersymmetric theories, we have shown a non-renormalization theorem that protects the saturation of the BPS bound at perturbative level: BPS black hole mass and entropy have exactly the same form as the classical one once re-expressed in terms of renormalized parameters. The non-renormalization theorem ensures, among others, that strong-coupling extrapolation of microscopic state counting for suitable D-brane configurations can be made to the macroscopic black hole mass and entropy configurations once short-distance quantum effects are correctly identified and included. We have shown that the perturbative non-renormalization theorem is valid not only for large T_2 limit but also everywhere inside the T, U moduli space so long as one stays away from the enhanced gauge symmetry points. The non-perturbative corrections are more involved. We have only estimated the leading-order corrections in the weak coupling, large T_2 limit. Clearly, a better understanding of non-perturbative effects is desirable.

We finally discuss the quantum heterotic BPS black holes from different string theory point of view that are related to the heterotic STU model by string-string duality. As mentioned already, the heterotic STU model is related to sequential T_2 compactification of the $D = 6$ type I orientifold model on K_3 orbifold $T^4/\mathbf{Z}_2$ [28]. At $D = 6$ the type I dilaton belongs to a hyper multiplet, while the Kaluza-Klein volume of the K_3 orbifold volume belongs to the tensor multiplet and controls the strength of the gauge interactions. While we have studied the rank-3 STU model as a concrete example, it is straightforward to generalize to higher-rank models. Of particularly interesting situation in the heterotic side is the non-perturbatively generated vector multiplets which originates from the non-perturbatively generated $D = 6$ tensor multiplets due to small instantons [53]. At $D = 4$ the gauge coupling corresponding these non-perturbatively generated vector-tensor multiplets is proportional to T-moduli field. As such, BPS black hole carrying these vector-tensor fields would be intrinsically quantum-mechanical. On the type I side, these vector-tensor multiplets are mapped to the vector fields that arise from the new open string sector associated with 5-branes in $D = 6$. Their interaction strength is determined by the volume of K_3 volume, hence, independent of type I dilaton. Thus, one expects that BPS black holes on the type I side provide a clean description of the intrinsically quantum-mechanical BPS black holes in the heterotic side.

Alternatively, one may use the heterotic-type II duality [9] and map the heterotic $N = 2$ BPS black hole to those on dual IIA string Calabi-Yau compactification. The prepotential of type IIA side encodes topological data of the Calabi-Yau three-fold under consideration. For example, the type IIA prepotential contains intersection numbers $d_{ABC} = \int J_A \wedge J_B \wedge J_C$, Euler number $\chi(CY)$ and infinitely many rational instanton numbers. The heterotic-type IIA duality maps these data to the heterotic couplings in a well-defined manner. The type II dilaton belongs to the hyper multiplets, hence, the prepotential arises classically only. Since, as we have shown, the $N = 2$ black hole configuration depends on details of the prepotential, the type IIA topological data are then map to various physical quantities associated with the BPS black hole such as mass and entropy. We have shown that, in the heterotic weak coupling limit, these quantities are protected by non-renormalization theorems. The corresponding BPS black holes on type IIA side should then be protected by a worldsheet non-renormalization theorem. Furthermore, there are other topological data of Calabi-Yau three-folds on type IIA side.

For example, gravitational curvature-squared coupling encodes informations of second Chern class $c_2(J_A) \equiv \int c_2 \wedge J_A$ and elliptic instanton numbers, while higher-order gravitational or gauge-gravitational couplings encode higher-genus instanton numbers of the Calabi-Yau three-fold. Again, they have precise map on the heterotic side as higer-order gauge and gravitational interaction couplings [54]. These couplings are also strongly constrained by the target-space duality symmetry, hence, the non-renormalization theorem we have proven in this work should be applicable to these couplings. Of particular interest on the heterotic side is the effect of the gravitational curvature-squared term since it appears already at the classical level[21]. Thus, already at heterotic classical level, the BPS black hole should carry information of the second Chern class $c_2(J_A)$ in addition to the classical intersection numbers of the corresponding Calabi-Yau three-fold on the type IIA side. Explicit constructions of heterotic BPS black holes and a generalization of the non-renormalization theorem will be reported elsewhere. Thus, plethora of topological data of the Calabi-Yau three-fold encoded through holomorphic couplings suggest that it would be extremely interesting to find other macroscopic physical quantities BPS black holes than the mass and entropy that are stable against extrapolation to the strong coupling regime. Like mass and entropy, such quantities will also depend on the topological data of Calabi-Yau space in a definite manner. Therefore, if one finds sufficiently many such physical quantities, it should then be possible to glean various topological data of Calabi-Yau space out of macroscopic black hole configurations.

I thank to S.R. Das, M.K. Gaillard, R. Kallosh, D. Lowe, S. Mathur, L. Susskind, S. Theisen and S. Yankielowicz for discussions on subjects related to this work. While preparing for this manuscript, we have received the preprint [55], which contains overlapping independent result with part of the present work.

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