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Crossing Angles In The Beam-Beam Interaction

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ABSTRACT

A Hamiltonian perturbation analysis of the beam-beam interaction with a horizontal crossing angle is performed. The beam-beam tune shifts and resonances that result from a crossing angle are determined.

I. INTRODUCTION

Many storage ring colliders are being designed to reach high luminosity through the use of a large number of closely spaced bunches. This introduces a potential problem of parasitic collisions near the interaction point, but these parasitic collisions can be avoided by having the beams cross at an angle rather than head-on (Figure 1). The contributions to the tune shifts and the beam-beam resonances introduced by a crossing angle are analyzed in this paper.

The beam-beam interaction with a crossing angle has been studied by a number of authors, [1] - [4]. This paper is closest to that of Sagan *et al* [2] which obtained some of the results presented here. The notation and method are discussed extensively in references [5] and [6].

II. PERTURBATION FORMALISM

The Hamiltonian of a particle in beam 1 is

$$H = H_0 - \frac{Nr_c}{\gamma}\tilde{V}_{BB}$$

where H_0 is the Hamiltonian of the transverse motion in the absence of the beam-beam interaction, $\tilde{V}_{BB}$ is the beam-beam potential, N is the number of particles in beam 2, r_c is the classical particle radius, and γ is the energy in units of rest energy. The betatron motions in the absence of the beam-beam interaction can be written in terms of the action-angle variables $\{I_x, \psi_x\}$ $\{I_y, \psi_y\}$ of the unperturbed Hamiltonian, H_0,

$$x_\beta = \sqrt{2I_x\beta_x}\cos\psi_x ; y_\beta = \sqrt{2I_y\beta_y}\cos\psi_y . \qquad (1)$$

These expressions are used in a perturbation analysis of $\tilde{V}_{BB}$.

The beam-beam potential is

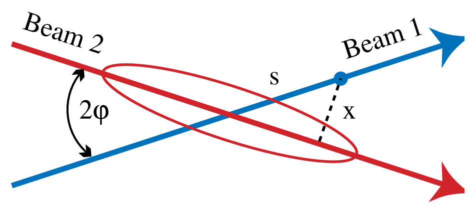

Figure 1: Beams crossing at an angle 2φ.

$$\tilde{V}_{BB} = \sqrt{\frac{2}{\pi\sigma_L^2}}\sum_{n=-\infty}^{\infty} V_F(x,y,s)\, e^{(-2(s-(nC+c\tau))^2/\sigma_L^2)} .$$

The sum is over all turns and the variables in this equation are: $\sigma_L \equiv$ the RMS bunch length of beam 2; s $\equiv$ coordinate along the reference orbit; C $\equiv$ the collider circumference; c $\equiv$ speed of light; and $\tau \equiv$ displacement of the collision point given in terms of the synchrotron oscillation amplitude, $\hat{\tau}$, and tune, Q_s, by

$$\tau = \frac{\hat{\tau}}{2}\cos(2\pi n Q_s) .$$

The potential V_F depends at the displacements of the particle from the center of beam 2

$$V_F = \int_0^\infty \frac{dq}{\sqrt{(2\sigma_x^2+q)(2\sigma_y^2+q)}} \exp\left\{-\left[\frac{x^2}{2\sigma_x^2+q} + \frac{y^2}{2\sigma_y^2+q}\right]\right\}$$

where σ_x and σ_y are the RMS transverse sizes of beam 2.

Using the expressions in eq. (1) as approximations for the betatron motions and assuming that the crossing is in the x-dimension, the potential can be rewritten

$$V_F = \int_0^\infty \frac{dq}{\sqrt{(2\sigma_x^2+q)(2\sigma_y^2+q)}} \exp\left\{-\frac{2\beta_y I_y\cos^2\theta_y}{2\sigma_y^2+q}\right\} \times \exp\left\{-\frac{(s\sin 2\varphi + \sqrt{2\beta_x I_x}\cos\theta_x)^2}{2\sigma_x^2+q}\right\} .$$

Fourier transforming the x expression with respect to s gives

$$V_F = \frac{\sqrt{\pi}}{2\pi}\frac{1}{\sin 2\varphi}\int_0^\infty \frac{dq}{\sqrt{(2\sigma_y^2+q)}} \exp\left\{-\frac{2\beta_y I_y\cos^2\theta_y}{2\sigma_y^2+q}\right\} \times \int_{-\infty}^{\infty} e^{i\omega s} d\omega \exp\left\{-\frac{\omega^2(2\sigma_x^2+q)}{4\sin^2 2\varphi} + \frac{i\omega\sqrt{2\beta_x I_x}\cos\theta_x}{\sin 2\varphi}\right\} .$$

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Following the usual procedure of Fourier transforming $\tilde{V}_{BB}$ with respect to ψ_x, ψ_y and s gives an expression for $\tilde{V}_{BB}$ in terms of Fourier coefficients each of which is related to the resonance

$$pQ_x + rQ_y + mQ_s = n$$

where p, r, m, and n are integers. Making a change of variables $\zeta = \omega/\sin 2\varphi$ this expression is

$$\begin{aligned}\tilde{V}_{BB} = \frac{1}{C}\sum_{m,n,p,r=-\infty}^{\infty}\int_{-\infty}^{\infty} d\zeta U_{pr}(I_x, I_y, \zeta) \\ \times\exp\left(-(k_{prm} + \zeta\sin 2\varphi)^2\sigma_L^2/8\right) \\ \times i^m J_m((k_{prm} + \zeta\sin 2\varphi)\hat{\tau}c/2) \\ \times\exp\left(i(p\psi_x + r\psi_y - 2\pi(n - mQ_s)s/C)\right)\end{aligned}$$

where

$$\begin{aligned}U_{pr} = \frac{\sqrt{\pi}}{(2\pi)^3}\int_0^{2\pi} d\theta_x e^{-ip\theta_x}\int_0^{2\pi} d\theta_y e^{-ir\theta_y}\int_0^{\infty}\frac{dq}{\sqrt{(2\sigma_y^2 + q)}} \\ \times\exp\left\{-\frac{2I_y\beta_y\cos^2\theta_y}{2\sigma_y^2 + q}\right\} \\ \times\exp\left\{-\frac{\zeta^2(2\sigma_x^2 + q)}{4} + i\zeta\sqrt{2\beta_x I_x}\cos\theta_x\right\}\end{aligned}$$

and

$$\begin{aligned}k_{prm} = 2\pi(n - mQ_s)/C + p(1/\beta_x^* - 2\pi Q_{x0}/C) \\ + r(1/\beta_y^* - 2\pi Q_{y0}/C).\end{aligned}$$

The quantities β_x^* and β_y^* are the β-functions at the collision point, and Q_{x0} and Q_{y0} are the tunes in the absence of the beam-beam interaction.

The resonance $pQ_x + rQ_y + mQ_s = n$ occurs for values of the tune where the phase is stationary, i.e.

$$\frac{d}{ds}(p\psi_x + r\psi_y - 2\pi(n - mQ_s)s/C) = 0,$$

and the average value of the beam-beam potential is given by the term in the series with p = r = m = n = 0.

Perform a Taylor expansion in powers of sin2φ

$$\tilde{V}_{BB} = \tilde{V}_{BB}\Big|_{\sin 2\varphi = 0} + \sin 2\varphi\frac{\partial\tilde{V}_{BB}}{\partial\sin 2\varphi}\Big|_{\sin 2\varphi = 0} + \ldots$$

The first term in the Taylor series is

$$\begin{aligned}\tilde{V}_{BB}\Big|_{\sin 2\varphi = 0} = \\ \frac{1}{C}\sum_{m,n,p,r=-\infty}^{\infty}\int_{-\infty}^{\infty} d\zeta U_{pr}(I_x, I_y, \zeta) \\ \times\exp\left(-k_{prm}^2\sigma_L^2/8\right)i^m J_m(k_{prm}\hat{\tau}c/2) \\ \times\exp\left(i(p\psi_x + r\psi_y - 2\pi(n - mQ_s)s/C)\right).\end{aligned} \tag{2}$$

The integral is

$$\begin{aligned}\int_{-\infty}^{\infty} d\zeta U_{pr}(I_x, I_y, \zeta) = \\ \frac{1}{(2\pi)^2}\int_0^{2\pi} d\theta_x e^{-ip\theta_x}\int_0^{2\pi} d\theta_y e^{-ir\theta_y}\int_0^{\infty}\frac{dq}{\sqrt{(2\sigma_y^2 + q)(2\sigma_x^2 + q)}} \\ \times\exp\left\{-\frac{2I_y\beta_y\cos^2\theta_y}{2\sigma_y^2 + q} - \frac{2I_x\beta_x\cos^2\theta_x}{2\sigma_x^2 + q}\right\}.\end{aligned} \tag{3}$$

The second term in the Taylor series is

$$\begin{aligned}\frac{\partial V_{BB}}{\partial\sin 2\varphi}\Big|_{\sin 2\varphi = 0} = \\ \frac{1}{C}\sum_{m,n,p,r=-\infty}^{\infty}\int_{-\infty}^{\infty}\zeta d\zeta U_{pr}(I_x, I_y, \zeta)\exp\left\{-\left(k_{prm}\sigma_L\right)^2/8\right\}i^m \\ \times\left[\frac{\hat{\tau}c}{2}J_{m-1}\left(\frac{k_{prm}\hat{\tau}c}{2}\right) - \left(\frac{k_{prm}\sigma_L^2}{4} + \frac{m}{k_{prm}}\right)J_m\left(\frac{k_{prm}\hat{\tau}c}{2}\right)\right] \\ \times\exp\left\{i(p\psi_x + r\psi_y - 2\pi(n - mQ_s)s/C)\right\}\end{aligned} \tag{4}$$

where

$$\begin{aligned}\int_{-\infty}^{\infty}\zeta d\zeta U_{pr}(I_x, I_y, \zeta) = \\ \frac{2i}{(2\pi)^2}\int_0^{2\pi} d\theta_x e^{-ip\theta_x}\int_0^{2\pi} d\theta_y e^{-ir\theta_y}\int_0^{\infty}\frac{dq}{\sqrt{(2\sigma_y^2 + q)(2\sigma_x^2 + q)^3}} \\ \times\sqrt{2I_x\beta_x}\cos\theta_x\exp\left\{-\frac{2I_y\beta_y\cos^2\theta_y}{2\sigma_y^2 + q} - \frac{2I_x\beta_x\cos^2\theta_x}{2\sigma_x^2 + q}\right\}.\end{aligned} \tag{5}$$

Equations (2) - (5) contain the results. These frightening looking expressions can be interpreted to give useful information about the beam-beam interaction.

III. DISCUSSION

A. Tune Shifts

The tune shifts as a function of amplitude are

$$\Delta Q_x = -\frac{CNr_c}{2\pi\gamma}\frac{\partial\left\langle \tilde{V}_{BB}\right\rangle}{\partial I_x};\ \Delta Q_y = -\frac{CNr_c}{2\pi\gamma}\frac{\partial\left\langle \tilde{V}_{BB}\right\rangle}{\partial I_y}$$

where $\left\langle \tilde{V}_{BB}\right\rangle$ is given by eqs. (2) - (5) evaluated with p = r = m = n = 0.

The tune shifts are the same as for head-on collisions because there are no contributions from the second term in the Taylor series, the term proportional to sin2φ. This follows from the parity of the θ_x integrands. In the case of ΔQ_y the integral is

$$\Delta Q_y \sim \int_0^{2\pi} d\theta_x \cos\theta_x \exp\left\{-\frac{2I_x\beta_x\cos^2\theta_x}{2\sigma_x^2+q}\right\} = 0$$

because the argument is an odd function of θ_x. The horizontal tune shift, ΔQ_x, is proportional to

$$\Delta Q_x \sim \sqrt{\frac{\beta_x}{2I_x}}\int_0^{2\pi} d\theta_x \cos\theta_x \exp\left\{-\frac{2I_x\beta_x\cos^2\theta_x}{2\sigma_x^2+q}\right\}$$
$$-\frac{\sqrt{2I_x\beta_x^3}}{2\sigma_x^2+q}\int_0^{2\pi} d\theta_x \cos^3\theta_x \exp\left\{-\frac{2I_x\beta_x\cos^2\theta_x}{2\sigma_x^2+q}\right\}.$$

Evaluating the integrals $\Delta Q_x = 0$ because the arguments of both integrals are odd functions of θ_x.

The tune shifts from the head-on collisions have been calculated in numerous references and are given here for completeness. They are

$$\frac{\Delta Q_y}{\xi_y} = \frac{1+R}{2}\int_0^1 \frac{d\eta}{\sqrt{\eta+R^2(1-\eta)}} I_0^e\left(\frac{B_x}{2}\right)\left\{I_0^e\left(\frac{B_y}{2}\right)-I_1^e\left(\frac{B_y}{2}\right)\right\},$$

and

$$\frac{\Delta Q_x}{\xi_x} = \frac{1+R}{2R}\int_0^1 \frac{d\eta}{\sqrt{\eta+(1-\eta)/R^2}} I_0^e\left(\frac{A_y}{2}\right)\left\{I_0^e\left(\frac{A_x}{2}\right)-I_1^e\left(\frac{A_x}{2}\right)\right\}.$$

The functions B_x, ..., A_y are

$$B_x(\eta) = \eta\frac{I_x/\varepsilon_x}{\eta+R^2(1-\eta)};\ B_y(\eta) = \eta I_y/\varepsilon_y;$$
$$A_y(\eta) = \eta\frac{I_y/\varepsilon_y}{\eta+(1-\eta)/R^2};\ A_x(\eta) = \eta I_x/\varepsilon_x.$$

The variables in these equations are $R = \sigma_y/\sigma_x$; ξ_x and ξ_y are the beam-beam strength parameters; and ε_x and ε_y are the emittances. The functions I_n^e are related to modified Bessel functions

$$I_n^e(x) = e^{-x}I_n(x).$$

B. Beam-Beam Resonances

Possible resonances can determined from the parity of the integrands in eqs. (3) and (5). The integrand of eq. (3) is an even function of θ_x and an even function of θ_y. The only allowed resonances for head-on collisions must have both p and r equal to even integers. The integrand of eq. (5) is an odd function of θ_x and an even function of θ_y. The allowed resonances must have r equal to and even integer and p equal to an odd integer.

The crossing angle has introduced new beam-beam resonances that have odd horizontal order and Fourier expansion coefficients proportional to sin 2φ. There are both betatron, m = 0, and synchrobetatron, m ≠ 0, resonances. The appearance of odd order betatron resonances can be understood because there is a phase shift of π across the interaction region. The synchrobetatron resonances arise from modulation introduced by the synchrotron oscillations. They depend on the synchrotron amplitude and have zero Fourier expansion coefficient when $\hat{\tau} = 0$.

C. Remarks

The crossing angle has not changed the tune shifts, so the beam-beam footprint, the area of the tune plane occupied by the beam, is the same as for head-on collisions. The effect of the crossing angle has been to introduce odd horizontal order resonances. These additional resonances could lower the beam-beam limit.

TeV33 is considering using both horizontal and vertical crossing angles. The vertical crossing angle will introduce odd order vertical resonances as well, and there is a still larger probability of a reduced beam-beam limit.

IV. REFERENCES

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