SLAC-PUB-95-7086
December 1995

Geometrical Wake of a Smooth Taper*

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Abstract

A transverse geometrical wake generated by a beam passing through a smooth taper is considered. Comparison of the existing theories for the impedance of smooth structures is performed to show that they agree in the frequency range determined by the dimensions of the transition region. A simple approach is developed for the calculation of the kick factor, which reduces finding of the kick factor to the solution of simple electrostatic and magnetostatic problems. Using results for the real part of the impedance of the taper, it is shown that the existing theory allows to accurately estimate the kick factor for the NLC Final Focus collimator.

Submitted to
Particles Accelerators

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*Work supported by Department of Energy contract DE-AC03-76SF00515

I. INTRODUCTION

Scrapers or collimators are often used in circular and linear colliders to eliminate halo particles from the beam. If the opening of the collimator is small, the wakefield excited by the collimator can perturb the beam motion and increase its emittance. There are two different physical mechanisms that produce the wakes. Firstly, for a perfectly conducting wall, an incomplete cancellation between electric and magnetic forces acting on the beam from the image charges generates a so called geometric wake [1]. Secondly, a finite wall conductivity adds a resistive wake to the geometric one [2]. In many situations, the two wakes can be computed independently and scale differently with the dimensions of the collimator.

To relax the wakefield effects one can try to taper the transition of the beam pipe to the collimator, and to find the length of the taper by minimizing the wake generated by the transition. To perform such an optimization one have to be able to compute the geometrical wake for a long, smooth structure. Unfortunately, the existing computer codes such as ABCI and MAFIA experience problems for smoothly tapered transitions.

Several attempts have been made to evaluate the geometric wake of a pipe with a slowly varying radius using analytical approaches based on the smallness of the angle between the pipe wall and the axis. Cooper, Krinsky and Morton developed a theory of the impedance for a periodic structure with a small but arbitrary variation Δb of the pipe radius b [3]. Using a different method, Warnock generalized their result for nonperiodic tapers [4]. The results of [3,4] assume a small ratio $\Delta b/b$ and cannot be applied to the scrapers, in which $\Delta b/b$ has to be large. The case of arbitrary $\Delta b/b$ has been studied by Yokoya [5] who found purely imaginary impedance and a δ-function wake.

A comparison of Refs. 4 and 5 for shallow collimators shows an apparent disagreement between the two papers, since Ref. 5 predicts $\operatorname{Re} Z = 0$ in contrast to nonvanishing $\operatorname{Re} Z$ of Ref. 4. We address this issue in Section II and show that this disagreement is due to the overestimation of the applicability range of the theory of Ref. 5. In Section III, a formula for the transverse kick factor is derived that includes contributions of both real and imaginary parts of the impedance. Using this formula we show that in the cases when the losses due to the wake are negligibly small, the kick factor is given by the value of the imaginary part of the impedance at the zero frequency. This observation provides a new method of calculation of the wake effect for smooth structures which is used in Section IV to find $\operatorname{Im} Z_t(0)$ for an axisymmetric pipe with a smoothly varying radius. In Section V, we study the transverse wake for the NLC Final Focus collimator. Section VI summarizes the results of the paper.

II. COMPARISON OF REFERENCES 4 AND 5

In Ref. 5, expressions has been derived for the longitudinal and transverse wakes and impedances caused by a smooth transitions in a circular pipe. According to this paper, the longitudinal impedance is given by

$$Z_l(\omega) = -\frac{i\omega Z_0}{4\pi c}\int_{-\infty}^{\infty} dz(b')^2, \qquad (1)$$

and the transverse impedance is

$$Z_t = -\frac{iZ_0}{2\pi}\int_{-\infty}^{\infty} dz\left(\frac{b'}{b}\right)^2, \qquad (2)$$

where $b(z)$ is the pipe radius as a function of *z,* and $Z_0 = 120\pi$ Ohm and the prime denotes the derivative with respect to z. Eq. (1) assumes that $b(-\infty) = b(\infty)$, otherwise an additional term $(Z_0/2\pi)\ln[b(\infty)/b(-\infty)]$ should be added to the right-hand side. Note that the impedance given by Eqs. (1) and (2) is purely imaginary. These formulas are claimed to be valid for the frequency range such that $kb \gg 1$ and $kbb' \ll 1$, where $k = \omega/c$. As we will show below, the validity region for these formulas is actually somewhat different.

Another formula for the longitudinal impedance has been derived in Ref. 4 based on the perturbation theory for small variations of $b(z)$, $b(z) = b_0 + \Delta b(z)$, where $\Delta b \ll b$. In this theory,

$$Z_l(\omega) = \frac{\omega Z_0}{2\pi c b_0^2}\sum_{s=1}^{\infty}\frac{1}{k_s(\omega)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dz du\, b'(z)b'(u)\exp(ik_s(\omega)|z-u| - i\omega(z-u)/c), \qquad (3)$$

where $k_s(\omega) = \left[(\omega/c)^2 - j_{0s}^2/b_0^2\right]^{1/2}$ and j_{0s} is the s*th* root of the Bessel function of zero order. Evidently, Eq. (3) has both real and imaginary parts.

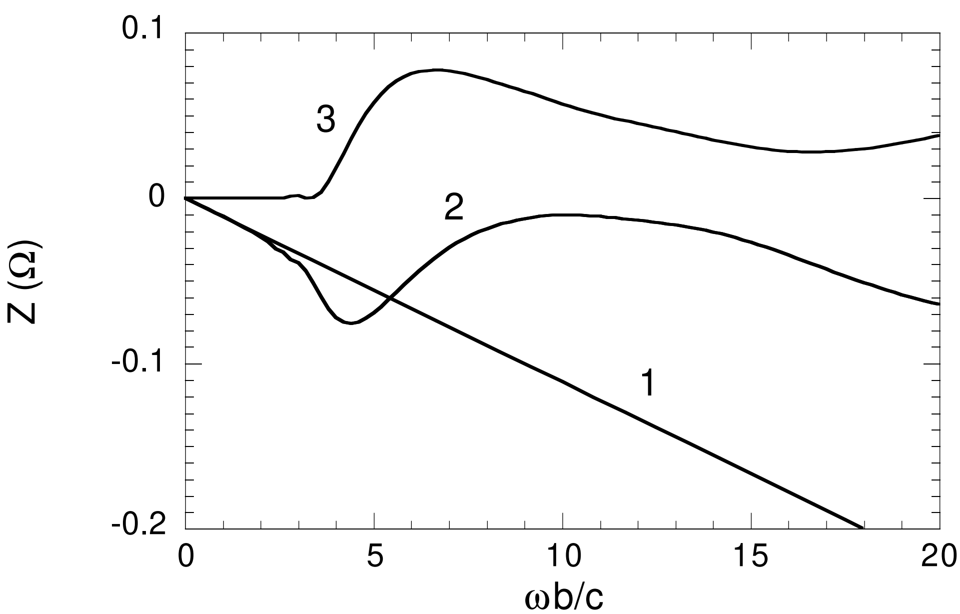

Fig.1. Comparison of longitudinal impedances Eq. (1) and (3) for a shallow collimator. *1* - imaginary part of Eq. (1), *2* and *3* - imaginary and real parts of Eq. (3), respectively.

Since either result is applicable to shallow collimators with small Δb, we compare Eqs. (1) and (3) for a particular example when the shape of the collimator is given by

$b(z) = b_0 + d\left(1+\cos(\pi z/l)\right)/2$ for $|z| < l$, and $b(z) = b_0$ for $|z| > l$ (cf. Ref. 4). For the parameters b_0, d and l we choose the values $b_0 = 1$ cm, d=0.03 cm and l=6 cm. Fig. 1 shows the impedances given by Eqs. (1) and (3).

The maximum value of b' in this example is equal to 7.8×10^{-3} and the condition $kbb' \ll 1$ is well satisfied within the region of k shown in the plot. However, as is seen from Fig. 1, the two equations agree only for relatively small values of k. Moreover, the imaginary parts of Eqs. (1) and (3) approach each other when $\omega \to 0$ where the real part of the impedance becomes negligibly small. This indicates that both theories agree if we limit Eq. (1) by a more stringent requirement than $kbb' \ll 1$. Indeed, it turns out that the upper limit of validity of equation (1) should be

$$kb^2/l \ll 1, \tag{4}$$

rather than $kbb' \ll 1$. We prove this in Appendix A by showing that only in this frequency range Eq. (3) reduces to Eq. (1). Since b' can be estimated as $\Delta b/l$, for large variation of the radius, $\Delta b \sim b$, inequality (4) is equivalent to $kbb' \ll 1$, but for small Δb (as that corresponding to Fig. 1) the condition of Ref. 5 strongly overestimates the validity region of Eq. (1). Furthermore, the requirement $kb \gg 1$ of Ref. 5 can also be omitted. This will be demonstrated in section IV by deriving Eq. (2) in the limit $\omega \to 0$.

III. TRANSVERSE KICK FACTOR AND IMPEDANCE

We now proceed to the transverse impedance and a kick factor associated with it. In application to collimators, the kick factor is usually used as an measure of beam emittance dilution caused by the collimation of the beam.

We define the transverse kick factor κ_t (which is sometimes called a transverse loss factor) such that an offset bunch will be deflected by the wake by an angle

$$y' = \frac{Nr_e}{\gamma} y\kappa_t, \tag{5}$$

where N is the number of particles in the bunch, γ is the relativistic factor, r_e is the classical electron radius and y is the displacement of the bunch. For a given transverse wake function $w_t(z)$, the kick factor can be calculated using

$$\kappa_t = \int f(s) f(s') w_t(s-s') ds ds', \tag{6}$$

where $f(s)$ is the longitudinal distribution function of particles normalized so that $\int f(s) ds = 1$.

Using the following relation between the wake function and the transverse impedance,

$$w_t(s) = \frac{i}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-is\omega/c} Z_t(\omega), \tag{7}$$

one can express the kick factor in terms of the imaginary part of the impedance,

$$\kappa_t = -\frac{1}{\pi} \int_0^{\infty} d\omega \left|\tilde{f}(\omega)\right|^2 \operatorname{Im} Z_t(\omega), \tag{8}$$

where $\tilde{f}(\omega)$ is the Fourier transform of the distribution function $f(s)$. For a Gaussian distribution, $\tilde{f}(k) = \exp\left(-\sigma_z^2 \omega^2 / 2c^2\right)$, and for the impedance (2), Eq. (8) yields,

$$\kappa_t = -\frac{c \operatorname{Im} Z_t}{2\sqrt{\pi}\sigma_z}. \tag{9}$$

One can also relate $w_t(z)$ to the real part of the impedance [6],

$$w_t(s) = \frac{2}{\pi}\int_0^\infty d\omega \sin\left(\frac{\omega s}{c}\right) \operatorname{Re} Z_t(\omega). \tag{10}$$

Using this equation, one can express the kick factor in terms of the real part of the impedance [7]:

$$\kappa_t = \int_0^\infty d\omega F\left(\frac{\omega}{c}\sigma_z\right) \operatorname{Re} Z_t(\omega), \tag{11}$$

where, for a Gaussian distribution function of the beam,

$$F(x) = -\frac{i}{\pi}\exp\left(-x^2\right)\operatorname{erf}(ix). \tag{12}$$

For large x, the function $F(x)$ asymptotically approaches $\pi^{-3/2} x^{-1}\left(1 + 2^{-1} x^{-2}\right)$.

Clearly equation (11) cannot be used if $\operatorname{Re} Z_t = 0$ as it is in Eq. (2). Firstly, it is important to understand that a purely imaginary impedance does not mean that the real part of Z_t vanishes identically; it only indicates that $\operatorname{Re} Z_t \ll \operatorname{Im} Z_t$ and, for that reason, $\operatorname{Re} Z_t$ has been neglected in Eq. (2). Secondly, Eq. (2) is approximately valid only for a limited range of frequencies, whereas integration in Eq. (11) runs up to infinity. Due to a slow roll off of the function $F(x)$, the integration in (11) is dominated by so high frequencies, where Eq. (2) is not valid any more. We can improve the convergence in the integral (11) at high frequencies by the following transformation.

Let us introduce a new function $G(x)$,

$$G(x) = F(x) - \frac{1}{\pi^{3/2} x}, \tag{13}$$

which decays much faster than $F(x)$ at large values of x, $G(x) \approx 2^{-1}\pi^{-3/2} x^{-3}$. The plot of the functions $G(x)$ and $F(x)$ is shown in Fig. 2. From Eq. (11) we find

$$\kappa_t = \int_0^\infty d\omega G\left(\frac{\omega}{c}\sigma_z\right) \operatorname{Re} Z_t(\omega) + \frac{c}{\pi^{3/2}\sigma_z}\int_0^\infty \frac{d\omega}{\omega} \operatorname{Re} Z_t(\omega). \tag{14}$$

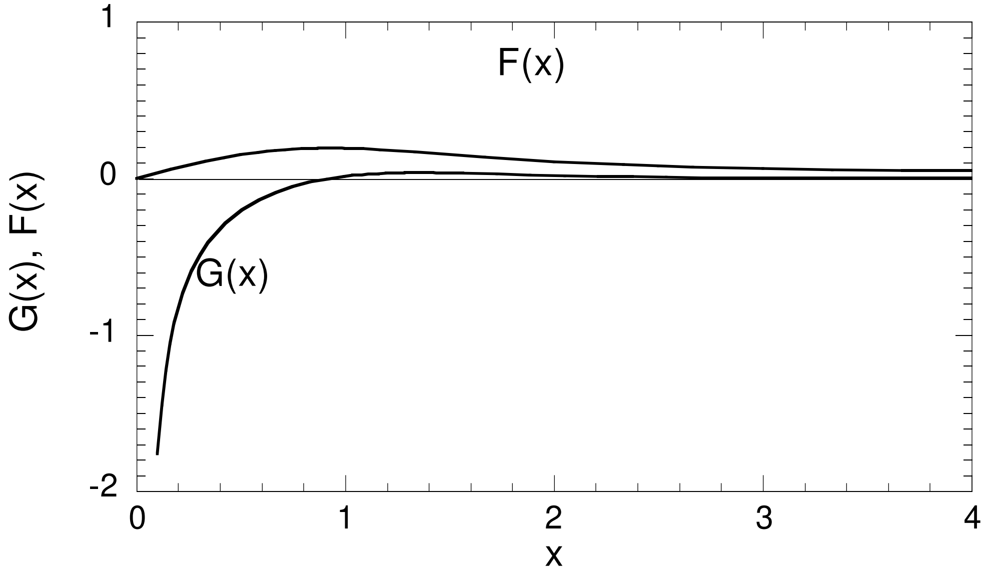

Fig. 2. Plot of functions $G(x)$ and $F(x)$.

The function $G(\omega\sigma_z/c)$ diverges at $\omega \to 0$, but the integrals in Eq. (14) converge because $\operatorname{Re} Z_t(\omega) = 0$ below the cutoff frequency. The second integral in this equation can be expressed in terms of the imaginary part of the impedance at the zero frequency through the Kramers-Kronig relation [8],

$$\kappa_t = \int_0^\infty d\omega G\left(\frac{\omega}{c}\sigma_z\right) \operatorname{Re} Z_t(\omega) - \frac{c}{2\pi^{1/2}\sigma_z} \operatorname{Im} Z_t(0). \tag{15}$$

This equation shows that if the real part of the impedance is much smaller that the imaginary part, the kick factor is proportional to the value of the imaginary part at the zero frequency. In many cases, calculation of $\operatorname{Im} Z_t(0)$ is much easier to perform than to find $Z_t(\omega)$ for arbitrary frequency, because $\operatorname{Im} Z_t(0)$ involves only solutions of Maxwell equations for static (time-independent) fields. In the next section we demonstrate how to find $\operatorname{Im} Z_t(0)$ for a smoothly varying axisymmetric structure.

Note that putting Eq. (2) into Eq. (15) immediately yields the correct result (8). Furthermore, using Eq. (15) will allow us to find in section V a correction to the kick factor due to nonvanishing real part of the impedance.

IV. $\boxed{\operatorname{Im} Z_t(0)}$ FOR A TUBE WITH A SMOOTHLY VARYING RADIUS

In this section we will solve magnetostatic and electrostatic problems and find the transverse impedance at the zero frequency for a tube with a smoothly varying radius.

A convenient approach to calculate the transverse impedance is based on consideration of a current $I(t,z) = \lambda c \exp(-i\omega t + ikz)$, where λ is the beam charge per unit length, displaced by an infinitesimally small distance Δ away from the pipe axis (see Fig.

3). In the limit of $\omega = 0$, $I(t,z)$ reduces to a dc current, $I = \lambda c$, with the transverse impedance given by the following formula:

$$Z_t(\omega = 0) = -\frac{i}{\lambda \Delta c} \int_{-\infty}^{\infty} \left(E_x - H_y \right) dz, \tag{16}$$

where E_x and H_y are the electric and magnetic fields on the axis of the pipe due to the presence of the walls.

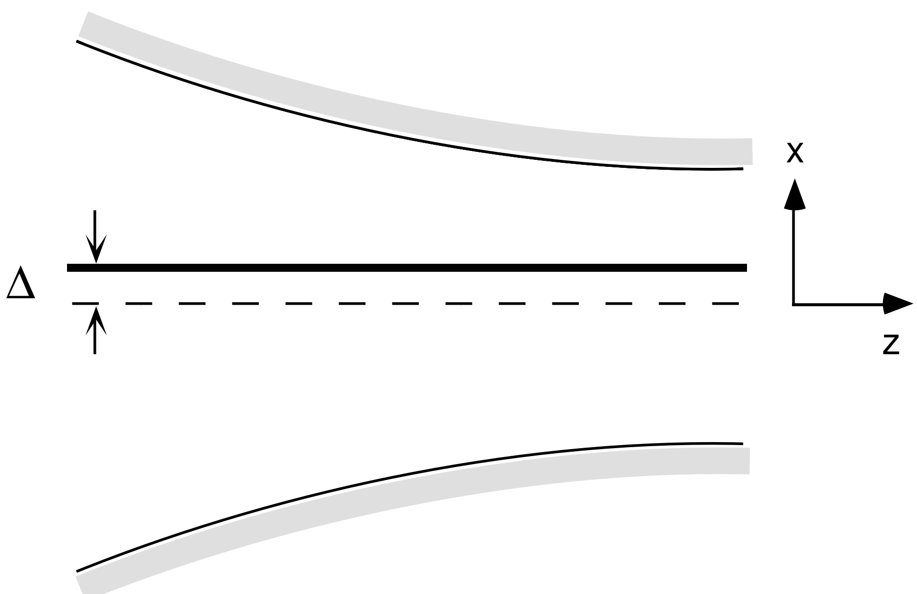

Fig. 3. An offset beam in a pipe with a smoothly varying radius.

A displaced beam with a charge density λ is equivalent to a linear dipole directed along x with a dipole moment $d_x = \lambda \Delta$ per unit length, and a linear magnetic moment directed along y with a dipole moment per unit length $m_y = -\lambda \Delta$. Hence, we need to find electrostatic and magnetostatic fields generated by these dipoles inside the pipe. The problem can be solved using a perturbation theory in the small parameter b'.

First, we will find the electrostatic potential $\varphi(r,\theta,z)$. It satisfies the Poisson equation with the dipole sources on the right hand side,

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} + \frac{\partial^2 \varphi}{\partial z^2} = -4\pi \Delta \lambda \delta'(x) \delta(y), \tag{17}$$

(δ' denotes the derivative of the delta function) with the boundary condition $\varphi = 0$ at $r = b(z)$.

In a pipe of a constant radius b, the potential of a linear dipole does not depend on z and is equal to

$$\varphi_0 = 2\lambda \Delta \left(\frac{1}{r} - \frac{r}{b^2} \right) \cos\theta. \tag{18}$$

To include the effect of the variation of $b(z)$, we assume that

$$\varphi = \varphi_0 + \varphi_1, \tag{19}$$

where $\varphi_1 \ll \varphi_0$ and φ_0 is given by Eq. (18) in which b is considered as a function of z. The first order correction φ_1 can be found from the equation

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \varphi_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi_1}{\partial \theta^2} = -\frac{\partial^2 \varphi_0}{\partial z^2}, \tag{20}$$

in which we neglected $\partial^2 \varphi_1 / \partial z^2$ in comparison with $\partial^2 \varphi_0 / \partial z^2$. The solution to this equation satisfying $\varphi_1 = 0$ at $r = b$ is

$$\varphi_1 = \frac{1}{4}\lambda\Delta\left(b^{-2}\right)''\left(r^3 - b^2(z)r\right)\cos\theta, \tag{21}$$

with the electric field on the axis,

$$E_x = -\frac{\partial \varphi_1}{\partial x} = \frac{1}{4}\lambda\Delta b^2\left(b^{-2}\right)''. \tag{22}$$

A similar approach can be used for the magnetic field $\mathbf{H}$. We introduce the magnetostatic potential $\psi(r,\theta,z)$ such that $\mathbf{H} = -\nabla\psi$. Note that a single-valued magnetic potential does exist in our problem because a total current in z-direction in a linear magnetic dipole equals zero. The function ψ satisfies the Poisson equation

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2} = -4\pi\Delta\lambda\delta(x)\delta'(y), \tag{23}$$

with the boundary condition corresponding to the vanishing normal component of the magnetic field at the pipe wall,

$$\frac{\partial \psi}{\partial r} = b'\frac{\partial \psi}{\partial z}. \tag{24}$$

Again, using the perturbation theory, one finds

$$\psi(r,\theta,z) = -2\lambda\Delta\sin\theta\left\{\left(\frac{1}{r} + \frac{r}{b^2}\right) - \frac{1}{8}\left(r^3 - 3b^2 r\right)\left(b^{-2}\right)'' + rbb'\left(b^{-2}\right)'\right\}, \tag{25}$$

with the magnetic field on the axis

$$H_y = -\frac{\partial \psi}{\partial y} = \lambda\Delta\left[\frac{3}{4}b^2\left(b^{-2}\right)'' + 2bb'\left(b^{-2}\right)'\right]. \tag{26}$$

Putting Eqs. (22) and (26) into Eq. (16) and integrating by parts using $b'(\pm\infty) = 0$, yields

$$Z_t(\omega = 0) = -\frac{iZ_0}{2\pi}\int_{-\infty}^{\infty} dz\left(\frac{b'}{b}\right)^2, \tag{27}$$

which is Yokoya's result (2). Thus we proved that Eq. (2) does not have a limitation from the side of low frequencies and is valid down to $\omega = 0$.

V. NLC FINAL FOCUS COLLIMATOR

Consider an axisymmetric taper shown in Fig. 4. The pipe radius gradually changes from b to g on a conical transition region of length l. The parameters of the taper are chosen to be b=0.5 cm, g=0.1 cm and l=28 cm; they correspond to that of the NLC Final Focus collimator [9]. We will assume the beam size $\sigma_z = 0.01$ cm.

Using Yokoya's formula (2) for the kick factor (9) we find

$$\kappa_t = \frac{\theta}{\sqrt{\pi}\sigma_z}\left(\frac{1}{g} - \frac{1}{b}\right) = 6.5 \text{ cm}^{-2} = 0.58 \times 10^3 \text{ V/pC/m}. \tag{28}$$

This result, however, should be checked against the applicability condition of Yokoya's formula. Estimating the highest frequency for which Eq. (2) is valid as $\omega = cl/b^2$, we find $\omega = 2\pi \times 540$ GHz, which is comparable to the frequency $c/\sigma_z = 2\pi \times 500$ GHz associated with the bunch length. This raises a question of how accurate Eq. (28) is.

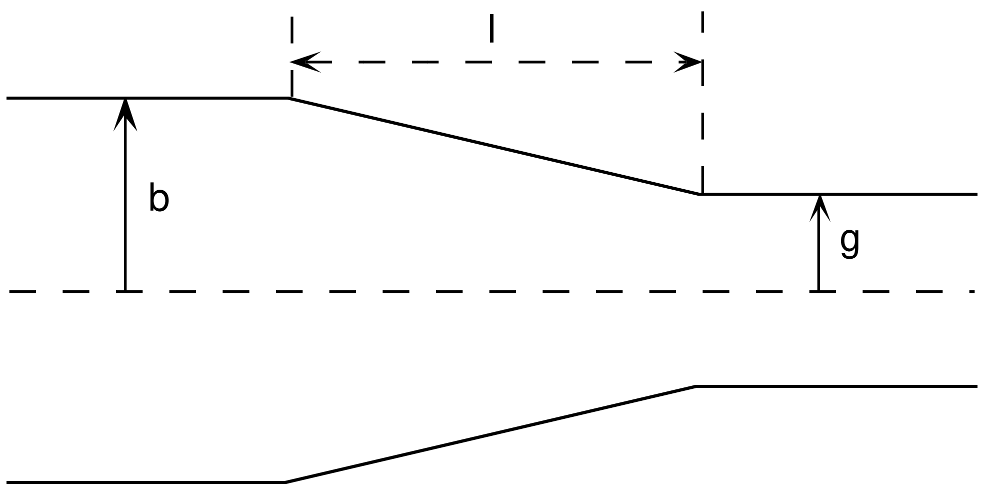

Fig. 4. A taper.

To find a correction to Eq. (28) due to the contribution of high frequencies where Eq. (2) is not accurate enough (and the real part of the impedance is not negligible), we used Eq. (15) in which the second term reproduces the result (28) and the first term represents a contribution caused by nonvanishing real part of the impedance. The latter has been numerically computed for the taper of Fig. 4 using the formulas of Ref. [8], and is shown in Fig. 5. Rapid oscillations of the impedance with the frequency are due to interference of the waves radiated by the beam in the transition region. For comparison, the imaginary part of the impedance for the taper is equal to 0.7 kΩ/m. Performing integration in Eq. (15) with $\mathrm{Re}\, Z_t(\omega)$ shown in Fig. 5 yields a correction $\Delta\kappa_t = 0.15 \text{ cm}^{-2}$ to the result of Eq. (28), that is less than 5%.

Note that the results of Sec. V for the real part of the impedance have about the same region of validity as Eq. (2) and are not accurate enough for high frequency. However, using them in Eq. (15), where the dominant contribution comes from the second term, and the high frequency range is suppressed by the rapid decay of the function G, allowed us to evaluate the accuracy of Yokoya's formula for the NLC-type collimator.

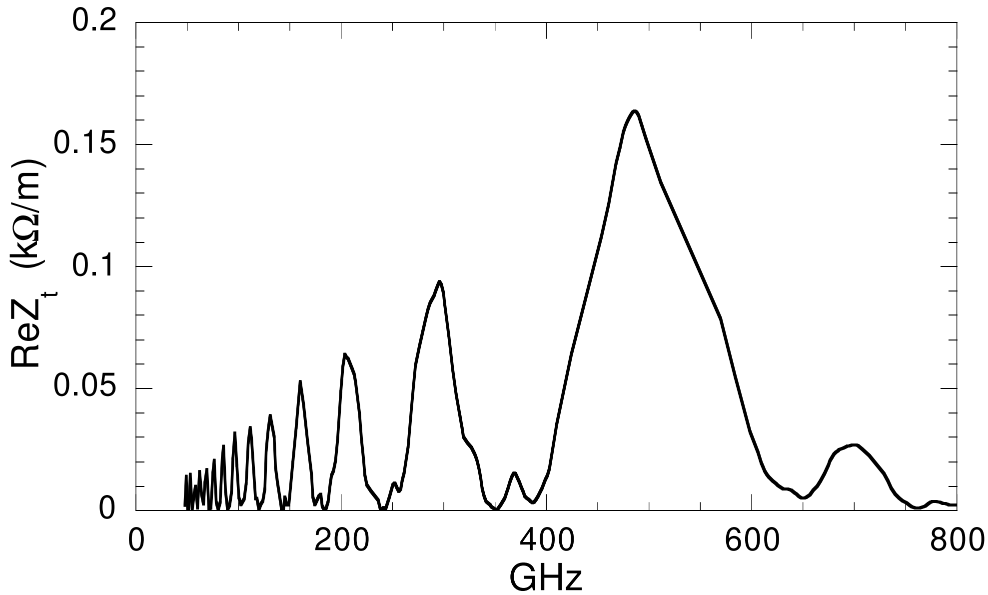

Fig. 5. Plot of $\mathrm{Re}\, Z_t(\omega)$ for the collimator shown in Fig. 4.

VI. CONCLUSION

We have shown that Yokoya's formula for the impedance of a smooth taper, if used within the right applicability range, gives an accurate estimation of the wake field and the transverse impedance. As shown in Section IV, it can actually be derived in a much simpler fashion, if one limits the consideration to a low frequency range. The method developed in this section can be rather easily applied to other geometries, e.g., for a collimator with plane jaws.

We estimated the error due to the nonvanishing real part of the impedance for the NLC Final Focus collimator. The prediction of Yokoya's formula for this collimator are correct within few percent.

ACKNOWLEDGMENTS

The author thanks A. Chao, S. Heifets, J. Irwin and R. Warnock for useful discussions.

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APPENDIX A

Consider the cernel of the integral in Eq. (3), $\exp\left(ik_s|\xi| - ik\xi\right)$, where $\xi = z - u$. For positive values of ξ, it is equal to $\exp\left(i\left(k_s - k\right)\xi\right)$, and for negative ξ, it reduces to $\exp\left(i\left(k_s + k\right)\xi\right)$. For large values of ξ, this function oscillates rapidly and does not contribute essentially to the integral. Characteristic value of $\xi = \xi^*$, below which the main contribution to the integral lies, can be estimated as follows,

$$\xi^* \approx \max\left(\frac{1}{k_s + k}, \frac{1}{k_s - k}\right). \tag{A1}$$

Remembering that $k_s = \left[k^2 - j_{0s}^2/b_0^2\right]^{1/2}$, one finds that the maximum value of ξ^*, in the limit of large frequencies, can be estimated as

$$\xi^* \approx \frac{2b_0^2 k}{j_{0s}^2}. \tag{A2}$$

Let us denote by l a typical distance on which $b(z)$ varies. If $l \gg \xi^*$, the product $b'(z)b'(u)$ varies slowly on the scale ξ^* associated with the function

$$\sum_{s=1}^{\infty} \frac{1}{k_s} \exp\left(ik_s|z-u| - ik(z-u)\right), \tag{A3}$$

and the latter can be substituted for by a delta function $A\delta(z-u)$,

$$\sum_{s=1}^{\infty} \frac{1}{k_s} \exp\left(ik_s|z-u| - ik(z-u)\right) \to A\delta(z-u), \tag{A4}$$

where the constant A is the area under the function (A3). We have

$$A = \sum_{s=1}^{\infty} \frac{1}{k_s} \int_{-\infty}^{\infty} \exp\left(ik_s|\xi| - ik\xi\right) d\xi = -2ib_0^2 \sum_{s=1}^{\infty} \frac{1}{j_{0s}^2} = -\frac{ib_0^2}{2}. \tag{A5}$$

Now equation (3) reduces to

$$Z_l(\omega) = -\frac{i\omega Z_0}{4\pi c} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dz du\, b'(z) b'(u) \delta(z-u) = -\frac{i\omega Z_0}{4\pi c} \int_{-\infty}^{\infty} dz \left(b'(z)\right)^2, \tag{A6}$$

in agreement with Eq. (1). The condition $l \gg \xi^*$, by the order of magnitude, is equivalent to Eq. (4).