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The Generalized Crewther Relation in QCD and its Experimental Consequences *

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Abstract

We use the BLM scale-fixing prescription to derive a renormalization-scheme invariant relation between the coefficient function for the Bjorken sum rule for polarized deep inelastic scattering and the R-ratio for the e^+e^- annihilation cross section. This relation provides a generalization of the Crewther relation to non-conformally invariant gauge theories. The derived relations allow one to calculate unambiguously without renormalization scale or scheme ambiguity the effective charges of the polarized Bjorken and the Gross-Llewellen Smith sum rules from the experimental value for the effective charge associated with R-ratio. Present data are consistent with the generalized Crewther relations, but measurements at higher precision and energies will be needed to decisively test these fundamental relations in QCD.

1. Introduction

In 1972 Crewther [1] derived a remarkable consequence of the operator product expansion for conformally invariant gauge theory. Crewther's relation has the form

$$3S = KR', \tag{1}$$

where S is the value of the anomaly controlling $\pi^0 \to \gamma\gamma$ decay, K is the value of the Bjorken sum rule in polarized deep inelastic scattering, and R' is the isovector part of the annihilation cross section ratio $\sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$.

The status of Crewther's relation within perturbative QCD where conformal invariance does not hold was recently analyzed in Ref. [2]. Using the existing multi-loop calculations for the coefficient function for $R(s) = \sigma_{\rm tot}(e^+e^- \to \text{hadrons})$ [3, 4, 5] and the polarized Bjorken sum rule [6, 7], the authors of Ref. [2] observed that all perturbative corrections of the type $C_{\rm F}^N \alpha_{\rm s}^N$, $1 \leq N \leq 3$ are mutually cancelled in the product of the coefficient function for the polarized Bjorken sum rule and the Adler's function for e^+e^- annihilation into hadrons. They also showed that the surviving corrections are suitably grouped in the two-loop QCD β-function and derived the QCD generalization of the Crewther relation at the three-loop order. It has been demonstrated in Ref. [8] that the cancellation of the $C_{\rm F}^N \alpha_{\rm s}^N$ type corrections (for arbitrary N) is related to the nonrenormalizability of the axial-vector-vector three-point Green function occurring when the appropriate normalization for the non-singlet axial current is chosen [9, 10] (for the discussions of this subject see e.g. the works from Ref. [11]). On the other hand, the authors of Ref. [12] have proposed to resolve the problem of the renormalization scale ambiguity by focusing on relations between experimentally measurable observables and using the BLM prescription [13]. In the present paper this idea is applied to the Crewther relation in QCD.

A helpful tool for relating physical quantities is the "effective charge" approach. Any perturbatively calculable physical quantity can be used to define an effective charge [14, 15, 16] by incorporating the entire radiative correction into its definition. An important result is that all effective charges $\alpha_A(Q^2)$ satisfy the renormalization group equation with the same first (β_0) and the second (β_1) coefficients of the β function. The renormalization group evolutions of the effective charges only differ through the third and higher coefficients of the effective β functions, which are process-dependent but scheme-independent. Thus, any effective charge can be used as a reference coupling of any renormalization scheme, including the $\overline{\rm MS}$-scheme. Each effective charge $\alpha_A(Q^2)$ is a special case of the universal coupling function $\alpha(Q^2, \beta_n)$, $n \geq 2$ (see e.g. [17]; for the recent theoretical studies of the "scheme-invariant" expansions see Ref. [18]).

For example, consider the Adler function [19] for the e^+e^- annihilation cross section

$$D(Q^2) = -12\pi^2 Q^2 \frac{d}{dQ^2}\Pi(Q^2), \;\; \Pi(Q^2) = -\frac{Q^2}{12\pi^2}\int_{4m_\pi^2}^{\infty} \frac{R_{e^+e^-}(s)ds}{s(s+Q^2)}. \tag{2}$$

The entire radiative correction to this function is defined as the effective charge $\alpha_D(Q^2)$:

$$\begin{aligned} D\left(Q^2/\mu^2, \alpha_{\rm s}(\mu^2/\Lambda^2_{\overline{\rm MS}})\right) &= D\left(1, \alpha_{\rm s}(Q^2/\Lambda^2_{\overline{\rm MS}})\right) \equiv \\ 3\sum_f Q_f^2\left[1 + \frac{\alpha_D(Q^2/\Lambda_D^2)}{\pi}\right] + (\sum_f Q_f)^2 C_{\rm L}(Q^2) &\equiv 3\sum_f Q_f^2 C_D(Q^2) + (\sum_f Q_f)^2 C_{\rm L}(Q^2), \end{aligned} \tag{3}$$

where Λ_D is the scheme-independent effective scale parameter. The coefficient $C_{\rm L}(Q^2)$ appears at the third order in perturbation theory and is related to the "light-by-light scattering type" diagrams. (Hereafter α_s will denote the $\overline{\rm MS}$ scheme strong coupling constant).

We can similarly define the entire radiative correction to the Bjorken sum rule as the effective charge $\alpha_{g_1}(Q^2)$ where Q is the corresponding momentum transfer:‡

$$\int_0^1 dx\left[g_1^{ep}(x,Q^2)-g_1^{en}(x,Q^2)\right]\equiv\frac{1}{6}\left|\frac{g_A}{g_V}\right|C_{\rm Bj}(Q^2)=\frac{1}{6}\left|\frac{g_A}{g_V}\right|\left[1-\frac{\alpha_{g_1}(Q^2/\Lambda_{g_1}^2)}{\pi}\right]. \quad (4)$$

It is straightforward to algebraically relate $\alpha_{g_1}(Q^2)$ to $\alpha_D(Q^2)$ using the known expressions to three loops in the $\overline{\rm MS}$ scheme. If one chooses new renormalization scales to re-sum all quark and gluon vacuum polarization corrections into $\alpha_D(Q^2)$, then the final result turns out to be remarkably simple and can be expressed in the following form [12]:

$$\widehat{\alpha}_{g_1}(Q)=\widehat{\alpha}_D(Q^*)-\widehat{\alpha}_D^2(Q^{**})+\widehat{\alpha}_D^3(Q^{***})+\cdots, \quad (5)$$

where $\widehat{\alpha}_{g_1}(Q)=\frac{3C_{\rm F}}{4\pi}\alpha_{g_1}(Q)$, $\widehat{\alpha}_D(Q)=\frac{3C_{\rm F}}{4\pi}\alpha_D(Q)$. This "commensurate scale relation" (CSR) was derived in Ref. [12] by using distinct commensurate scales Q^*, Q^{**} and Q^{***}.

These commensurate scales were related in Ref. [12] through the mean value theorem to the mean virtuality of the momenta of the gluon propagators which appear in each respective order of the perturbation theory. (For discussions and various applications of this physical language, see Refs. [12, 13, 21, 22, 23].)

In this paper we will show that it is also possible and convenient to choose one unique mean scale $\overline{Q}^*$ in $\alpha_D(Q)$ so that the corrections at the right-hand side will also reproduce the coefficients of the geometric progression. The possibility of using a single scale in the generalization of the BLM prescription beyond the next-to-leading order (NLO) was first considered in Ref. [24]. The new single-scale Crewther relation has the form:

$$\widehat{\alpha}_{g_1}(Q)=\widehat{\alpha}_D(\overline{Q}^*)-\widehat{\alpha}_D^2(\overline{Q}^*)+\widehat{\alpha}_D^3(\overline{Q}^*)+\cdots, \quad (6)$$

The coefficients in the CSR relating $\alpha_{g_1}(Q)$ to $\alpha_D(Q)$ (aside for a factor of $C_{\rm F}$, (in QCD $C_{\rm F}=4/3$) which can be absorbed in the definition of the couplings) are actually independent of color and are the same in Abelian, non-Abelian, and conformal

‡For a detailed analysis of the available experimental data for this sum rule, see Ref. [20].

gauge theory. The non-Abelian structure of the theory is reflected in the expression for the scale $\overline{Q}^*$. Note that the $\overline{\mathrm{MS}}$ renormalization scheme is used here for calculational convenience; it serves simply as an intermediary between observables. This is equivalent to the group property defined by Peterman and Stückelberg [25] which ensures that the forms of the CSR relations in perturbative QCD are independent of the choice of an intermediate renormalization scheme or renormalization scale μ. (The renormalization group method was developed by Gell-Mann and Low [26] and by Bogoliubov and Shirkov [27].)

2. Derivation of Commensurate Scale Relations and the Generalized Crewther Relation

Let us now discuss in more detail the derivation of the relation between observables in QCD. Any effective charge in perturbative QCD can be written in the following form:

$$\begin{aligned}\frac{\alpha_1(Q^2/\Lambda^2_{1\mathrm{eff}})}{4\pi} &= \frac{\alpha_{\mathrm{s}}(Q^2/\Lambda^2_{\overline{\mathrm{MS}}})}{4\pi} + \Big(A_1 + B_1\beta_0\Big)\left(\frac{\alpha_{\mathrm{s}}(Q^2/\Lambda^2_{\overline{\mathrm{MS}}})}{4\pi}\right)^2 \\ &+ \Big(C_1 + D_1\beta_0 + E_1\beta_0^2 + B_1\beta_1\Big)\left(\frac{\alpha_{\mathrm{s}}(Q^2/\Lambda^2_{\overline{\mathrm{MS}}})}{4\pi}\right)^3 + \cdots, \qquad (7)\end{aligned}$$

where $\Lambda^2_{1\mathrm{eff}} = \Lambda^2_{\overline{\mathrm{MS}}}\exp(\frac{A_1+B_1\beta_0}{\beta_0})$ and β_0, β_1 are the scheme-invariant first two coefficients of QCD β-function defined as $Q^2 d(\alpha_{\mathrm{s}}/4\pi)/dQ^2 = \beta(\alpha_{\mathrm{s}}) = -\sum_{i\geq 0}\beta_i(\alpha_{\mathrm{s}}/4\pi)^{i+2}$. Within our normalization conditions the first two scheme-independent coefficients of the QCD β-function read: $\beta_0 = 11 - (2/3)f$, $\beta_1 = 102 - (38/3)f$. In the MS-like schemes the third coefficient was calculated in Ref. [28] and has the following form $\beta_2 = 2857/2 - (5033/18)f + (325/54)f^2$.

Notice that the appropriate adjustment of the D_1 multiplier allows one to obtain the B_1 coefficient in front of β_1 in the next-next-to-leading order (NNLO) (it coincides with the coefficient in front of β_0 in the NLO).

Similarly, given a second effective charge $\alpha_2(Q)$, we can put it in the form

$$\frac{\alpha_2(Q^2/\Lambda^2_{2\mathrm{eff}})}{4\pi} = \frac{\alpha_{\mathrm{s}}(Q^2/\Lambda^2_{\overline{\mathrm{MS}}})}{4\pi} + \Big(A_2 + B_2\beta_0\Big)\left(\frac{\alpha_{\mathrm{s}}(Q^2/\Lambda^2_{\overline{\mathrm{MS}}})}{4\pi}\right)^2$$

$$+ \quad \left(C_2 + D_2\beta_0 + E_2\beta_0^2 + B_2\beta_1\right)\left(\frac{\alpha_s(Q^2/\Lambda^2_{\overline{\rm MS}})}{4\pi}\right)^3 + \cdots, \tag{8}$$

with $\Lambda^2_{\rm 2eff} = \Lambda^2_{\overline{\rm MS}}\exp(\frac{A_2+B_2\beta_0}{\beta_0})$. The two effective charges $\alpha_1(Q)$ and $\alpha_2(Q)$ are related through the following series,

$$\begin{aligned}\frac{\alpha_1(Q^2/\Lambda^2_{\rm 1eff})}{4\pi} &= \frac{\alpha_2(Q^2/\Lambda^2_{\rm 2eff})}{4\pi} + \left(A_{12} + B_{12}\beta_0\right)\left(\frac{\alpha_2(Q^2/\Lambda^2_{\rm 2eff})}{4\pi}\right)^2 + \\ &+ \left(C_{12} + D_{12}\beta_0 + E_{12}\beta_0^2 + B_{12}\beta_1\right)\left(\frac{\alpha_2(Q^2/\Lambda^2_{\rm 2eff})}{4\pi}\right)^3 + \cdots, \end{aligned}\tag{9}$$

where the coefficients $A_{12}, B_{12}, C_{12}, D_{12}$ and E_{12} are given by: $A_{12} = A_1 - A_2$, $B_{12} = B_1 - B_2$, $C_{12} = C_1 - C_2 - 2(A_1 - A_2)A_2$, $D_{12} = D_1 - D_2 - 2(A_1B_2 + A_2B_1) + 4A_2B_2$, $E_{12} = E_1 - E_2 - 2(B_1 - B_2)B_2$.

In the multiple-scale CSR approach derived in Ref. [12], one absorbs order-by-order the coefficients which depend on the number of flavors into the redefinition of the commensurate scales. As discussed in Ref. [12], this feature is analogous to analyses in QED, where the skeleton diagrams of different orders can have different renormalization scales [13, 29]. After absorbing the running coupling effects order-by-order, the authors of Ref. [12] obtained

$$\frac{\alpha_1(Q^2/\Lambda^2_{\rm 1eff})}{4\pi} = \frac{\alpha_2(Q^{*2}/\Lambda^2_{\rm 2eff})}{4\pi} + A_{12}\left(\frac{\alpha_2(Q^{**2}/\Lambda^2_{\rm 2eff})}{4\pi}\right)^2 + C_{12}\left(\frac{\alpha_2(Q^{***2}/\Lambda^2_{\rm 2eff})}{4\pi}\right)^3 + \cdots, \tag{10}$$

where in the definitions of this work the relations between the renormalization scales read

$$\ln\left(Q^{*2}/Q^2\right) = -B_{12} + \beta_0\left(B_{12}^2 - E_{12}\right)\left(\frac{\alpha_2(Q^{*2}/\Lambda^2_{\rm 2eff})}{4\pi}\right), \ln\left(Q^{**2}/Q^2\right) = -\frac{D_{12}}{2A_{12}}, \tag{11}$$

and Q^{***} can be chosen as Q^{**} if the NLO coefficient is non-zero, or as Q^* if the NLO coefficient is zero. As discussed in Ref. [12], the use of different renormalization scales at different orders of perturbation theory can be related to the different mean momenta in the skeleton graphs contributing at each order. Notice also that since the coefficient before β_1-term in the NNLO contribution to Eq. (9) is equal to the coefficient B_{12} of the NLO correction to Eq. (9), the absorption of the proportional to β_0 NLO term in Eq. (9) into the scale Q^* automatically leads to the nullification of the NNLO contributions into Eq. (9), which is proportional to the β_1-coefficient of the QCD β-function. The remaining NNLO terms in Eq. (9), which turn out to be proportional to β_0^2 and β_0, were absorbed in Ref. [12] into the scales Q^* and Q^{**}. An alternative approach is to use a single renormalization scale for all orders in the right-hand side of Eq. (9). This approach for generalizing the BLM procedure has been considered in Ref. [24]. In the single-scale approach we have

$$\frac{\alpha_1(Q^2/\Lambda_{1\text{eff}}^2)}{4\pi} = \frac{\alpha_2(\overline{Q}^{*2}/\Lambda_{2\text{eff}}^2)}{4\pi} + A_{12}\left(\frac{\alpha_2(\overline{Q}^{*2}/\Lambda_{2\text{eff}}^2)}{4\pi}\right)^2 + C_{12}\left(\frac{\alpha_2(\overline{Q}^{*2}/\Lambda_{2\text{eff}}^2)}{4\pi}\right)^3 + \cdots, \tag{12}$$

where the expansion coefficients in this series are identical to those of the multiple-scale case in Eq. (9). However, there is only one single renormalization scale

$$\ln\left(\overline{Q}^{*2}/Q^2\right) = -B_{12} + \left[\beta_0\ \left(B_{12}^2 - E_{12}\right) + 2A_{12}B_{12} - D_{12}\right]\left(\frac{\alpha_2(\overline{Q}^{*2}/\Lambda_{2\text{eff}}^2)}{4\pi}\right). \tag{13}$$

Hereafter we will simply drop the dependence of the coupling constants on the QCD parameters Λ_{eff}.

We now apply the above general procedures to derive a single-scale CSR between the effective couplings for the coefficient function of Bjorken polarized sum rule and the Adler's function for the e^+e^- annihilation. The perturbative series for $\alpha_{g_1}(Q)/4\pi$ using dimensional regularization and the $\overline{\text{MS}}$ scheme with the renormalization scale fixed at $\mu = Q$ has been computed at the NNLO in [7]. The effective charge for the annihilation cross section has been computed in the $\overline{\text{MS}}$ scheme at the NNLO with the renormalization scale fixed at $\mu = Q$ in Ref. [4, 5]. After eliminating the $\overline{\text{MS}}$-scheme and applying the BLM procedure, the single scale result has the following simple form:

$$\widehat{\alpha}_{g_1}(Q) = \widehat{\alpha}_D(\overline{Q}^*) - \widehat{\alpha}_D^2(\overline{Q}^*) + \widehat{\alpha}_D^3(\overline{Q}^*) + \cdots, \tag{14}$$

where $\widehat{\alpha}_{g_1} = (3C_{\text{F}}/4\pi)\alpha_{g_1}$ and $\widehat{\alpha}_D = (3C_{\text{F}}/4\pi)\alpha_D$. Recalling that $\widehat{\alpha}_{g_1}(Q) = C_{\text{Bj}}(Q) - 1$, $\widehat{\alpha}_D(\overline{Q}^*) = C_D(\overline{Q}^*) - 1$, we get the simple relation

$$C_{\text{Bj}}(Q)C_D(\overline{Q}^*) = 1. \tag{15}$$

Here

$$\ln\left(\frac{\overline{Q}^{*2}}{Q^2}\right) = \frac{7}{2} - 4\zeta(3) + \left(\frac{\alpha_D(\overline{Q}^*)}{4\pi}\right)\left[\left(\frac{11}{12} + \frac{56}{3}\zeta(3) - 16\zeta^2(3)\right)\beta_0 \right.$$
$$\left. + \frac{26}{9}C_{\rm A} - \frac{8}{3}C_{\rm A}\zeta(3) - \frac{145}{18}C_{\rm F} - \frac{184}{3}C_{\rm F}\zeta(3) + 80C_{\rm F}\zeta(5)\right]. \quad (16)$$

where in QCD $C_{\rm A} = 3$, $C_{\rm F} = 4/3$. The relations (15) and (16) show how the coefficient functions for these two different processes are related to each other at their respective commensurate scales. The evaluation of one of them at the appropriate physical scale gives us information about the second one at the different physical scale. Notice also that all the $\zeta(3)$ and $\zeta(5)$ dependencies have been absorbed into the renormalization scale $\overline{Q}^*$. The explicit forms for the corresponding multiple scales are given in Ref. [12].

3. The Generalized Crewther Relation and its Experimental Consequences

The simple form of Eqs. (5), (6), (15) points to the existence of a "secret symmetry" between $\alpha_D(Q)$ and $\alpha_{g_1}(Q)$ which is revealed after the application of the BLM scale setting procedure. In the conformally invariant limit, *i.e.*, for vanishing beta functions, the generalized Crewther relation derived in Ref. [2] becomes

$$(1 + \widehat{\alpha}_D^{\rm eff})(1 - \widehat{\alpha}_{g_1}^{\rm eff}) = 1, \quad (17)$$

which is equivalent to Eqs. (5), (6), (15). Thus Eqs. (5), (6) or (15) can be regarded as the extension of the Crewther relation to non-conformally invariant gauge theory in which all effects of the non-vanishing QCD β-function are absorbed into the scale of the coupling and each physical quantity has its appropriate scale of momenta or energy.

We can also write down analogous equations for the polarized Bjorken sum rule or for the Gross-Llewellyn Smith sum rule, defined as

$$\frac{1}{2}\int_0^1 F_3^{\nu p+\overline{\nu}p}(x,Q^2)dx = 3C_{\rm GLS}(Q^2) = 3[1 - \frac{\alpha_{\rm GLS}(Q^2)}{\pi}] \quad (18)$$

and the Adler's function for the e^+e^- annihilation process. Taking into account the additional three-loop "light-by-light-type" diagrams, the needed relations can be expressed in the form

$$\frac{1}{3\sum_f Q_f^2} D(\overline{Q}^*) C_{\rm Bj}(Q) = 1 + \varepsilon_1(Q) + \cdots, \tag{19}$$

and

$$\frac{1}{3\sum_f Q_f^2} D(\overline{Q}^*) C_{\rm GLS}(Q) = 1 + \varepsilon_2(Q) + \cdots, \tag{20}$$

where the dots denote higher order corrections and $\varepsilon_1(Q)$, $\varepsilon_2(Q)$ are related to the light-by-light scattering type diagrams:

$$\begin{aligned}
\varepsilon_1(Q) &\equiv \left(\frac{44}{9} - \frac{32}{3}\zeta(3)\right) \frac{d^{abc}d^{abc}}{C_{\rm F} N_c} \frac{\left(\sum_f Q_f\right)^2}{\sum_f Q_f^2} \left(\frac{\alpha_D(Q)}{4\pi}\right)^3, \\
\varepsilon_2(Q) &\equiv \left(\frac{44}{9} - \frac{32}{3}\zeta(3)\right) \frac{d^{abc}d^{abc}}{C_{\rm F} N_c} \frac{\left(\sum_f Q_f\right)^2 - f\sum_f Q_f^2}{\sum_f Q_f^2} \left(\frac{\alpha_D(Q)}{4\pi}\right)^3.
\end{aligned} \tag{21}$$

The scales Q and $\overline{Q}^*$ are defined above, $N_c = 3$ is the number of colors, f is the number of active flavors of quarks with the charges Q_f and in QCD $d^{abc}d^{abc} = 40/3$. Strictly speaking, the right hand sides of these equations depend on a scale variable $\overline{Q}^*$; however recalling that we are not able to control the fourth order corrections to these relations we just replace $\overline{Q}^*$ by Q in the expressions for the third order corrections. Notice that the numerical values of $\varepsilon_1(Q)$ and $\varepsilon_2(Q)$ terms are very small.

We now address the question whether one can extract phenomenologically useful consequences from these relations. To do this we shall express the perturbatively calculated Adler's function through the experimentally measurable R-ratio for the e^+e^- annihilation cross section. As is well known, the perturbatively defined Adler's function is computed for the space-like transfer momenta. In order to obtain the expression for the measurable R-ratio one has to analytically continue from the space-like to the time-like momenta. If one is sufficiently far from the lowest resonance production thresholds, it is possible to perform this continuation using the following simple formula

$$R_{e^+e^-}(s) = \frac{1}{2\pi i}\int_{-s-i\varepsilon}^{-s+i\varepsilon} \frac{d\tau}{\tau} D(\alpha_{\rm s}(\tau)) = 3\sum_f Q_f^2 [1 + \frac{\alpha_R(\sqrt{s})}{\pi}]. \tag{22}$$

In general, this procedure results in the appearance of π^2 like terms in the coefficient functions. More precisely, in our case the following shifts of numerical coefficients of Eq. (8) and Eq. (9) take place: $E_2 \to E_2 - \frac{1}{3}\pi^2$; $E_{12} \to E_{12} + \frac{1}{3}\pi^2$. In addition it is necessary also to perform the replacements: $\overline{Q}^* \to \sqrt{s}$, $\alpha_D \to \alpha_R$, where s is defined through the value of Q^2

$$\begin{aligned} \ln\left(\frac{Q^2}{s}\right) &= -\frac{7}{2} + 4\zeta(3) - \left(\frac{\alpha_R(\sqrt{s})}{4\pi}\right)\left[\left(\frac{11}{12} + \frac{56}{3}\zeta(3) - 16\zeta^2(3) - \frac{\pi^2}{3}\right)\beta_0 \right. \\ &\left. + \frac{26}{9}C_{\rm A} - \frac{8}{3}C_{\rm A}\zeta(3) - \frac{145}{18}C_{\rm F} - \frac{184}{3}C_{\rm F}\zeta(3) + 80C_{\rm F}\zeta(5)\right]. \end{aligned} \quad (23)$$

Now the generalized Crewther relation takes the form

$$\frac{1}{3\sum_f Q_f^2} R_{e^+e^-}(s) C_{\rm Bj}(Q^2) = 1 + \varepsilon_1(Q^2), \quad (24)$$

and

$$\frac{1}{3\sum_f Q_f^2} R_{e^+e^-}(s) C_{\rm GLS}(Q^2) = 1 + \varepsilon_2(Q^2), \quad (25)$$

with ε_1 and ε_2 defined above.

In what follows we will neglect the small values of ε_1 and ε_2 by simply assuming that $C_{\rm Bj} \simeq C_{\rm GLS}$, which is valid up to the NLO approximation of perturbation theory and is almost valid at the NNLO (note that the additional light-by-light-type contribution to the NNLO correction to $C_{\rm GLS}$, which is included in the expression for $\varepsilon_2(Q)$, is very small [7]). The experimental measurements of the R-ratio above the thresholds for the production of $c\overline{c}$-bound states, together with the theoretical fit performed in Ref. [30], provide the constraint

$$\frac{1}{3\sum_f Q_f^2} R_{e^+e^-}(\sqrt{s} = 5.0 \text{ GeV}) \simeq \frac{3}{10}(3.6 \pm 0.1) = 1.08 \pm 0.03. \quad (26)$$

and thus

$$\frac{\alpha_R^{\rm exp}(\sqrt{s} = 5.0 \text{ GeV})}{\pi} \simeq 0.08 \pm 0.03. \quad (27)$$

As a consequence, from Eqs. (24) and (25) we obtain the following estimate for $C_{\rm Bjp}(Q) \simeq C_{\rm GLS}(Q)$:

$$C_{\rm Bj}(Q = 12.33 \pm 1.20 \text{ GeV}) \simeq C_{\rm GLS}(Q = 12.33 \pm 1.20 \text{ GeV}) = 0.926 \mp 0.026, \quad (28)$$

where the error bars for Q and $C_{\rm Bj}(Q)$ are calculated using the Eqs. (23) and (24) respectively. The error bars correspond to the uncertainties in the empirical value of α_R in Eq. (27).[§]

The corresponding expression for the effective coupling constants is

$$\frac{\alpha_{g_1}^{\rm exp}(Q=12.33\pm1.20\ {\rm GeV})}{\pi} \simeq \frac{\alpha_{\rm GLS}^{\rm exp}(Q=12.33\pm1.20\ {\rm GeV})}{\pi} \simeq 0.074\pm0.026; \quad (29)$$

The predictions given above for $\alpha_{g_1} \simeq \alpha_{\rm GLS}$ are the predictions which can be tested experimentally. The recent measurements for the Gross-Llewellyn Smith sum rule are performed only at relatively small values of Q^2 [31, 32]; however, one can use the results of the theoretical extrapolation [33] of the experimental data presented in [34] and turn to the domain of large values of Q^2. Notice that the results of Ref. [33] for the Gross-Llewellyn Smith sum rule at $Q^2 = 3$ GeV^2 are in good agreement with the well-determined experimental results (see Refs. [31, 32]). Thus it is not difficult to extract the value for $\alpha_{\rm GLS}(Q)/\pi$ from Table 1 of Ref. [33]:

$$\frac{\alpha_{\rm GLS}^{\rm extrapol}(Q=12.25\ {\rm GeV})}{\pi} \simeq 0.093\pm0.042. \quad (30)$$

This interval overlaps with the result previously derived in Eq. (29): 0.074 ± 0.026. This gives empirical support for the generalized Crewther relation derived in Eqs.(14)-(16) and in Eqs.(24), (25). The relation of the results Eqs.(27)-(30) to the commonly used language of the $\overline{\rm MS}$-scheme will be presented elsewhere [35].

4. Discussions.

It is worthwhile to point out that all of the results presented here are derived within the framework of perturbation theory and do not involve the nonperturbative contributions to the Adler's function $D(Q^2)$ [36] and the R-ratio, as well as to the polarized Bjorken and the Gross-Llewellyn Smith sum rules [37, 38]. These nonperturbative contributions are expected to be significant at small energies and momentum transfer.

[§]Keeping the third significant digit after the decimal point in these equations is an overestimate of the available theoretical accuracy, but we will keep it nevertheless in order to have better control of the real values of the second significant digit in the r.h.s. of Eqs.(28)-(30).

In order to make these contributions comparatively negligible, we have chosen relatively large values for s and Q^2 in our numerical study. Even at the current level of understanding we can conclude that our numerical analysis demonstrates reasonable agreement between the theoretical predictions and experimental results. In order to put the analysis of the experimental data for lower energies on more solid ground, it will be necessary to understand whether there exist any Crewther-type relations between non-perturbative order $O(1/Q^4)$-corrections to the Adler's D-function [36] and the order $O(1/Q^2)$ higher twist contributions to the deep-inelastic sum rules [37, 38].

The generalized Crewther relation written in the form of CSR provides an important test of QCD. Since the Crewther formula written in the form of the CSR relates one observable to another observable, the predictions are independent of theoretical conventions, such as the choice of renormalization scheme. It is clearly very interesting to test these fundamental self-consistency relations between the polarized Bjorken sum rule or the Gross-Llewellyn Smith sum rule and the e^+e^--annihilation R-ratio. Present data are consistent with the generalized Crewther relations, but measurements at higher precision and energies will be needed to decisively test these fundamental connections in QCD.

In order to check the consequence of the generalized Crewther relation at a higher confidence level, it will be necessary, first, to reduce the experimental error of the measurement of $R_{e^+e^-}$ at $\sqrt{s} \approx 5$ GeV and, secondly, to have more precise information on the value of the Gross-Llewellyn Smith sum rule at $Q^2 \approx 150\ GeV^2$ or to measure the polarized Bjorken sum rule at this momentum transfer. We hope that the first problem can be attacked after the start of the operation of a $c-\tau$-factory. Moreover, the possible future study of the deep inelastic scattering with both polarized electron and proton beams can open the window for the direct measurements of the polarized Bjorken sum rule at high momentum transfer [39]. Keeping this in mind we propose direct measurements of the polarized Bjorken sum rule at the scales $Q^2 = 150$ GeV^2 and $Q^2 = 25$ GeV^2. As was shown in the previous section, the related measurements at the energy scale $Q^2 = 150$ GeV^2 can be helpful for checking the generalized Crewther relation, written down in the form of the single-scale CSR, derived in Eq. (14).

The direct measurements of the polarized Bjorken sum rule (or of the Gross-Llewellyn Smith sum rule) at the scale $Q^2 = 25$ GeV^2 can be useful for the study of the intriguing question whether the experimental data can "sense" the violation of the initial conformal invariance caused by the renormalization procedure. In the language of the Crewther relation this question can be reformulated in the following manner: what will happen if we put the scales of $R_{e^+e^-}$ and the corresponding sum rules to be equal to each other? Will the experimental data produce the unity on the r.h.s. of Eq. (17), related to the conformal invariant limit, if we put $s = |Q^2|$? Recall, that in this case, the theoretical expression for the generalized Crewther relation will differ from the conformal invariant result starting from the proportional to the well-known factor $\beta(\alpha_s)/\alpha_s$ the α_s^2-order corrections [2], which presumably reflects the violation of the conformal symmetry by the procedure of renormalization [1, 40] (for discussions see Ref. [8]). Notice, however, that the size of the perturbative contribution proportional to the QCD β - function is rather small [2].

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