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A MAP FOR THE THICK BEAM-BEAM INTERACTION†

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We give a closed-form expression for the thick beam-beam interaction for a small disruption parameter, as typical in electron-positron storage rings. The dependence on transverse angle and position of the particle trajectory as well as the longitudinal position of collision and the waist-modified shape of the beam distribution are included. Large incident angles, as are present for beam-halo particles or for large crossing-angle geometry, are accurately represented. The closed-form expression is well approximated by polynomials times the complex error function. Comparisons with multi-slice representations show even the first order terms are more accurate than a five slice representation, saving a factor of 5 in computation time.

I. INTRODUCTION

S. Krishnagopal and R. Siemann[1] have established that it is essential to retain the finite longitudinal thickness of the beam when simulating the beam-beam interaction. Present beam-beam simulation codes[2] represent the thick interaction as a set of thin beam-beam kicks. Since the thin beam-beam kick for a bi-Gaussian distribution is usually represented by the Erskine-Bassetti[3] formula involving the evaluation of two complex error functions, the calculation is computer intensive and the representation of the thick beam-beam interaction as a set of slices slows down the simulation by a factor roughly equal to the number of slices. To find a thick beam-beam generator, we represent the thin beam-beam kick by an exponential Lie map, and the thick beam-beam interaction by an infinite product of thin beam-beam kicks. We then use the Cambell-Baker-Hausdorff formula to compose this product to a single exponential map.

II. THE THIN BEAM-BEAM KICK

S. Kheifets[4] has shown that the potential for the Erskine-Bassetti thin beam-beam kick is given by

$$dV(x,y,\sigma_x^2,\sigma_y^2) = dN\frac{2r_e}{\gamma} I(x,y,\sigma_x^2,\sigma_y^2) \tag{1}$$

where dN is the number of electrons in the thin sheet, r_e is the classical electron radius, γ is the Lorentz factor of the particle, and $I(x,y,\sigma_x^2, \sigma_y^2)$ is the (dimensionless) integral

$$I(x,y,\sigma_x^2,\sigma_y^2) = \int_0^\infty d\lambda \cdot g(x,\sigma_x^2,\lambda)\cdot g(y,\sigma_y^2,\lambda) \tag{2}$$

The function

$$g(x,\sigma_x^2,\lambda) = \frac{1}{\sqrt{\pi}}\cdot\frac{e^{-\frac{x^2}{(2\sigma_x^2+\lambda)}}}{\sqrt{2\sigma_x^2+\lambda}} \tag{3}$$

is a normalized Gaussian distribution of rms width $(\sigma_x^2+\lambda/2)^{1/2}$ and a solution of the diffusion equation,

$$\frac{1}{2}\frac{\partial^2}{\partial x^2} g(x,\sigma_x^2,\lambda) = \frac{\partial}{\partial\sigma_x^2} g(x,\sigma_x^2,\lambda) \cdot \tag{4}$$

The vertical and horizontal kicks from the potential dV are given by the Erskine-Basseti formula[3] as

$$\Delta y' + i\Delta x' = dN\frac{4r_e}{\gamma}\frac{\sqrt{\pi}}{\sigma}\left[w(z_1) - e^{-(\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2})} w(z_2)\right] \tag{5}$$

where

$$\sigma = \sqrt{2(\sigma_x^2-\sigma_y^2)},\quad z_1 = \frac{1}{\sigma}(x+iy),\quad z_2 = \frac{1}{\sigma}\left(\frac{\sigma_y}{\sigma_x}x + i\frac{\sigma_x}{\sigma_y}y\right)$$

and $w(z) = e^{-z^2}\left(1+\frac{2i}{\pi}\int_0^z du\cdot e^{u^2}\right)$ (6)

is the complex error function. The map for this kick may be represented as an exponential Lie map by[5] $M = e^{-:dV:}$. (7)

III. THE THICK BEAM-BEAM MAP

We assume the longitudinal particle density is also Gaussian and given by

$$\rho(s-s_0,\sigma_s) = \frac{1}{\sqrt{2\pi\sigma_s^2}} e^{-\frac{(s-s_0)^2}{2\sigma_s^2}}. \tag{8}$$

The potential for a small slice of thickness ds at position s will be given by

$$dV = \rho(s-s_o,\frac{\sigma_s}{2})ds\,\frac{2Nr_e}{\gamma} I(x(s),y(s),\sigma_x^2(s),\sigma_y^2(s)) \tag{9}$$

since the beam distribution as seen by the counter-moving particle is fore-shortened by a factor of 2. The coordinate $s_0 = c\,\tau/2$ is the location of the center of the bunch as experienced by a counter-moving particle whose time of arrival at the interaction point (IP) is delayed by τ. The map through a set of n slices is given by

$$M = \prod_{j=1}^n e^{-\left[:\rho(s_j-s_o,\frac{\sigma_s}{2})ds_j\frac{2Nr_e}{\gamma} I(x(s_j),y(s_j),\sigma_x^2(s_j),\sigma_y^2(s_j)):\right]} \tag{10}$$

where s_j is the longitudinal position of the center of the slice of width ds_j.

The first-order Cambell-Baker-Hausdorff (CBH) formula can be invoked to obtain, in the limit of very small slices,

$$M \approx e^{-:G_1:} = \prod_{j=1}^n e^{-\left[\frac{2Nr_e}{\gamma}\cdot\int ds\,\rho(s-s_o,\frac{\sigma_s}{2})\cdot ds\cdot I(x(s),y(s),\sigma_x^2(s),\sigma_y^2(s)):\right]} \tag{11}$$

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The transverse position of the particle may be written $x(s) = x_0 + (s-s_0)\, x'_0$ and $y(s) = y_0 + (s-s_0)\, y'_0$, where x_0, x'_0, y_0 and y'_0 are the particle's position and slope at $s=s_0$. Also $\sigma_x(s)^2=\sigma_x^2 + s^2\sigma_{x'}^2$ and $\sigma_y(s)^2=\sigma_y^2 + s^2\sigma_{y'}^2$ describe the longitudinal shape of the waist with σ_x, $\sigma_{x'}$, σ_y, and $\sigma_{y'}$ being the rms values of the position and slope of the beam distribution at the interaction point (IP).

The first-order CBH formula implicitly assumes that the trajectory of the particle is not changed by the interaction, and hence it represents an integrated impulse approximation. The accuracy of this impulse approximation may be estimated by noting that for electron storage rings we have

$$\frac{\Delta y'_{\max}}{\sigma_{y'}} = \Delta\theta_{y,\max} = 4\pi\xi_y \le 0.4$$

where we have taken the beam-beam tune-shifts $\xi_y = 0.03$. The second-order correction can be estimated through a two-slice approximation as:

$$\Delta y'_2 = \Delta y \cdot \frac{\partial}{\partial y}\left[\frac{1}{2}\Delta y'_1\right] \cong \sigma_s\left[\frac{1}{2}\Delta y'_1\right]\cdot\frac{\partial}{\partial y}\left[\frac{1}{2}\Delta y'_1\right] \le \frac{1}{4}\sigma_s \Delta y'_1 \cdot \frac{\Delta y'_{\max}}{\sigma_y}$$

from which we deduce that

$$\frac{\Delta y'_2}{\Delta y'_1} \le \frac{1}{4}\frac{\Delta y'_{\max}}{\sigma_{y'}} \le 0.1 \qquad \text{implying} \qquad \frac{\Delta y'_2}{\sigma_{y'}} \le 0.04$$

To achieve better than 4% accuracy it will be necessary to include second-order effects.

1. The first-order CBH integral

To find the generator for the thick beam-beam interaction we must perform the integral in equation (11). If we begin by assuming $x'_0 = \sigma_{x'} = \sigma_{y'} = 0$ the integral to be performed, after changing the order of the s and λ integration, is

$$Q_1 = \int ds \cdot \rho(s - s_0, \frac{\sigma_s}{2}) \cdot g(y_0 + (s - s_0)y'_0, \sigma_y^2, \lambda) \tag{12}$$

For simplicity we introduce the variables

$$\hat{s} = \frac{(s - s_0)}{(\frac{\sigma_s}{2})} \quad \text{and} \quad \hat{y}'_0 = y'_0 \cdot (\frac{\sigma_s}{2}).$$

We evaluate this integral, in a way that will later allow us to manage the full s dependence, by introducing a translation operator to represent

$$g(y_0 + \hat{s}\cdot\hat{y}', \sigma_y^2, \lambda) = e^{\hat{s}\cdot\hat{y}'\frac{\partial}{\partial y_0}} g(y_0, \sigma_y^2, \lambda). \tag{13}$$

We combine the $\hat{s}$ dependence of the translation operator $e^{\hat{s}\cdot\hat{y}'\frac{\partial}{\partial y_0}}$ with the $\hat{s}$ dependence of $\rho(\hat{s},1)$ and then complete the square, to obtain

$$\rho(\hat{s},1)\cdot e^{\hat{s}\hat{y}'\frac{\partial}{\partial y_0}} = \rho(\hat{s} - \hat{y}'\frac{\partial}{\partial y_0}, 1)\cdot e^{\frac{1}{2}\hat{y}'^2\frac{\partial^2}{\partial y_0^2}} \tag{16}$$

The integral over $d\hat{s}$ can now be performed by translating ρ:

$$\int d\hat{s}\cdot\rho(\hat{s} - \hat{y}'\frac{\partial}{\partial y_0}, 1) = \int d\hat{s}\cdot e^{-\hat{y}'\frac{\partial}{\partial y_0}\frac{\partial}{\partial s}}\rho(\hat{s},1)$$

$$= \int d\hat{s}\cdot\sum\frac{1}{n!}(-\hat{y}'\frac{\partial}{\partial y_0})^n\frac{\partial^n}{\partial\hat{s}^n}\rho(\hat{s},1) = 1 \tag{17}$$

Hence we have

$$Q_1 = e^{\frac{1}{2}\hat{y}'^2\frac{\partial^2}{\partial y_0^2}}\cdot g(y_0, \sigma_y^2, \lambda) \tag{18}$$

Since g is a solution of the diffusion equation

$$Q_1 = e^{\hat{y}'^2\frac{\partial^2}{\partial\sigma_y^2}}\cdot g(y_0, \sigma_y^2, \lambda) = g(y_0.\sigma_y^2 + \hat{y}'^2, \lambda) \tag{19}$$

This result for Q_1 implies that the thick beam-beam potential for the case $x'_0 = \sigma_{x'} = \sigma_{y'} = 0$ is just the original potential with σ_y^2. replaced by $\sigma_y^2 + \hat{y}'^2$.

Next we perform the integral with only $\sigma_{x'} = \sigma_{y'} = 0$.

$$Q_2 = \int d\hat{s}\cdot\rho(\hat{s},1)\cdot g(x_0 + \hat{s}\hat{x}', \sigma_x^2, \lambda)\cdot g(y_0 + \hat{s}\hat{y}', \sigma_y^2, \lambda) \tag{20}$$

$$= e^{\hat{x}'\hat{y}'\frac{\partial}{\partial x_0}\frac{\partial}{\partial y_0}} g(x_0, \sigma_x^2 + \hat{x}'^2, \lambda)\cdot g(y_0, \sigma_y^2 + \hat{y}'^2, \lambda)$$

The integral with only $\sigma_{x'} = 0$ is:

$$Q_3 = \int ds\cdot\rho(s - s_0, \frac{\sigma_s}{2})\cdot g(x_0 + (s - s_0)x'_0, \sigma_x^2, \lambda). \tag{21}$$

$$\cdot g(y_0 + (s - s_0)y'_0, \sigma_y^2 + s^2\sigma_{y'}^2, \lambda)$$

In changing s to $\hat{s}$ it is convenient to introduce

$$\hat{s}_0 = \frac{s_0}{(\frac{\sigma_s}{2})}, \quad \sigma_{\hat{y}'} = \frac{\sigma_s}{2}\sigma_{y'} \text{ and } \sigma_{y_0}^2 = \sigma_y^2 + s_0^2\sigma_{y'}^2.$$

Using these definitions

$$g(y_0 + \hat{s}\hat{y}', \sigma_{y_0}^2 + \sigma_{\hat{y}'}^2(2\hat{s}_0\hat{s} + \hat{s}^2), \lambda)$$

$$= e^{\frac{\sigma_{\hat{y}'}^2}{2}(2\hat{s}_0\frac{\partial}{\partial\hat{y}'}\frac{\partial}{\partial y_0} + \frac{\partial^2}{\partial\hat{y}'^2})} e^{\hat{s}\hat{y}'\frac{\partial}{\partial y_0}} g(y_0, \sigma_{y_0}^2, \lambda) \tag{22}$$

The first factor has no $\hat{s}$ dependence and can be taken out of the integral. The remaining integral, after translating x_0, is just Q_2. We can now write down an expression for the double integral in the generator of eq. 11 with no assumptions on x'_0, y'_0, $\sigma_{x'}$, and $\sigma_{y'}$:

$$G_1 = \frac{2Nr_e}{\gamma} e^{\frac{\sigma_{\hat{x}'}^2}{2}(2\hat{s}_0\frac{\partial}{\partial\hat{x}'}\frac{\partial}{\partial x_0} + \frac{\partial^2}{\partial\hat{x}'^2})} e^{\frac{\sigma_{\hat{y}'}^2}{2}(2\hat{s}_0\frac{\partial}{\partial\hat{y}'}\frac{\partial}{\partial y_0} + \frac{\partial^2}{\partial\hat{y}'^2})} e^{\hat{x}'\hat{y}'\frac{\partial}{\partial x_0}\frac{\partial}{\partial y_0}} \tag{23}$$

$$I(x_0, y_0, \sigma_{x_0}^2 + \hat{x}'^2, \sigma_{y_0}^2 + \hat{y}'^2)$$

2. First-order symplectification

To obtain a symplectic evaluation of G_1 one can introduce a mixed-variable generator

$$F_1(x, y, X', Y') = xX' + yY' + G_1(x, y, X', Y'). \tag{24}$$

The implicit equations can be solved approximately by expanding F_1 in a Taylor series about the point (x,x',y,y').

3. Evaluation of the derivatives of the generator

The calculation can be arranged so that the derivatives of G_1 required to compute coordinate changes can be found from derivatives of the complex error function. Derivatives of $I(x_0, y_0, \sigma_{x_0}^2 + \hat{x}'^2, \sigma_{y_0}^2 + \hat{y}'^2)$, with respect to $\hat{x}'$ or $\hat{y}'$ may be replaced by derivatives with respect to $\sigma_{x0}{}^2$ and $\sigma_{y0}{}^2$ which may in turn be replaced by derivatives with respect to x_0 and y_0. Thus the computation reduces to derivatives of the Erskine-Bassetti formula with respect to x_0 and y_0. These derivatives become polynomials times the complex error function by using the relationship

$$\frac{\partial w(z)}{\partial z} = -2z\,w(z) + \frac{2i}{\sqrt{\pi}} \qquad (25)$$

The order of this polynomial depends on the accuracy with which the waist effects are represented. The largest corrections to the thin beam-beam kick, arising from the slope y', are accounted for by replacing $\sigma_y{}^2$ by $\sigma_y^2 + \hat{y}'^2$.

4. The second-order CBH integral

The second order CBH may also be computed. The result is

$$G_2 = \frac{1}{2}\left(\frac{2Nr_e}{\gamma}\right)^2 \frac{1}{\sqrt{\pi}}\left[\left(\frac{\partial I}{\partial y}\right)^2 + \hat{y}'^2\left(\frac{\partial^3 I}{\partial y^3}\frac{\partial I}{\partial y} - \left(\frac{\partial^2 I}{\partial y^2}\right)^2\right) + \dots\right]$$

where $I = I(x, y, \sigma_x^2, \sigma_y^2 + \frac{\hat{y}'^2}{2})$. (26)

G_2 can be inserted into equation

$$\begin{aligned} F_{12}(x,y,X',Y') &= xX' + yY' + G_1(x,y,X',Y') \\ &+ \frac{1}{2}\frac{\partial G_1(x,y,X',Y')}{\partial x}\frac{\partial G_1(x,y,X',Y')}{\partial X'} \\ &+ \frac{1}{2}\frac{\partial G_1(x,y,X',Y')}{\partial y}\frac{\partial G_1(x,y,X',Y')}{\partial Y'} \end{aligned} \qquad (27)$$

to obtain a second order implicit generator.

IV. NUMERICAL RESULTS

We compare the results calculated by the thick beam-beam interaction with the multiple slice approach for the vertical plane, since the horizontal plane has little thick beam-beam effect. We scanned the range of y=0 to 10 σ, y'=-10 to 10 σ, in 1/4 σ steps. In Table 1, the maximum and r.m.s. differences from a 21-slice results are listed.

The thick beam-beam integral may be calculated for various orders. The 1st order includes the lowest order of correction of G_1—up to the 1st order derivatives of single beam-beam kick, and the 5th order includes up to 5th order derivatives. Then, 2nd order CBH correction, G_2, is included respectively. Finally, both 2nd order Poisson Bracket of G_1 and CBH corrections are included.

One can see that even the 1st order thick beam-beam calculation gives better result than 5 slices. Notice that in this approximation, the y' kick requires essentially the same computation time as a single slice, implying a factor of 5 can be saved in beam-beam simulation. In addition, the maximum errors occur at about 1σ in thick beam-beam, unlike the multi-slices methods which have larger error at large y'.

Table 1. Comparison of beam-beam kick with s_0=0

		location of maximum error (σ)		Errors: * $(\Delta z - \Delta z_{21})/\max(\Delta z_{21})$ z=y or y'	
		y	y'	max (%)	r.m.s.(%)
1 slice	y	0	9.25	100	.032
	y'	1.75	-10	60	.233
3 slices	y	4.5	-10	34	.009
	y'	4.75	-10	14	.05
5 slices	y	5.75	-10	20	.005
	y'	5.75	-10	7.4	.024
thick BB 1st order	y	1.5	-1.5	4.9	.001
	y'	1	-1	5.1	.017
thick BB 5th order	y	.75	5.5	5.5	.002
	y'	1	1.25	3.5	.015
thick BB w/ 2nd CBH†	y	2	4.75	4.9	.002
	y'	1.25	1.75	3.0	.015
thick BB w/2ndPB& CBH†	y	7.75	10	4.8	.001
	y'	1.25	1.75	3.0	.014

* z_{21} indicates the Δz calculated with a 21-slice method.
† These results should be considered as preliminary.

V. CONCLUSION

We represent the thick beam-beam interaction as the infinite product of thin kicks. When the disruption parameter is small it is valid to compose this product to a single map defined by a generator given by the CBH formula. The largest effect by far in typical beam-beam situations is the dependence on the vertical slope. The first order computation for this case results in the replacement of $\sigma_y{}^2$ by $\sigma_y{}^2 + (\sigma_s/2\ y')^2$ in the Erskine-Bassetti formula. The numerical comparison shows the thick beam-beam method, especially the 1st order approach, can replace the multi-slice method which has been used in most of beam-beam simulations, and save substantial CPU time.

VI. REFERENCES

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