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Transverse instabilities for multiple nonrigid bunches in a storage ring *

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Abstract

We describe a method for computing eigenfrequencies for a system consisting of several highly relativistic bunches of charged particles circulating in a storage ring. The bunches are treated as nonrigid, smooth distributions of charge. A bunch feels forces due to fields induced in the vacuum chamber by the dipole line density of itself and other bunches. It also feels forces due to magnets designed to guide the beam. We also include effects of feedback systems designed to correct these dipole motions. We include an example from a real storage ring design.

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We consider a storage ring in which there are several bunches of charged particles circulating at a frequency ω_0. The charged particles are confined so as to keep them bunched and moving near an ideal orbit. The charges also induce electromagnetic fields which can then act back on the charges, potentially causing the beam to be unstable. These instabilities can often be corrected by making a feedback system which senses the dipole moment of the beam at a given position and acts back on the beam at a different position based on the dipole moment detected.

The particles are bunched in the following sense: Parameterize the distance along the ideal orbit by s, and consider a particle's position to be the position along the ideal orbit of a plane perpendicular to that orbit that the particle lies in. Then at any given time, all particles in the bunch lie in a closed interval in s. Also, the intervals for any two bunches do not overlap, nor does any interval extend more than the length of the ring.

Previous work has considered several aspects of this effect. We will confine our description to work considering the force due to transverse wakefields (forces depend on the dipole line density of the charge distribution). Authors have considered how the various eigenmodes involving transverse and longitudinal motion in a single bunch can couple together and create instability (transverse mode coupling) [1–6]. Authors have also shown how to calculate multibunch eigenfrequencies where the bunches are considered to be point particles [7] or where internal degrees of freedom are included but are not considered to be coupled [8–11].

Here, we show how to compute eigenfrequencies for modes of multiple bunches where the internal degrees of freedom are considered and allowed to couple. The technique we use here is a straightforward extension of the formalism of J. M. Wang [8]. The single-bunch case for this has been worked out by R. D. Ruth [3,4].

We will describe the force that a single particle feels by a Hamiltonian [12]. The Hamiltonian is written with s, the position along an ideal orbit through the ring, as the independent variable. Because of the bunched nature of the beam described above, the Hamiltonian may be written as a sum of the Hamiltonians H_r that describe each bunch. Each bunch Hamiltonian will be the sum of two terms.

The first term describes the forces due to magnets, as well as any "potential well distortion" due to longitudinal wakefield forces [13–15], which may be different for different bunches. We assume that the motion due to these forces is at least approximately integrable, so we can write this term as a function of the action variables only [12]. We call this term $H_{r0}(\boldsymbol{J})$, where r is the bunch index.

The second term in the bunch Hamiltonian describes the forces due to electromagnetic fields generated by the bunches. The fields induced at a position s in the ring are proportional to the transverse displacement and charge of a particle passing that position. The transverse force a time τ later at the same position in the ring will then be proportional to this dipole moment of the source charge times a transverse wakefield $W_\perp(\tau)$. Taking into account the fact that a storage ring is periodic with length L, this second term for the rth bunch is

$$-\frac{r_0}{\beta_0^2\gamma_0 L} y_r(\boldsymbol{\theta},\boldsymbol{J}) \sum_{kn} N_n \int y_n(\boldsymbol{\theta}',\boldsymbol{J}')\Psi_n(\boldsymbol{\theta}',\boldsymbol{J}',s-kL) \\ W_\perp[\tau_r(\boldsymbol{\theta},\boldsymbol{J}) - \tau_n(\boldsymbol{\theta}',\boldsymbol{J}') + kT_0]\,d^2\boldsymbol{\theta}'d^2\boldsymbol{J}'. \qquad (1)$$

Here r_0 is the classical radius of the electron; β_0 is the average velocity of a reference particle divided by the speed of light; $\gamma_0 = (1-\beta_0^2)^{-1/2}$; $\beta_0 c T_0 = L$ where c is the speed of light; N_n is the number of particles in bunch n; $y_r(\boldsymbol{\theta},\boldsymbol{J})$ is the transverse displacement for particles in bunch r; $\tau_r(\boldsymbol{\theta},\boldsymbol{J})$ is the arrival time of a particle in bunch r minus the arrival time of a synchronous particle; and $\Psi_n(\boldsymbol{\theta},\boldsymbol{J},s)$ is the phase space distribution function for bunch n at a position s. The summation over k takes into account the circulation of the particles around the ring. Note that y_r and τ_r represent the two of the four phase space coordinates, and these functions describe part of a canonical transformation to the coordinates y and τ from the action-angle coordinates $\boldsymbol{\theta}$ and $\boldsymbol{J}$.

The quantity $y_n(\boldsymbol{\theta},\boldsymbol{J})\Psi_n(\boldsymbol{\theta},\boldsymbol{J},s)$ is a dipole density about the "central orbit" in the ring as a function of phase space position and longitudinal position in the ring. $W_\perp$ couples this dipole density to a transverse force.

We have assumed here that the ring is uniform, in the following senses: First of all, the transverse displacement y_r and time-of-flight displacement τ_r, written in terms of action-angle variables, are independent of s. Secondly, the wake field $W_\perp$ is also independent of s.

We can also consider the effect of adding a feedback system to this formalism. Such a system is designed to damp the motion of multibunch modes by sensing the position of each bunch and giving it a kick a distance from the pickup so as to reduce the oscillation amplitude of the bunch in question.

We can describe such a system through an extra term in the Hamiltonian similar to Eq. (1) above. Simply replace kL in that equation with $kL+\Delta s$ (and thus kT_0 by $kT_0+\Delta s/\beta_0 c$), and rename W to W_{FB}.

We have once again assumed that the ring is uniform; in this case, this corresponds to the fact that a distribution at *any* point s in the ring generates a feedback kick a distance $s+\Delta s$ further down in the ring. The real system generally consists of a pickup and kicker at fixed positions in the ring. The approximation here is that the feedback kick received by the beam is small compared to other forces in the system.

We now consider a distribution $\Psi_n'(\boldsymbol{\theta}, \boldsymbol{J}, s) = \Psi_{n0}(\boldsymbol{J}) + \Psi_n(\boldsymbol{\theta}, \boldsymbol{J}, s)$, where Ψ_n is a small perturbation to Ψ_{n0}. We further assume that

$$\int y \Psi_{n0}[\boldsymbol{J}(y, p_y, \tau, p_\tau)] dy\, dp_y\, dp_z = 0, \tag{2}$$

where y and τ are the canonical phase space coordinates corresponding to the transformation described above, and p_y and p_τ are the corresponding canonical momenta. This is true if $\boldsymbol{J}$ is only a function of y^2, which is the case for "midplane symmetry" in accelerator ring designs (usually true in the vertical direction only). If this is not the case, we may still do an approximate normalization of the Hamiltonian H_{r0} plus the term from Eq. (1) with the Ψ_n replaced with Ψ_{n0}. We then change to the action-angle variables of this system for subsequent calculations. This is similar to the process used to handle potential well distortion in the longitudinal case [13–15].

The time evolution of the distribution Ψ of particles where the motion of any individual particle is described by a Hamiltonian H is given by the Vlasov equation [13]

$$\frac{\partial \Psi}{\partial s} + [\Psi, H] = 0, \tag{3}$$

where [] are Poisson brackets [12]. Note that since the distribution itself appears in our Hamiltonian, this is not strictly true. However, even with interactions, the Vlasov equation turns out to be the first equation in the BBGKY hierarchy of equations, and we can truncate that hierarchy under the assumption that the two-particle distribution function factors into a product of one-particle distribution functions [16].

Because of the bunched nature of the beam, we may write an individual Vlasov equation for each bunch. The Vlasov equations will end up being coupled through the wakefield term in each Hamiltonian. The Vlasov equation for bunch r is

$$\frac{\partial \Psi_r'}{\partial s} + [\Psi_r', H_r] = 0. \tag{4}$$

Note that Ψ_{r0} is independent of s. Second of all, note that from Eq. (2), H_r does not contain Ψ_{n0}. Because Ψ_r is a perturbation to Ψ_{r0}, we will also ignore terms that are second order in Ψ_r.

It is useful to define the impedance, which is the Fourier transform of the wakefield:

$$Z_\perp(\omega) = i \int_{-\infty}^{\infty} W_\perp(\tau) e^{i\omega\tau} d\tau. \tag{5}$$

Next we define the Fourier transform of the perturbation to the bunch distribution:

$$\Psi_r(\boldsymbol{\theta}, \boldsymbol{J}, \Omega) = \frac{1}{\beta_0 c} \int \Psi_r(\boldsymbol{\theta}, \boldsymbol{J}, s) e^{i\Omega s/\beta_0 c} ds. \tag{6}$$

Since the phase space variables have already taken into account the motion of the bunch around the ring, the frequency Ω refers to the frequency of oscillation in the bunch frame. In the lab frame, the frequencies seen will be Ω plus all multiples of the revolution frequency ω_0. We can thus define the dipole moment due to the entire train of bunches seen in the lab frame at frequency $p\omega_0 + \Omega$ to be

$$D_p(\Omega) = \sum_r N_r \int e^{i(p\omega_0+\Omega)\tau_r(\boldsymbol{\theta},\boldsymbol{J})} y_r(\boldsymbol{\theta}, \boldsymbol{J}) \Psi_r(\boldsymbol{\theta}, \boldsymbol{J}, \Omega) d^2\boldsymbol{\theta} d^2\boldsymbol{J}. \tag{7}$$

The Vlasov equation then becomes

$$\left[\Omega + i\boldsymbol{\omega}_r(\boldsymbol{J})\cdot\frac{\partial}{\partial\boldsymbol{\theta}}\right]\Psi_r(\boldsymbol{\theta},\boldsymbol{J},\Omega) = -\frac{r_0c^2}{\gamma_0L^2}\sum_p D_p(\Omega)$$
$$Z_\perp(\Omega+p\omega_0)\frac{\partial\Psi_{r0}}{\partial\boldsymbol{J}}\cdot\frac{\partial}{\partial\boldsymbol{\theta}}\left[y_r(\boldsymbol{\theta},\boldsymbol{J})e^{-i(p\omega_0+\Omega)\tau_r(\boldsymbol{\theta},\boldsymbol{J})}\right] \tag{8}$$

where $\boldsymbol{\omega}_r = \partial H_{r0}/\partial\boldsymbol{J}$.

Adding a feedback system simply adds a term to the right hand side of this where $Z_\perp(\Omega+p\omega_0)$ is replaced by $Z_{\mathrm{FB}}(\Omega+p\omega_0)e^{-ip\Delta s/R}$. Z_{FB} is the Fourier transform of W_{FB} in the same way as $Z_\perp$ was done.

To find the eigenfrequencies Ω in Eq. (8), we must first turn the left hand side of that equation into a D_p. To do this, we need to invert the operator

$$\Omega + i\boldsymbol{\omega}_r(\boldsymbol{J})\cdot\frac{\partial}{\partial\boldsymbol{\theta}} \tag{9}$$

After doing so, and applying Eq. (7), we get an eigenvalue equation $D_q(\Omega) = \sum_p A_{qp}(\Omega)D_p(\Omega)$.

We now assume that we have a lattice consisting only of linear focusing and defocusing magnets. We also assume that all bunches are identical and equally spaced. We also assume that the distribution $\Psi_{r0}(\boldsymbol{J})$ depends on $\boldsymbol{J}$ only in the Hamiltonian H_{r0}, and is Gaussian in τ and y. Since all the bunches are identical, all the r indices in the above equations disappear, except in τ_r. The linear lattice leads to $\boldsymbol{\omega}$ being independent of $\boldsymbol{J}$. Thus, we get

$$\Psi_{r0}(\boldsymbol{J}) = \left|\frac{\omega_y\alpha_c^2\beta_0^2c^2}{4\pi^2\sigma_l^4\omega_z^3}\right| e^{-(\omega_yJ_y+\omega_zJ_z)\alpha_c\beta_0c/\omega_z^2\sigma_l^2}, \tag{10}$$

$$y_r(\boldsymbol{\theta},\boldsymbol{J}) = \sqrt{2J_y\beta_y}\cos\theta_y, \tag{11}$$

$$\tau_r(\boldsymbol{\theta},\boldsymbol{J}) = \frac{2\pi r}{M\omega_0} + \sqrt{\frac{2J_z\alpha_c}{\beta_0c\omega_z}}\cos\theta_z \tag{12}$$

where α_c is the momentum compaction factor, β_y is the average betatron function [17], σ_l is the RMS bunch length, and M is the number of bunches. Also, ω_{ry} and ω_{rz} are the y and τ components of $\boldsymbol{\omega}_r$ respectively.

There is a summation over bunch number resulting from the application of Eq. (7) described above. Under our approximations, this summation can be performed; it gives zero unless p and q are separated by M, and M in that case. Thus, we get M separate eigenvalue systems, indexed by p_0, with $p = p_0 + \alpha M$ and $q = p_0 + \beta M$. Thus, α and β are now the indices on A. The resulting A is

$$A^{p_0}_{\beta\alpha}(\Omega) = i\frac{r_0 c^2 \beta_y N M}{4\gamma_0 L^2 \omega_y} Z_\perp(p\omega_0 + \Omega) e^{-\sigma_l^2\left[(q\omega_0+\Omega)^2+(p\omega_0+\Omega)^2\right]/2\beta_0^2c^2}$$
$$\sum_m (\delta_{m1} + \delta_{m,-1}) \Bigg[2e^{\sigma_l^2(q\omega_0+\Omega)(p\omega_0+\Omega)/\beta_0^2c^2}$$
$$-\frac{\Omega}{\omega_z} \csc\pi\frac{\Omega - m\omega_y}{\omega_z} \int_{-\pi}^{\pi} \cos\frac{(\Omega - m\omega_y)\theta'}{\omega_z} e^{-\sigma_l^2(q\omega_0+\Omega)(p\omega_0+\Omega)\cos\theta'/\beta_0^2c^2}\, d\theta' \Bigg] \quad (13)$$

We take the series for small σ_l of the bracketed term in Eq. (13). Next, we make a change of basis to

$$\phi_n(\Omega) = \left(\frac{\sigma_l}{\beta_0 c}\right)^n \sum_\alpha Z_\perp[\Omega + (p_0 + M\alpha)\omega_0] D_{p_0+M\alpha}(\Omega)$$
$$e^{-\sigma_l^2[\Omega+(p_0+M\alpha)\omega_0]^2/2\beta_0^2c^2} [\Omega + (p_0 + M\alpha)\omega_0]^n. \quad (14)$$

Our system then becomes

$$\phi_m(\Omega) = \sum_{n=0}^{\infty} K_{m+n}(\Omega + p_0\omega_0) F_n(\Omega; \omega_y, \omega_z) \phi_n(\Omega) \quad (15)$$

$$F_n(\Omega; \omega_y, \omega_z) = \frac{1}{2^n n!} \sum_{k=0}^{n} \binom{n}{k} \frac{[\omega_y + (n-2k)\omega_z]^2}{\Omega^2 - [\omega_y + (n-2k)\omega_z]^2} \quad (16)$$

$$K_k(\Omega) = -i\frac{r_0 c^2 \beta_y N M}{\gamma_0 L^2 \omega_y} \sum_\alpha Z_\perp(\omega + M\alpha\omega_0) e^{-\sigma_l^2(\omega+M\alpha\omega_0)^2/2\beta_0^2c^2} \left(\frac{\sigma_l}{\beta_0 c}\right)^k (\omega + M\alpha\omega_0)^k \quad (17)$$

We consider the PEP-II B-Factory [18] low-energy ring as an example. We will only calculate the effects involving the $m = 0$ and $m = 1$ terms in Eq. (15). If we consider only a single bunch, we find that the $m = 0$ and $m = 1$ modes couple, as expected (Fig. 1). If we consider multiple bunches, but don't consider coupling between the modes, we would see growth rates and frequency shifts approximately proportional to current. If we now allow

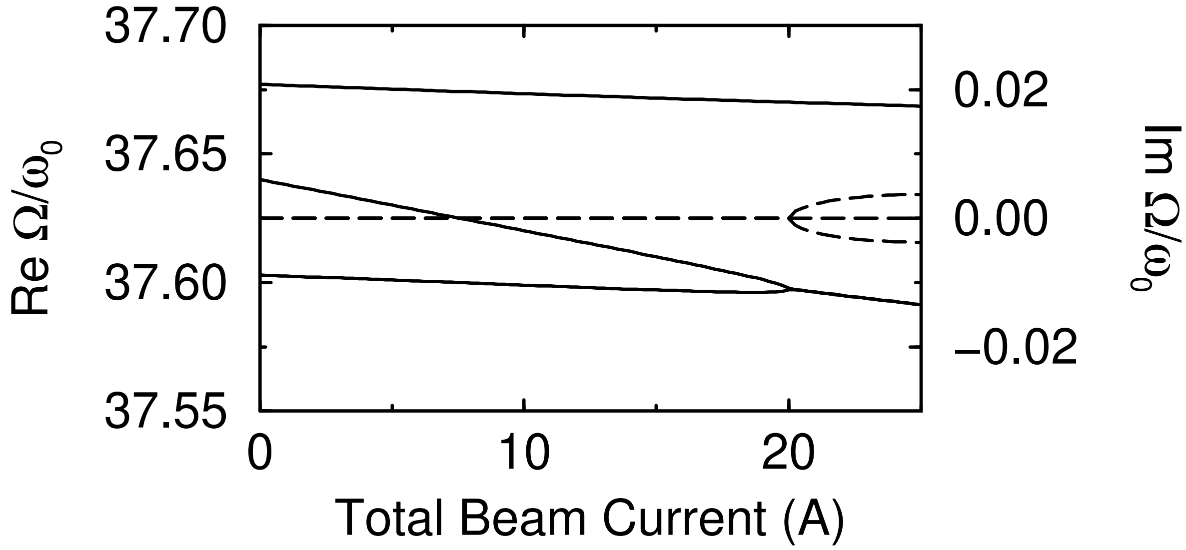

FIG. 1. Single bunch mode coupling. Solid lines are the real part of the mode frequencies, dashed lines are the imaginary parts (growth rates).

the multibunch modes to couple, we see a strong mode coupling effect, as frequencies much smaller than the mode coupling threshold we see for a single bunch (Fig. 2; compare Fig. 1). The growth rates of the $m = 0$ modes which have large growth rates when mode coupling is ignored are only slightly affected by mode coupling. Those which had small growth rates essentially mirror the single bunch behavior of Fig. 1.

The major effect is on the $m = 1$ modes. If mode coupling is ignored, these typically have small growth rates which increase linearly with current. Mode coupling affects these modes in two ways. First, the $m = 1$ growth rates no longer increase linearly with current, and so for even small currents, can become very large. Second, the sudden increase in growth rate as a function of current that occurs in the single bunch case (see Fig. 1) now occurs at much lower currents. The approximate location of this increase can be seen from the point where the real frequencies coincide (see Fig. 2), similarly to the single bunch case.

While the growth rates are still typically dominated by the $m = 0$ modes, there is still reason to be concerned with the $m = 1$ modes. Typical feedback systems damp only the $m = 0$ modes. Damping the $m = 1$ modes requires pickups and kickers that operate at very

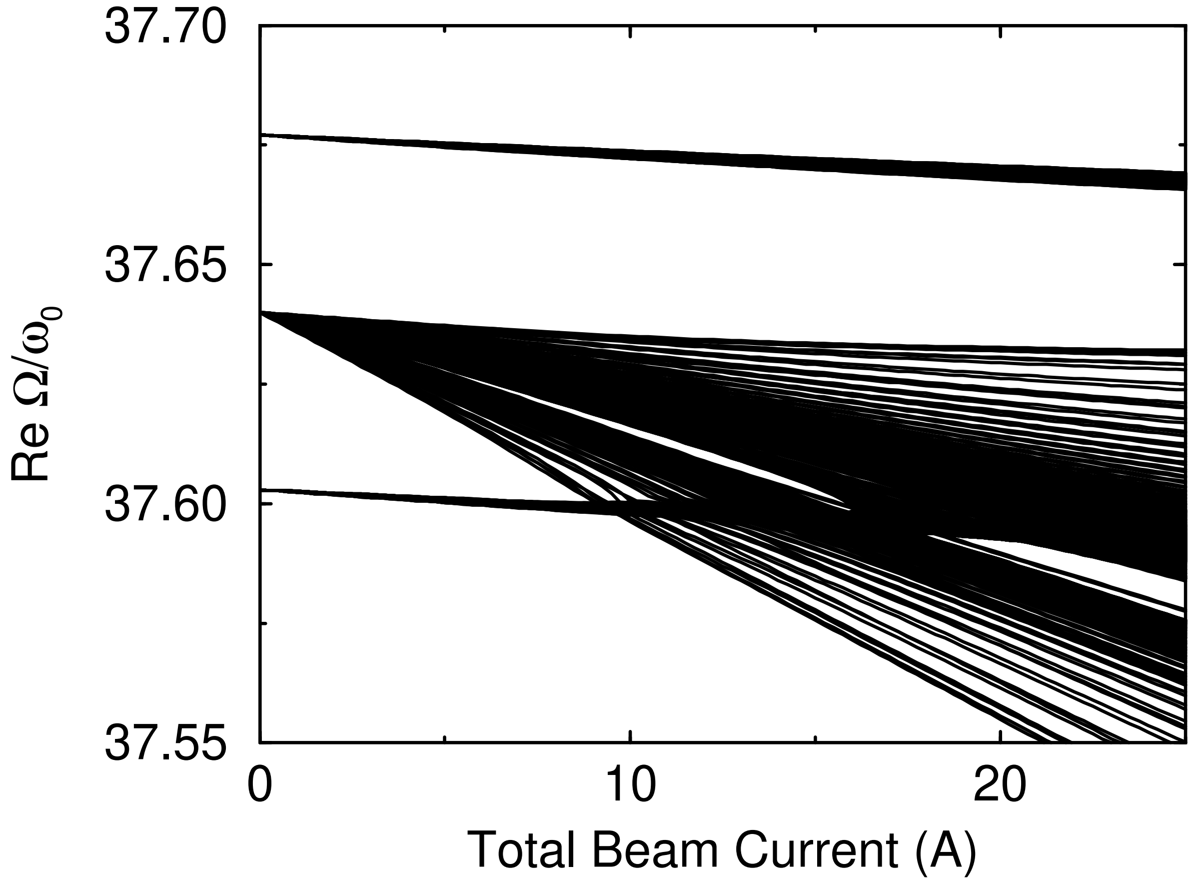

FIG. 2. Multibunch with mode coupling, real part of frequencies.

high frequencies, as can be seen by examining the form of K_k in Eq. (17). Thus, the strong effect of mode coupling on the $m = 1$ modes can produce instabilities that are difficult to correct, and are at much lower currents than the single-bunch mode coupling threshold (see Fig. 3).

We have demonstrated a method for calculating the eigenfrequencies for a storage ring containing multiple bunches, which allows us to include coupling between the internal degrees of freedom in the bunches. We show that this can have a similar effect to what is seen with single-bunch mode coupling, but at significantly lower currents. This is an important effect to consider in the design of the next generation electron-positron storage rings such as the PEP-II B-Factory.

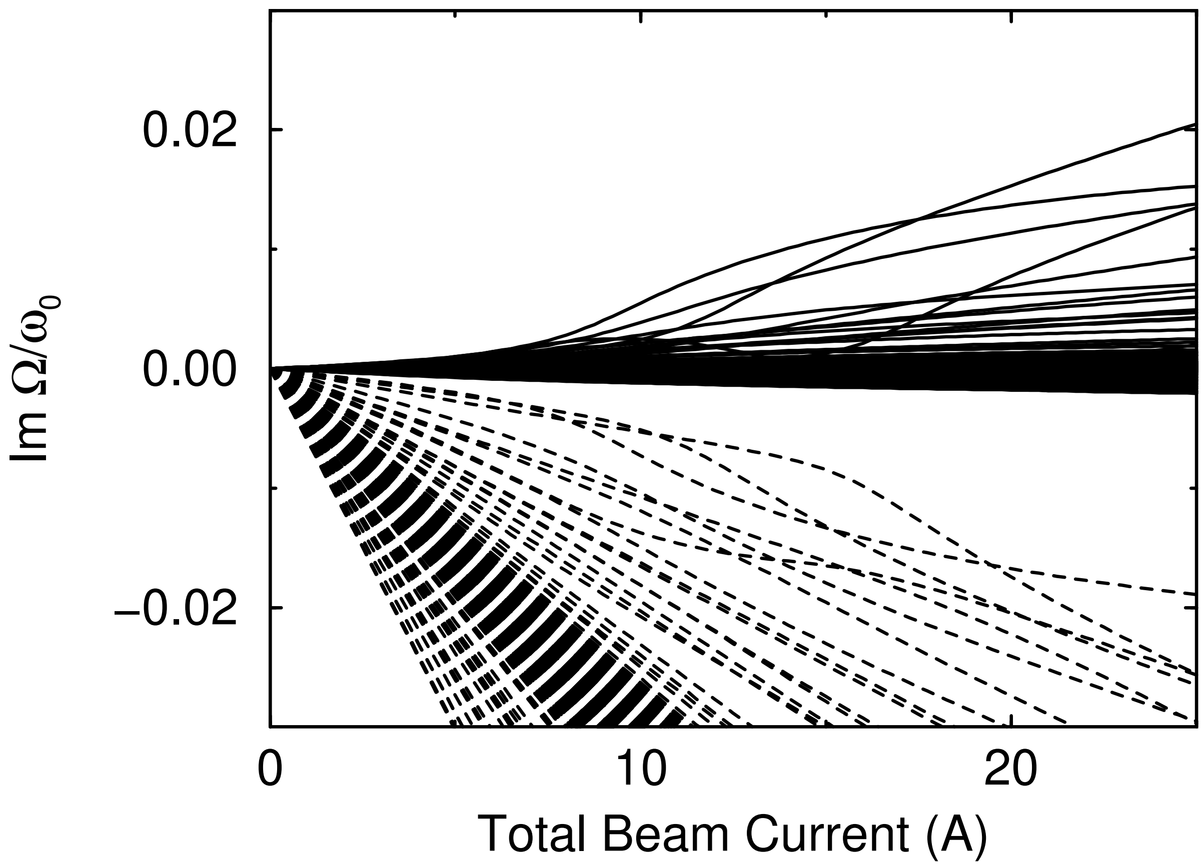

FIG. 3. Multibunch growth rates with feedback. Dashed lines are the $m = 0$ modes, other lines are $m = 1$.

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