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MEAN COSINE FOR THERMAL ENERGY ELASTIC SCATTERING*

K. D. Lathrop

Stanford Linear Accelerator Center, Stanford University
Stanford, CA 94309

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ABSTRACT

The cosine of the laboratory scattering angle is derived for a neutron elastically scattering from a nucleus moving with a specified velocity. Assuming scattering is isotropic in the center of mass system, the mean cosine of the laboratory scattering angle is calculated and shown to agree with the first Legendre moment of a scattering probability function derived by Blackshaw and Murray. Further assuming isotropic neutron-nucleus encounters, a second average is taken to calculate a mean cosine as a function of the neutron-nuclear speed ratio. This mean cosine approaches 2/(3m), where m is the nucleus mass relative to the neutron mass, as the neutron speed becomes large compared to the speed of the nucleus; but for m > 1, the scattering becomes more anisotropic as this speed ratio decreases before approaching isotropy at small neutron/nucleus speed ratios.

INTRODUCTION

The mean cosine of the scattering angle, or synonymously, the first Legendre moment of the differential scattering cross section, is a useful measure of the anisotropy of the scattering process. Qualitatively, the angular distribution of elastically scattered neutrons is isotropic in the center of mass (CM) system for neutron energies below 100 keV and markedly anisotropic for energies above 1 MeV.[1] Even when scattering is isotropic in the CM system, it is anisotropic in the laboratory (L) system, and the dynamics of elastic scattering are a feature of most transport theory texts as well as classical mechanics books.[2,3] These analyses often determine the mean cosine of the L scattering angle, but they invariably assume the scattering nucleus is at rest. Thus, at thermal neutron energies, where the motion of scattering nuclei is important, there is no simple measure of the relative importance of anisotropic scattering. Last year, while preparing an exam for sophomore physics majors taking classical mechanics, I thought to provide this measure by asking them to calculate the mean cosine of the L scattering angle when the scattering nucleus was moving with a known speed and direction, given isotropic CM scattering. Attempting this analysis myself, I concluded that for an in-class exam the problem was too time consuming but probably not too difficult for Stanford students. Below I outline the determination of the elastic scattering mean cosine when the scattering nucleus is moving.

Blackshaw and Murray[4] derived very general scattering probability functions for thermal-energy classical elastic scattering, and I will show subsequently how the mean cosine derived below can be obtained from their scattering probability functions. However, I first take a more direct approach by expressing the L scattering angle cosine in terms of CM variables and then averaging this cosine over CM angles, using the assumed isotropy of CM scattering.

DETERMINATION OF THE LABORATORY SCATTERING ANGLE COSINE

Figure 1 shows the L system geometry of the velocity vectors for a neutron of initial speed $\vec{u}$ scattering from a nucleus of initial speed $\vec{p}$ with the angle between $\vec{u}$ and $\vec{p}$ being $\theta = \cos^{-1}\mu$. After the collision, the neutron has velocity $\vec{v}$ and the nucleus has velocity $\vec{q}$. The L scattering angle, that is, the angle between $\vec{u}$ and $\vec{v}$, is $\theta_0 = \cos^{-1}\mu_0$. The same labeling of velocity vectors, with a subscript c, is used in the CM system. The vectors $\vec{u}$ and $\vec{p}$ define a vertical plane which has an azimuthal angle ϕ, and after scattering the azimuthal angle of $\vec{v}$ is ϕ_0.

If the mass of a nucleus relative to the mass of the neutron is m, then from conservation of momentum the CM velocity, $\vec{w}_c$, is given by

$$\vec{w}_c = \frac{\vec{u} + m\vec{p}}{m+1} \quad , \tag{1}$$

and its magnitude is

$$w_c = \frac{\sqrt{u^2 + m^2 p^2 + 2ump\mu}}{m+1} \quad . \tag{2}$$

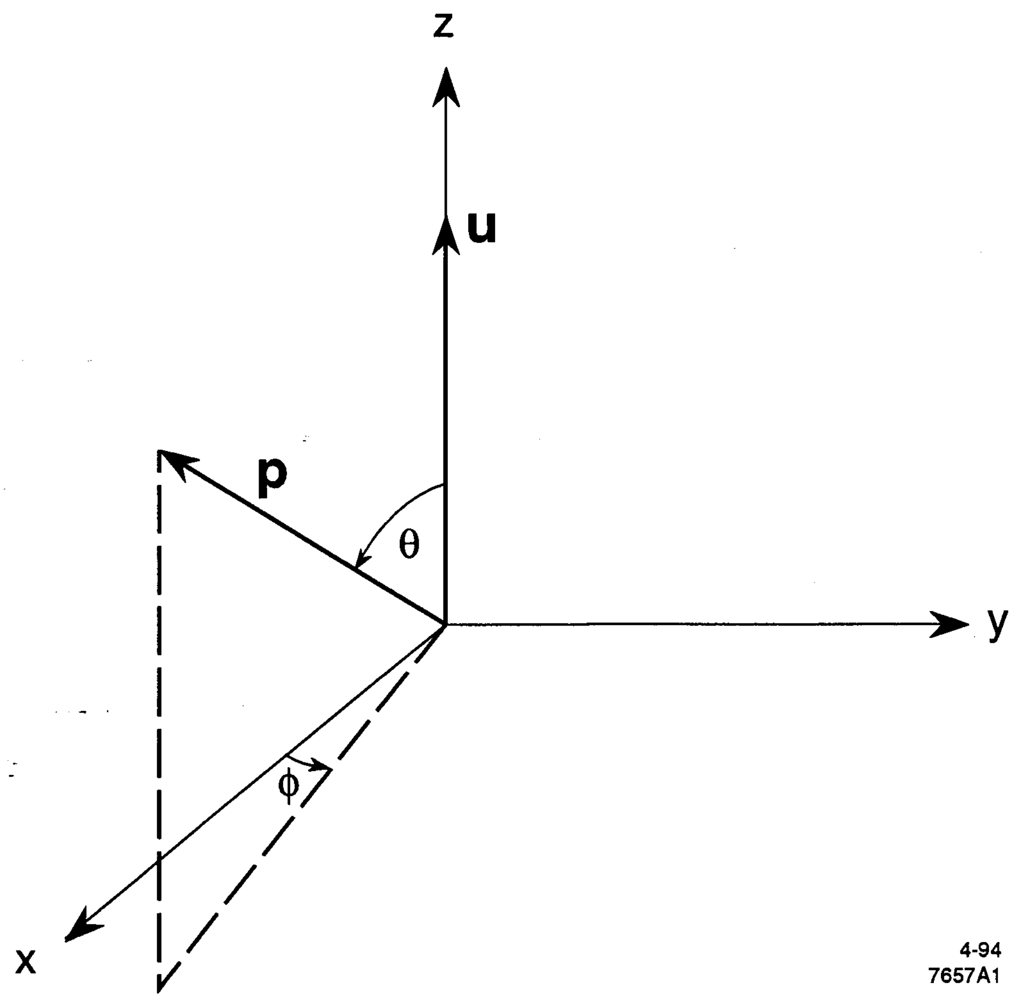

Fig. 1. L system orientation of neutron velocity vector $\vec{u}$ and nucleus velocity vector $\vec{p}$ before a scattering event.

Transforming to the system where the CM is at rest amounts to subtracting $\vec{w}_c$ from the L system velocities. The velocity of the neutron in the CM system is thus

$$\vec{u}_c = \vec{u} - \vec{w}_c = \frac{m}{m+1}(\vec{u} - \vec{p}) \equiv \frac{m}{m+1}\vec{v}_r \quad , \tag{3}$$

where $\vec{v}_r$ is the neutron-nucleus relative velocity, with magnitude

$$v_r = \sqrt{u^2 + p^2 - 2up\mu} \quad . \tag{4}$$

Similarly, the nucleus speed in CM is

$$\vec{p}_c = \vec{p} - \vec{w}_c = \frac{-\vec{v}_r}{m+1} \quad , \tag{5}$$

so that by design

$$m\vec{p}_c + \vec{u}_c = 0 \quad , \tag{6}$$

and by conservation of momentum

$$m\vec{q}_c + \vec{v}_c = 0 \quad . \tag{7}$$

Using these two momentum equations (6) and (7) to eliminate p_c and q_c from the conservation of energy equation

$$mp_c^2 + u_c^2 = mq_c^2 + v_c^2 \tag{8}$$

shows directly the well-known result

$$u_c = v_c \quad . \tag{9}$$

Because $\vec{w}_c$ is in the same plane as $\vec{u}$ and $\vec{p}$, the relations between velocity vectors in the L and CM systems before the collision can be shown simply as illustrated in Figure 2. From this diagram, the angle α between $\vec{w}_c$ and $\vec{u}$ is seen to be determined from

$$(m+1)w_c \cos\alpha = u + mp\mu \quad . \tag{10}$$

The assumption of isotropic CM scattering means that after the collision $\vec{v}_c$ is oriented with equal probability in any direction. To be definite, the orientation of $\vec{v}_c$ is measured in an orthogonal rectangular coordinate system (x', y', z') with the z' axis in the direction of $\vec{w}_c$ and the x' axis in the $\vec{p}-\vec{u}$ plane. This defines a CM coordinate system. The angle $\vec{v}_c$ makes with $\vec{w}_c$ (the z' axis) is $\theta_c = \cos^{-1}\mu_c$ and the azimuthal angle about $\vec{w}_c$ measured from the $z'-x'$ plane is ϕ_c. The neutron velocity after the collision in L is obtained by adding $\vec{w}_c$ to $\vec{v}_c$,

$$\vec{v} = \vec{w}_c + \vec{v}_c \quad . \tag{11}$$

These relations are shown in Figure 3 for an arbitrary but specific choice of μ_c and ϕ_c.

There are now several different ways to relate the cosine of the L scattering angle, μ_0, to CM angular variables. I find the approach of Blackshaw and Murray to be most appealing. Note that the CM system (x', y', z') is related to the L system (x, y, z) by two rotations. The first is a rotation about z by an angle ϕ with rotation matrix

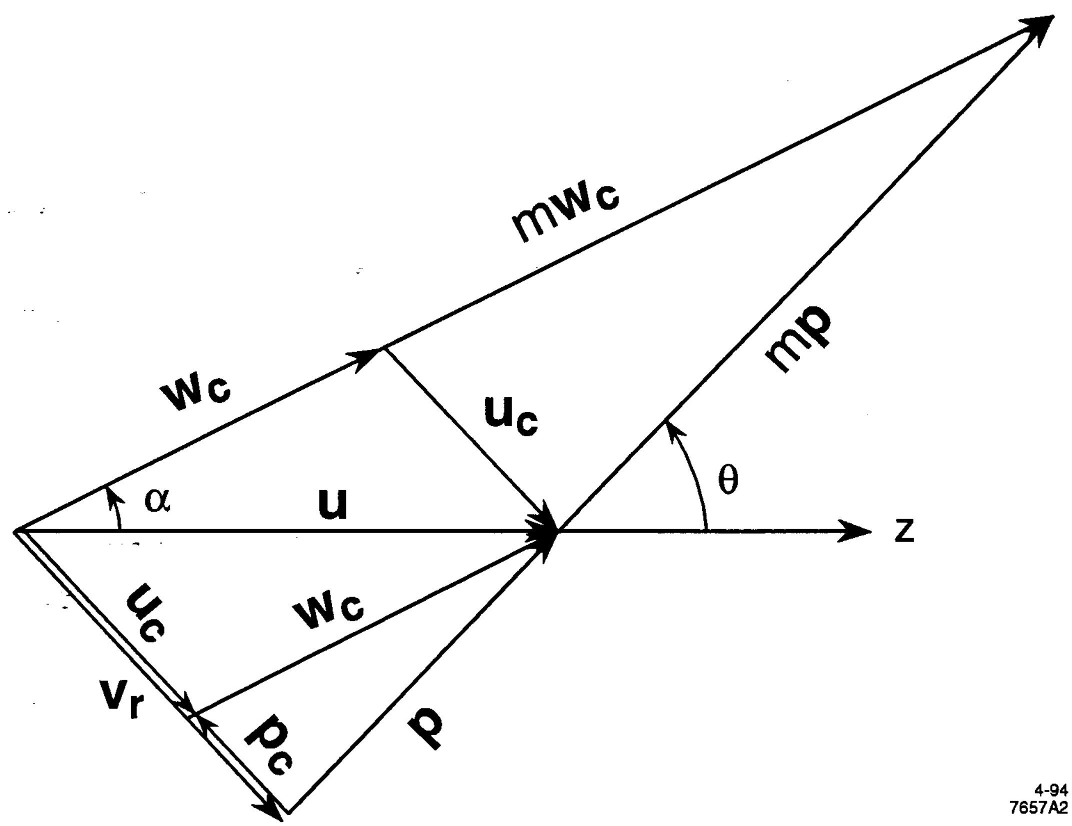

Fig. 2. Geometric relations between L and CM velocity vectors.

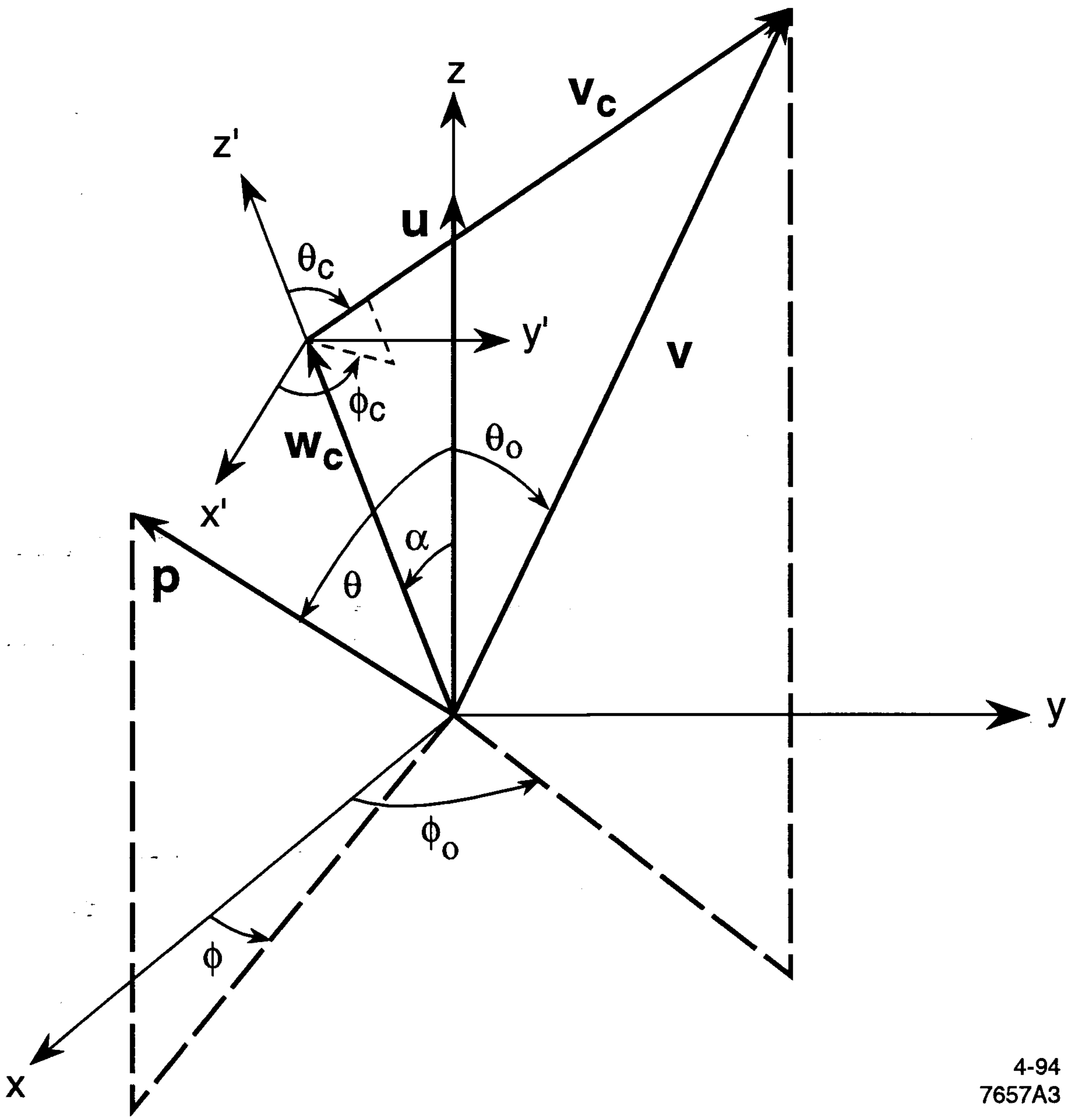

Fig. 3. Relation of the velocity vectors in the L system after a scattering event. $\vec{u}$, $\vec{p}$ and $\vec{w}_c$ are coplanar.

$$R_\phi = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{12}$$

and the second is a rotation about the intermediate y axis by an angle α

$$R_\alpha = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix} , \tag{13}$$

so that the combined rotation $R_\alpha R_\phi$ connnects the two systems. The components of the vector $\vec{v}$ in each system are

$$\begin{aligned} (\vec{v})_L &= \left(v \sin\theta_0 \cos\phi_0,\ v \sin\theta_0 \sin\phi_0,\ \cos\theta_0\right) , \\ (\vec{v})_{CM} &= \left(v_c \sin\theta_c \cos\phi_c,\ v \sin\theta_c \sin\phi_c,\ w_c + v_c \cos\theta_c\right) , \end{aligned} \tag{14}$$

where the z' component includes the addition of w_c. Thus we have

$$\begin{aligned} (\vec{v})_{CM} &= R_\alpha R_\phi (\vec{v})_L , \\ \text{or} & \\ (\vec{v})_L &= \left(R_\alpha R_\phi\right)^{-1} (\vec{v})_{CM} . \end{aligned} \tag{15}$$

Remembering that the inverse of the rotation matrices is their transpose, the second of these gives

$$\begin{pmatrix} v \sin\theta_0 \cos\phi_0 \\ v \sin\theta_0 \sin\phi_0 \\ v \cos\theta_0 \end{pmatrix} = \begin{pmatrix} \cos\alpha\cos\phi & -\sin\phi & \sin\alpha\cos\phi \\ \cos\alpha\sin\phi & \cos\phi & \sin\alpha\sin\phi \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} v_c \sin\theta_c \cos\phi_c \\ v_c \sin\theta_c \sin\phi_c \\ w_c + v_c \cos\theta_c \end{pmatrix} . \tag{16}$$

The last of these three relations gives directly, using Eq. (11),

$$\mu_0 = \frac{\cos\alpha\left(w_c + v_c\mu_c\right) - \sin\alpha\ v_c\left(1-\mu_c^2\right)^{1/2}\cos\phi_c}{\sqrt{w_c^2 + v_c^2 + 2w_c v_c \mu_c}} \quad . \tag{17}$$

When p=0, i.e., when the nucleus is at rest, this reduces to the familiar[5] expression

$$\mu_0 = \frac{1+m\mu_c}{\sqrt{1+m^2+2m\mu_c}} \quad . \tag{18}$$

Eq. (17) makes clear through the presence of the ϕ_c term the fact, first made explicit by Blackshaw and Murray, that when there is nuclear motion the nonplanar nature of isotropic CM scattering must be accounted for.

While v_c [equal to u_c and thus given by Eq. (3)], w_c and $\cos\alpha$ are functions of u, p, and μ, they do not depend on μ_c or ϕ_c, so the average of μ_0,

$$\left\langle\mu_0\right\rangle = \frac{\int_{-1}^{1} d\mu_c \int_0^{2\pi} d\phi_c \mu_0}{4\pi} \quad , \tag{19}$$

can be determined directly with the result

$$\begin{aligned} \left\langle\mu_0\right\rangle &= \cos\alpha\frac{2a}{3} & \qquad a = \frac{w_c}{v_c} < 1 \\ &= \cos\alpha\left(1-\frac{1}{3a^2}\right) & \qquad a > 1 \quad . \end{aligned} \tag{20}$$

When $p = 0$, $\cos\alpha = 1$ and $a = 1/m$ with the usual result that $\left\langle\mu_0\right\rangle = 2/3m$.

LEGENDRE MOMENTS OF THE SCATTERING PROBABILITY

The result of Eq. (20) can also be obtained by taking the first Legendre moment of the elastic scattering probability function given by Blackshaw and Murray. They derive the probability of an initial neutron velocity $\vec{u}$ resulting in a final velocity $\vec{v}$ after isotropic CM elastic scattering to be

$$P(\vec{u} \to \vec{v}) = \frac{v}{2\pi v_c w_c} \frac{\delta(\phi_0 - \phi_{01})}{\sqrt{-\mu_0^2 + 2s\cos\alpha\,\mu_0 + \sin^2\alpha - s^2}} , \tag{21}$$

where ϕ_{01} is the particular azimuthal angle corresponding to $u_c = v_c$ and

$$s = \frac{v^2 - v_c^2 + w_c^2}{2vw_c} . \tag{22}$$

Because of the delta function, the average probability that the initial and final speeds are u and v with an angle of scatter $\theta_0 = \cos^{-1}\mu_0$ is simply

$$P(u \to v, \mu_0) = \int P(\vec{u} \to \vec{v}) d\phi_0 = \frac{v}{2\pi v_c w_c \sqrt{-\mu_0^2 + 2s\cos\alpha\mu_0 + \sin^2\alpha - s^2}} . \tag{23}$$

The zeroth Legendre moment is the probability of initial speed u becoming v or

$$P_0(u \to v) = \int_{\mu_{0-}}^{\mu_{0+}} d\mu_0\, P(u \to v, \mu_0) , \tag{24}$$

where the limits are those required to keep the radical real

$$\mu_{0\pm} = s\cos\alpha \pm \left(1 - s^2\right)^{1/2} \sin\alpha . \tag{25}$$

Performing the integration of Eq. (24) gives the result obtained[6] by many authors in thermalization studies.

$$P_0(u \to v) = \frac{v}{2v_c w_c} = \frac{2v}{v_{max}^2 - v_{min}^2} \qquad v_{min} \le v \le v_{max} \quad , \tag{26}$$

where

$$\begin{aligned} v_{max} &= v_c + w_c \qquad (\mu_c = +1) \\ v_{min} &= |v_c - w_c| \quad (\mu_c = -1) \quad . \end{aligned} \tag{27}$$

Finally, integrating Eq. (26) over all possible final speeds shows the total probability is unity, as it should be.

Following the same sequence of operations for the P_1 moment of the elastic scattering probability function of Eq. (21), that is for

$$\langle \mu_0 \rangle = \int_{\mu_{0-}}^{\mu_{0+}} \int_0^{2\pi} d\mu_0 d\phi_0 \mu_0 P(\vec{u} \to \vec{v}) \quad , \tag{28}$$

gives the less frequently seen $P_1(u \to v)$ as

$$P_1(u \to v) = \frac{\cos\alpha\left(v^2 - v_c^2 + w_c^2\right)}{w_c\left(v_{max}^2 - v_{min}^2\right)} \qquad v_{min} \le v \le v_{max} \quad , \tag{29}$$

which, when integrated over the allowable range of v, gives Eq. (20) for $\langle \mu_0 \rangle$.

AVERAGING OVER NEUTRON-NUCLEUS ENCOUNTER ANGLES

Through $\cos\alpha$ and a, Eq. (20) is still a function of four parameters, namely m, u, p, and μ, and as such is not yet a useful measure of anisotropic scattering. It is frequently assumed in thermalization studies that the orientation of neutron-nucleus encounters is isotropic in L. Using this, I define a second average

$$\langle\langle\mu_0\rangle\rangle = \frac{\int d\mu \int d\phi \, \langle\mu_0\rangle}{4\pi} \quad . \tag{30}$$

Because $\langle\mu_0\rangle$ is independent of ϕ, that integration is trivial; the integration over μ requires more care because the form of $\langle\mu_0\rangle$ depends on the ratio a which in turn depends on μ. Considering this constraint gives

$$\begin{aligned} \langle\langle\mu_0\rangle\rangle &= \int_{-1}^{1} \frac{d\mu}{2} \cos\alpha \frac{2a}{3} & p &< \frac{(m-1)u}{2m} \quad , \\ &= \int_{-1}^{\frac{u(m-1)}{2m}} \frac{d\mu}{2} \cos\alpha \frac{2a}{3} + \int_{\frac{u(m-1)}{2m}}^{1} \frac{d\mu}{2} \cos\alpha \left(1 - \frac{1}{3a^2}\right) & p &> \frac{(m-1)u}{2m} \quad . \end{aligned} \tag{31}$$

Executing these somewhat tedious integrations (a perfect job for automatic symbolic integrators) gives, when the result is expressed in terms of the neutron-nuclear speed ratio, $\frac{u}{p} \equiv b$,

$$\langle\langle\mu_0\rangle\rangle = \frac{2b(m+3)}{9m} + (m+1)^2\left(\frac{2-m}{6m} + \frac{m-5}{18m}\sqrt{1+\frac{b^2}{m}}\right)$$

$$+ (m+1)^2\frac{(2m-1)}{9b^2}\left(\sqrt{1+\frac{b^2}{m}} - 1\right) \qquad b \le \frac{2m}{m-1}\,, \tag{32a}$$

$$\langle\langle\mu_0\rangle\rangle = \frac{2}{3m} + \frac{2}{9b^2} \qquad b > \frac{2m}{m-1}\,. \tag{32b}$$

This result could further be averaged over a Maxwellian nucleus speed distribution as is often done in thermalization studies, but while I have done this numerically, it does not add further insight. As it stands, the equation describes the situation when all nuclei are assumed to be moving with the same speed. One can think of the scattering medium properties as being characterized by the average nuclear speed, say.

When the neutron speed is large compared to the nuclear speed ($b >> 1$) and $m \neq 1$, the mean cosine is given by Eq. (32b) and approaches $2/(3m)$, the usual result for no nuclear motion. When $m = 1$, e.g. for neutron-proton scattering, and the same condition ($b >> 1$) pertains, Eq. (32a) approaches $2/3$, also the expected result. When the neutron speed is small compared to the nuclear speed ($b << 1$), the mean cosine approaches zero linearly with b. One might therefore expect that the mean cosine is monotone between isotropic scattering at $b = 0$ and $2/3m$ at large b. However, evaluation of either Eq. (32a) or Eq. (32b) at their common point gives

$$\langle\langle\mu_0\rangle\rangle = \frac{2}{3m}\left[1+\frac{(m-1)^2}{12m}\right] \qquad b = \frac{2m}{m-1}\,, \tag{33}$$

which is greater than $2/3m$ except for $m = 1$. Indeed, Figure 4 shows $\langle\langle\mu_0\rangle\rangle$ as a

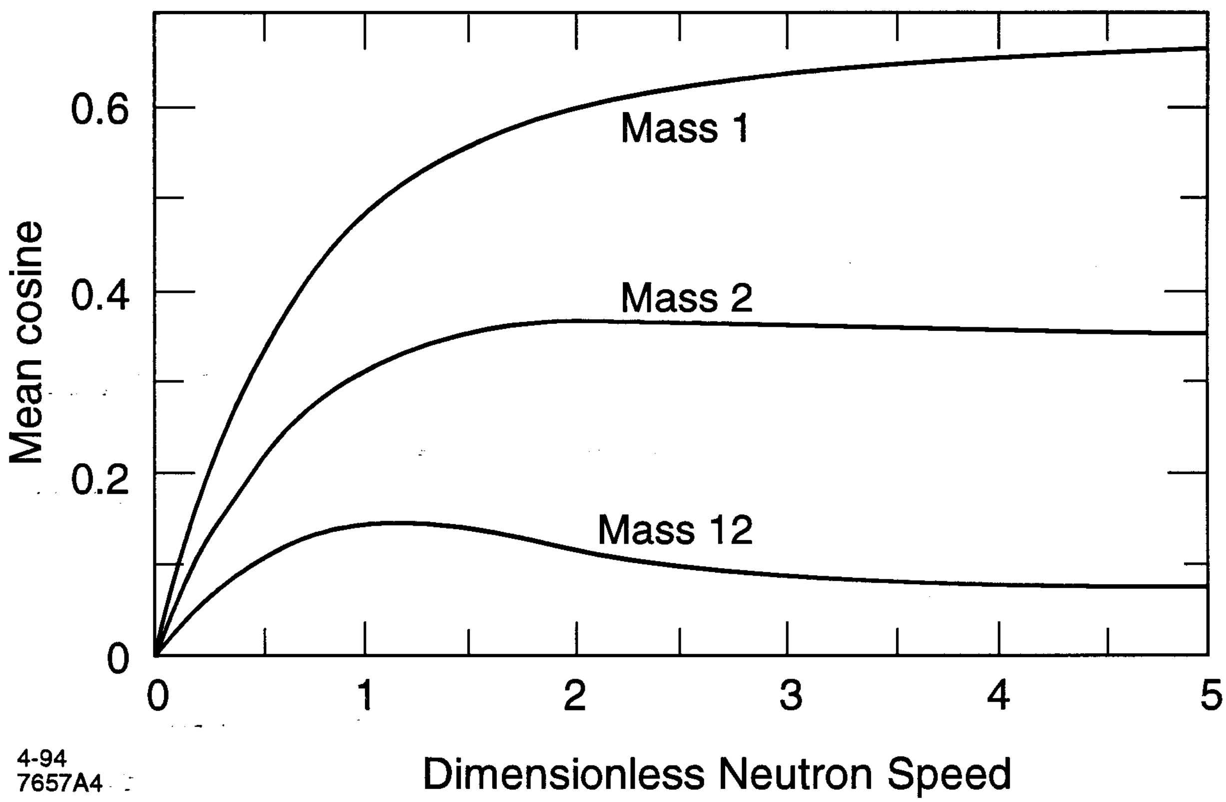

Fig. 4. Mean neutron scattering cosine as a function of the speed of the neutron relative to the speed of the scattering nucleus.

function of b for several values of m, and although the mean cosine decreases as m increases, the scattering is more anisotropic at lower b values than might be expected when $m > 1$.

This interesting result is also somewhat surprising. When there is no nuclear motion and m increases, the CM velocity decreases so that the CM system is very nearly the same as the L system. Thus, if scattering is isotropic in CM it is isotropic in L. However, if there is nuclear motion, as m increases the CM system does not become the L system; rather it becomes the system in which the nucleus is at rest and the CM velocity approaches $\vec{p}$. In this system the neutron approaches with velocity $\vec{v}_r$, which depends strongly on the encounter angle $\theta = \cos^{-1}\mu$. Hence isotropy in CM is not isotropy in L. As m increases this anisotropy peaks near $u = p$ and approaches a value of $\langle\langle\mu_0\rangle\rangle = 15/144 = 0.104$.

The results shown in Figure 4 are plotted for the variation of the cosine depending on the speed of the neutron relative to the (single) nuclear speed. In an ensemble of materials at the same temperature, the speed characterizing different mass nuclei would not be the same, and this effect must be accounted for in comparing anisotropy of scattering.

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VECTORS AND MATRICES

I. VECTORS

$\mathbf{u}$

$\mathbf{v}$

$\mathbf{p}$

$\mathbf{q}$

$\mathbf{u_c}$

$\mathbf{v_c}$

$\mathbf{p_c}$

$\mathbf{q_c}$

$\mathbf{w_c}$

$\mathbf{v_r}$

II MATRICES

R_ϕ

R_α