6454
March, 1994; rev. August, 1994
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# Violation of CPT and Quantum Mechanics in the $K_0$–$\bar{K}_0$ System

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## ABSTRACT

We reconsider the model of quantum mechanics violation in the $K_0$–$\bar{K}_0$ system, due to Ellis, Hagelin, Nanopoulos, and Srednicki, in which $CP$- and $CPT$-violating signatures arise from the evolution of pure states into mixed states. We present a formalism for computing time-dependent asymmetries in this model and show that present data constrains its parameters significantly. In the future, this model will be put to very stringent tests at a $\phi$ factory. We present the theory of these tests and show the relation between particular $\phi$ decay correlations and the parameters of quantum mechanics violation.

Submitted to *Nuclear Physics B*

⋆ Work supported by the Department of Energy, contract DE–AC03–76SF00515.

# 1. Introduction

The propagation and decays of $K_0$ and $\bar{K}_0$ mesons has provided elementary particle physicists with the most fruitful system for probing the fundamental discrete symmetries of Nature. The paradoxes of this system led Lee and Yang to postulate the violation of parity in the weak interactions.[1] More detailed exploration led to the discovery of $CP$ violation,[2] and this system remains the only place in which $CP$ violation and $T$ violation have been observed. New experiments at CPLEAR are now strengthening the precision of our knowledge of neutral kaon physics,[3] and beautiful new experiments on the discrete symmetries are planned for future $\phi$ factories.[21,5]

One of the goals of current and planned experiments on the $K_0$–$\bar{K}_0$ system is to search for the violation of $CPT$. To interpret these experiments, one should have some idea of the source of $CPT$ violation. Within the context of local quantum field theory, $CPT$ conservation is a theorem.[6] Thus, theories of $CPT$ violation must necessarily step outside the standard assumption that particle physics is governed by a local quantum field theory. In the 1960's, one could plausibly build a theory of $CPT$ violation in the context of a nonlocal model of the strong interactions. However, with the triumph of the standard model, that route is no longer open.

On the other hand, developments in the quantum theory of gravity have opened another possible avenue to theories of $CPT$ violation. Building from his results on the spectrum of radiation from black holes, Hawking has proposed that the generalization of quantum mechanics which encompasses gravity allows the evolution of pure states into mixed states.[7] Page then showed that any such dynamics also leads to conflict with $CPT$ conservation.[8] These ideas raised the interesting possibility that one could find observable $CPT$ violation due to a mechanism that lies not only beyond a local quantum description but also beyond quantum mechanics altogether. The notion that gravitation effects beyond quantum mechanics can affect elementary particle physics is controversial. For example, in theories which allow the evolution of pure states into mixed states, there is a serious conflict between energy-momentum conservation and locality.[10] In this paper, however, we will use a formalism which avoids the issue of locality. Our main goal will be to discuss how the issue of the violation of quantum mechanics can be addressed experimentally.

To our knowledge, the first attempt at a phenomenological analysis of this type of quantum mechanics violation was made by P.H. Eberhard in the early 70s',[11] as an attempt to probe the postulates of quantum field theory. Eberhard suggested various tests of the existence of a unitary S-matrix. His ideas motivated an experimental test of quantum mechanics in the $K_0$–$\bar{K}_0$ system by Carithers *et al.*.[12] The suggestion that Hawking's idea could be tested experimentally in the $K_0$–$\bar{K}_0$ system was put forward independently by Ellis, Hagelin, Nanopoulos, and Srednicki (EHNS).[13] EHNS set up an evolution equation for the $K_0$–$\bar{K}_0$ system in the space of density matrices. Their equation contains three new $CPT$ violating parameters $\alpha$, $\beta$, and $\gamma$. These parameters have the dimensions of mass and might be expected to be of order $m_K^2/m_{\rm Pl} \sim 10^{-19}$ GeV.

These authors were attracted to the $K_0$–$\bar{K}_0$ system by the fact that it contains phenomena which depend on quantum coherence over macroscopic distances. Though there are other phenomena that depend on macroscopic quantum correlations, for example, the persistence of superfluid and superconducting currents, the $K_0$–$\bar{K}_0$ system gives a controlled setting, involving only one particle, in which precision experiments can reveal small deviations from the predictions of quantum mechanics. A second such system, also discussed by EHNS, is found in macroscopic neutron interferometry. Here, experiments of Zeilinger, Horne, and Shull have constrained a similar dimensionful parameter of quantum mechanics violation to a level less than $0.8 \times 10^{-25}$ GeV,[14] under the assumption that quantum mechanics violation can randomize the neutron spin.

Recently, Ellis, Mavromatos, and Nanopoulos (EMN)[15,16] reconsidered the analysis of EHNS for the $K_0$–$\bar{K}_0$ system and presented experimentally allowed regions for the parameters $\alpha$, $\beta$, and $\gamma$ which were consistent with nonzero values of the magnitude of the earlier estimates. Their analysis suggested that this new source of $CPT$ violation might fully account for the observed $CP$ violation in the $K_0$–$\bar{K}_0$ system. EMN also presented a microscopic theory of $\alpha$, $\beta$, and $\gamma$, based on string theory, which gives values of the size of the estimate above. While we do not accept their detailed microscopic arguments, we believe that their suggestion that quantum mechanics violation might be observable in present or planned experiments is an exciting one which deserves further consideration. This is especially true because the analysis of EMN was incomplete, as these authors themselves pointed out, in that it did not fully take into account constraints from the time evolution of the $K_0$–$\bar{K}_0$ state.

It is the purpose of this paper to analyze the dynamics of the $K_0$–$\bar{K}_0$ system taking into account the possibility of $CPT$ violation from mechanisms both within and outside quantum mechanics. We will use this framework to discuss the implications of past, present, and future experiments on the $K_0$–$\bar{K}_0$ system. Our formulae will include both the conventional $CPT$ violation within quantum mechanics and the $CPT$ violation outside quantum mechanics of EHNS. Our goal is to explain how to disentangle these two possible sources of $CPT$ violation from one another, and how to distinguish both from conventional $CP$ violation.

Our discussion will proceed as follows: In Section 2, we will review briefly the conventional parametrization of $CP$ and $CPT$ violation within quantum mechanics by the parameters $\epsilon$ and $\Delta$. We will then rewrite this discussion in the language of density matrices. In Section 3, we will review the formalism of EHNS and generalize the density matrix equations to include their parameters for quantum mechanics violation, $\alpha$, $\beta$, and $\gamma$.

In Section 4, we will work out expressions for the observable quantities in experiments on single neutral kaons in terms of these parameters. We will show that the parameter $\alpha$ is not significantly constrained by these experiments, but that $\beta$ and $\gamma$ are tightly constrained by current data. Using the classic results of the Carithers *et al.*[12] and CERN-Heidelberg experiments[17–19] and recent results from CPLEAR,[3] we determine the values $\beta = (0.32 \pm 0.29) \times 10^{-18}$ GeV, $\gamma =$

$(-0.2 \pm 2.2) \times 10^{-21}$ GeV. These bounds are much stronger than those of EMN and lead to the simple but important conclusion that the $CP$ violation observed in the $K_0$–$\bar{K}_0$ system is dominantly quantum mechanical in nature and of $CPT$ conserving origin. The analysis of Section 4 includes provision for $CPT$ violation from within quantum mechanics, but it ignores the possibility of $CPT$ violation in decay matrix elements. In Section 5, we introduce this possible additional complication and explain how it potentially weakens our results.

In Sections 6–8, we will discuss precision tests for $CPT$ violation and violation of quantum mechanics at the future $\phi$ factories. A $\phi$ factory produces $K$ meson pairs in an antisymmetric pure state. In ref. 4, Peccei and collaborators showed that, within the context of quantum mechanical $CP$ and $CPT$ violation, the full time dependence of the decays into identical states, for example, to $\{\pi^+\pi^-\,,\ \pi^+\pi^-\}$, is independent of the $CP$ violation parameters and so provides an accurate measurement of the mass difference and lifetimes of the kaon eigenstates. When we allow for a departure from quantum mechanics, this is no longer true. We find an additional time-dependent oscillation which, if observed experimentally, would provide direct evidence of time evolution outside quantum mechanics. Section 6 provides the general formalism for describing this effect. Section 7 gives formulae from which $\alpha$, $\beta$, and $\gamma$ can be constrained independently of one another in specific $\phi$ factory experiments. In Section 8, we discuss various processes in which the two kaons decay asymmetrically. In particular, we consider the decay into $\{\pi^+\pi^-\,,\ \pi^0\pi^0\}$, which provides a measurement of Re $\epsilon'/\epsilon$, and we display corrections to the standard formulae which appear when $\beta$ and $\gamma$ are nonzero.

## 2. Quantum mechanics of a neutral kaon beam

In quantum mechanics, a kaon state evolves according to the action of a Hamiltonian. Even if the kaon decays, its evolution is correctly described by an effective Hamiltonian which includes the natural width of the state.

One conventionally writes this effective Hamiltonian as

$$H = M - \frac{i}{2}\Gamma, \tag{2.1}$$

where $M$ and $\Gamma$ are Hermitian $2\times2$ matrices acting on states in the basis $(|K_0\rangle\,, |\bar{K}_0\rangle)$. These states are alternatively described using the basis of $CP$ eigenstates,

$$|K_1\rangle = \frac{1}{\sqrt{2}}\big(|K_0\rangle + |\bar{K}_0\rangle\big)\ ; \qquad |K_2\rangle = \frac{1}{\sqrt{2}}\big(|K_0\rangle - |\bar{K}_0\rangle\big)\ , \tag{2.2}$$

or the basis of eigenstates of $H$, $|K_S\rangle$ and $|K_L\rangle$.

It is easiest to express predictions for the $K_0$–$\bar{K}_0$ system by trading the matrix elements of $H$ for parameters which express the eigenvalues and eigenvectors of this matrix. We will use the conventions of Maiani,[20] which express the eigenstates of $H$ in terms of parameters $\epsilon_M$, which is odd under $CP$ but even under $CPT$, and $\Delta$, which is odd under both $CP$ and $CPT$. Explicitly,

$$\begin{aligned} |K_S\rangle &= \frac{N_S}{\sqrt{2}}\big((1+\epsilon_S)\,|K_0\rangle + (1-\epsilon_S)\,|\bar{K}_0\rangle\big) \\ |K_L\rangle &= \frac{N_L}{\sqrt{2}}\big((1+\epsilon_L)\,|K_0\rangle - (1-\epsilon_L)\,|\bar{K}_0\rangle\big), \end{aligned} \tag{2.3}$$

where

$$\epsilon_S = \epsilon_M + \Delta\ , \qquad \epsilon_L = \epsilon_M - \Delta, \tag{2.4}$$

and $N_S$, $N_L$ are real, positive normalization factors. We write the corresponding eigenvalues as

$$
\begin{aligned}
m_S - \frac{i}{2}\Gamma_S &= (\bar{m} - \Delta m/2) - \frac{i}{2}(\bar{\Gamma} + \Delta\Gamma/2) \\
m_L - \frac{i}{2}\Gamma_L &= (\bar{m} + \Delta m/2) - \frac{i}{2}(\bar{\Gamma} - \Delta\Gamma/2),
\end{aligned}
\tag{2.5}
$$

so that $\Delta m$ and $\Delta\Gamma$ are positive.

From this description of the kaon dynamics in terms of states, it is easy to construct the description in terms of density matrices. With the complex Hamiltonian (2.1), the density matrix evolves according to

$$
\rho_K(\tau) \;=\; e^{-iH\tau}\,\rho_K(0)\,e^{iH^\dagger\tau}. \tag{2.6}
$$

The eigenmodes of this equation are

$$
\begin{aligned}
\rho_L &= |K_L\rangle\,\langle K_L| \\
\rho_S &= |K_S\rangle\,\langle K_S| \\
\rho_I &= |K_S\rangle\,\langle K_L| \\
\rho_{\bar{I}} &= |K_L\rangle\,\langle K_S|\,.
\end{aligned}
\tag{2.7}
$$

A generic kaon beam can be decomposed into these modes. If we write the expansion coefficients at time $\tau = 0$ as $A_L$, $A_S$, $A_I$, $A_{\bar{I}}$, we find the following general form for the time evolution:

$$
\rho_K(\tau) \;=\; A_L\,\rho_L\,e^{-\Gamma_L\tau} \;+\; A_S\,\rho_S\,e^{-\Gamma_S\tau} \;+\; A_I\,\rho_I\,e^{-\bar{\Gamma}\tau}\,e^{-i\Delta m\tau} \;+\; A_{\bar{I}}\,\rho_{\bar{I}}\,e^{-\bar{\Gamma}\tau}\,e^{+i\Delta m\tau}\,. \tag{2.8}
$$

When $\rho_K$ describes a pure quantum state, $\rho_{L,S}$ and $\rho_{I,\bar{I}}$ are as in (2.7) and the coefficient of the interference term $A_I$ is correlated with $A_L$ and $A_S$. If we write the pure state as $\rho_K = |K\rangle\,\langle K|$, with $|K\rangle = a_L\,|K_L\rangle + a_S\,|K_S\rangle$, then $A_L = |a_L|^2$, $A_S = |a_S|^2$ and $A_I = A_{\bar{I}}^* = a_L a_S^\dagger$. When $\rho_K$ is a mixed state, we will find that these properties no longer hold.

In the $|K_1\rangle, |K_2\rangle$ basis, eq. (2.2), the four eigenmodes of the density matrix take the form:

$$\begin{aligned} \rho_L &= \begin{pmatrix} |\epsilon_L|^2 & \epsilon_L \\ \epsilon_L^* & 1 \end{pmatrix} \qquad \rho_S = \begin{pmatrix} 1 & \epsilon_S^* \\ \epsilon_S & |\epsilon_S|^2 \end{pmatrix} \\ \rho_I &= \begin{pmatrix} \epsilon_L^* & 1 \\ \epsilon_L^* \epsilon_S & \epsilon_S \end{pmatrix} \qquad \rho_{\bar{I}} = \begin{pmatrix} \epsilon_L & \epsilon_S^* \epsilon_L \\ 1 & \epsilon_S^* \end{pmatrix} . \end{aligned} \tag{2.9}$$

Here and henceforth, we normalize the eigenmodes $\rho_i$ so that the largest element is 1; the normalization factors $N_S$, $N_L$ in (2.3) can be absorbed into the coefficients $A_i$ in (2.8).

Any physical property of the kaon beam can be extracted from the density matrix by tracing with a suitable Hermitian operator. To extract the value of the observable $\mathcal{P}$, we write

$$\langle \mathcal{P} \rangle = \mathrm{tr}\big[\rho_K \mathcal{O}_{\mathcal{P}}\big]. \tag{2.10}$$

This expression for expectation values will remain true in the generalization of quantum mechanics described in Section 3. In the remainder of this section, we will write the operators $\mathcal{O}_{\mathcal{P}}$ associated with the most important observables of the neutral kaon system.

In principle, $CP$ and $CPT$ violation can show up not only in the neutral kaon mass matrix but also in the kaon decay amplitudes. There is already some experimental evidence for a nonzero value of the parameter $\epsilon'$ which characterizes direct $CP$ violation in decays to two pions.[22,23] However, the $CP$ violation associated with mass mixing is much more important. In models such as the superweak model, in which $CP$ violation is the result of new physics at a scale $M$ much greater than $m_W$, direct $CP$ violation typically results from higher-dimension local operators and so is suppressed by extra powers of $(m_W/M)$. Thus, the observation of direct $CP$ violation is evidence for an origin of $CP$ violation at the weak interaction scale, for example, by the Kobayashi-Maskawa mechanism.

Although we know strictly that $CPT$ violation cannot be the result of a local quantum field theory, we believe it is nevertheless reasonable to use the dimensional analysis rules of local quantum field theory to restrict possible sources of $CPT$ violation. This is certainly true in a model such as that of ref. 16, in which violation of quantum mechanics arises from applying an unusual averaging procedure to quantum mechanical amplitudes. When we consider models of quantum mechanics violation, the scale $M$ which suppresses multipoint amplitudes must be very large. The appearance of $CPT$ violation within quantum mechanics will signal a breakdown of local field theory, and this must take place at very short distances compared to those probed in the LEP experiments at the $Z^0$. The $CPT$ violation outside quantum mechanics that we will describe in the next section has the Planck scale as its characteristic mass scale. Combining these ideas, we expect that $CPT$ violation is associated with processes with the minimum number of external particles. Thus, in our main analysis, we will include $CPT$ violation only in the time development of the neutral kaon state, and we will ignore possible $CPT$-violating contributions to the decay vertices.

Despite this argument, many phenomenological analyses include the possibility of $CPT$-violating decay amplitudes. In this case, the experimental effects of these $CPT$-violating terms cannot be unambiguously disentangled from those of quantum mechanics violation. In the standard discussion without quantum mechanics violation, there is a similar difficulty in disentangling $CPT$ violation in the decay amplitudes from that in the $K_L$–$K_S$ mass matrix. In that case, one can at least present constraints on combinations of $CPT$-violating parameters. Then one can argue that either the individual parameters obey similar bounds or there are unnatural large cancellations.[20] We will present the generalization of these constraints to the theory with quantum mechanics violation in Section 5.

By the same argument as that which eliminates $CPT$-violating decay amplitudes, we will ignore the possibility that these new processes violate the $\Delta S = \Delta Q$ rule for leptonic $K$ decays. In his review article, ref. 20, Maiani has demonstrated that conventional contributions to $\Delta S \neq \Delta Q$ amplitudes are also negligible. Thus, we will ignore $\Delta S \neq \Delta Q$ effects throughout this paper.

Within this set of assumptions, we now construct the operators $\mathcal{O}_{\mathcal{P}}$ associated with the semileptonic and 2-pion decays of the neutral kaons. If we ignore violations of the $\Delta S = \Delta Q$ rule, only the $|K_0\rangle$ state can decay to $\pi^-\ell^+\nu$. Then, using the basis $(|K_1\rangle, |K_2\rangle)$ of eq. (2.2), the decay rate to this final state corresponds to the operator

$$\mathcal{O}_{\ell^+} = |a|^2 |K_0\rangle \langle K_0| = \frac{|a|^2}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \tag{2.11}$$

Similarly, the decay rate to $\pi^+\ell^-\bar{\nu}$ is given by

$$\mathcal{O}_{\ell^-} = |a|^2 \left|\bar{K}_0\right\rangle \left\langle\bar{K}_0\right| = \frac{|a|^2}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{2.12}$$

In our expressions for the decay rate of neutral kaons to two pions, we will ignore the possibility of direct $CPT$ violation, but we must include an allowance for direct $CP$ violation. Then the amplitudes for the decay of $|K_0\rangle$ and $\left|\bar{K}_0\right\rangle$ to $\pi^+\pi^-$ are

$$\begin{aligned} \mathcal{M}(K_0 \to \pi^+\pi^-) &= A_0 e^{i\delta_0} + \frac{1}{\sqrt{2}} A_2 e^{i\delta_2} \\ \mathcal{M}(\bar{K}_0 \to \pi^+\pi^-) &= A_0^* e^{i\delta_0} + \frac{1}{\sqrt{2}} A_2^* e^{i\delta_2}. \end{aligned} \tag{2.13}$$

We choose the Wu-Yang convention in which $A_0$ is real. Then the decay amplitudes from states of definite $CP$ are:

$$\begin{aligned} \mathcal{M}(K_1 \to \pi^+\pi^-) &= \sqrt{2} A_0 e^{i\delta_0} + \mathrm{Re} A_2 e^{i\delta_2} \\ \mathcal{M}(K_2 \to \pi^+\pi^-) &= i \mathrm{Im} A_2 e^{i\delta_2}. \end{aligned} \tag{2.14}$$

Then, the decay rate to $\pi^+\pi^-$ is given, in the basis (2.2), by the operator

$$\mathcal{O}_{+-} = \begin{pmatrix} |\sqrt{2}A_0 + \mathrm{Re}A_2 e^{i\delta}|^2 & (\sqrt{2}A_0 + \mathrm{Re}A_2 e^{-i\delta})(i\mathrm{Im}A_2 e^{i\delta}) \\ (\sqrt{2}A_0 + \mathrm{Re}A_2 e^{i\delta})(-i\mathrm{Im}A_2 e^{-i\delta}) & |i\mathrm{Im}A_2 e^{i\delta}|^2 \end{pmatrix}, \tag{2.15}$$

where $\delta = \delta_2 - \delta_0$. One can check this result by tracing with the density matrices for the eigenstates $K_L$ and $K_S$, as given in eq. (2.9). We find

$$\begin{aligned} \mathrm{tr}\big[\mathcal{O}_{+-}\rho_L\big] &= \big|\epsilon_L(\sqrt{2}A_0 + \mathrm{Re}A_2 e^{i\delta}) + i\mathrm{Im}A_2 e^{i\delta}\big|^2 \\ \mathrm{tr}\big[\mathcal{O}_{+-}\rho_S\big] &= \big|\sqrt{2}A_0 + \mathrm{Re}A_2 e^{i\delta}) + \mathcal{O}(\epsilon_s)\big|^2, \end{aligned} \tag{2.16}$$

from which we recover, to leading order in $CP$ violation and $\Delta I = 1/2$ enhancement, the familiar result[20]

$$|\eta_{+-}|^2 = \left|\frac{\mathcal{M}(K_L \to \pi^+\pi^-)}{\mathcal{M}(K_S \to \pi^+\pi^-)}\right|^2 = \big|\epsilon_L + \frac{1}{\sqrt{2}}\frac{i\mathrm{Im}A_2}{A_0}e^{i\delta}\big|^2. \tag{2.17}$$

The analogous argument for the neutral pion decay leads to the operator

$$\mathcal{O}_{00} = \begin{pmatrix} |\sqrt{2}A_0 - 2\mathrm{Re}A_2 e^{i\delta}|^2 & (\sqrt{2}A_0 - 2\mathrm{Re}A_2 e^{-i\delta})(-2i\mathrm{Im}A_2 e^{i\delta}) \\ (\sqrt{2}A_0 - 2\mathrm{Re}A_2 e^{i\delta})(2i\mathrm{Im}A_2 e^{-i\delta}) & |-2i\mathrm{Im}A_2 e^{i\delta}|^2 \end{pmatrix}. \tag{2.18}$$

From this expression, in the context of purely quantum-mechanical evolution, we find

$$|\eta_{00}|^2 = \left|\frac{\mathcal{M}(K_L \to \pi^0\pi^0)}{\mathcal{M}(K_S \to \pi^0\pi^0)}\right|^2 = \big|\epsilon_L - \sqrt{2}\frac{i\mathrm{Im}A_2}{A_0}e^{i\delta}\big|^2. \tag{2.19}$$

The quantities $\eta_{+-}$ and $\eta_{00}$ are parametrized in terms of $CP$-violating observables $\epsilon$ and $\epsilon'$:

$$\eta_{+-} = \epsilon + \epsilon' \ , \qquad \eta_{00} = \epsilon - 2\epsilon' \ . \tag{2.20}$$

When we ignore all effects outside quantum mechanics and also ignore the possibility of $CPT$ violation in decay amplitudes, we find[20]

$$\epsilon = \epsilon_L = \epsilon_M - \Delta \ , \qquad \epsilon' = \frac{i}{\sqrt{2}}\frac{\mathrm{Im}A_2}{A_0}e^{i\delta}. \tag{2.21}$$

From this review of the effects of purely quantum-mechanical evolution on the $K_0$–$\bar{K}_0$ system, we now procede to the case of propagation outside quantum mechanics.

## 3. Kaon evolution outside quantum mechanics

We now wish to enlarge the framework of our kaon beam equations of motion to allow the possibility that pure states can evolve into mixed states. In this section, we add terms to the density matrix equation to allow this possibility. Our discussion will follow the arguments of Ellis, Hagelin, Nanopoulos, and Srednicki (EHNS).[13] We will extend their work in providing formulae for time-dependent interference phenomena.

In conventional quantum mechanics, the density matrix obeys the evolution equation

$$i\frac{d}{d\tau}\rho = \big[H\,,\,\rho\big]. \tag{3.1}$$

This equation guarantees the conservation of probability

$$\frac{d}{d\tau}\,\mathrm{tr}[\rho] = 0, \tag{3.2}$$

and also a set of higher identities

$$\frac{d}{d\tau}\,\mathrm{tr}[\rho^n] = 0, \tag{3.3}$$

which imply that the purity of the state is not changed by quantum mechanical evolution.

In a two-state system such as the neutral kaon system, the density matrix can be expanded

$$\rho = \rho^0\mathbf{1} + \rho^i\sigma^i, \tag{3.4}$$

where $i = 0, 1, 2, 3$ and $\sigma^i$ is a Pauli sigma matrix. The statement of conservation of probability $\mathrm{tr}[\rho] = 1$ becomes $\rho^0 = \frac{1}{2}$, and the statement that $\rho$ has positive eigenvalues becomes

$$(\rho^0)^2 \geq \sum_{i=1}^{3} (\rho^i)^2 \tag{3.5}$$

If the Hamiltonian is expanded in the same way, the equation of motion can be written

$$\frac{d}{d\tau}\rho = 2\epsilon^{ijk} H^i \rho^j \sigma^k. \tag{3.6}$$

The quantum mechanical description of the neutral kaon system is only slightly more complicated. Here we work with the non-Hermitian Hamiltonian (2.1) which includes a provision for the kaon states to decay. The evolution equation (2.6) leads to the equation

$$\frac{d}{d\tau}\rho = 2\epsilon^{ijk} M^i \rho^j \sigma^k - \Gamma^0 \rho - \Gamma^i (\rho^0 \sigma^i + \rho^i \mathbf{1}). \tag{3.7}$$

The value of $\rho^0 = 2\,\mathrm{tr}[\rho]$ will now decrease. However, the inequality (3.5) must still be satisfied, and it will be as long as the matrix $\Gamma$ has two positive eigenvalues.

Hawking[7] proposed that the modifications of quantum-mechanical evolution due to quantum gravity effects could be described by writing a more general linear equation for the density matrix. The most general term that we could add to (3.7) is

$$-h^{0j} \rho^j \mathbf{1} - h^{j0} \sigma^j - h^{ij} \sigma^i \rho^j \tag{3.8}$$

There are two natural restrictions on these terms. First, they should be consistent with probability conservation. Second, they should not decrease the entropy of the density matrix; though pure states can evolve into mixed states, mixed states should not evolve into pure states. The first of these requirements sets $h^{0j} = 0$. The second requirement eliminates any $h^{j0}$ terms, since these would order the completely random distribution with $\rho = \frac{1}{2}\mathbf{1}$, and also requires that the remaining submatrix $h^{ij}$ be positive definite.[13] This leads to the following set of equations for the components of the density matrix:

$$\begin{aligned} \frac{d}{d\tau}\rho^0 &= -\Gamma^0\rho^0 - \Gamma^i\rho^i \\ \frac{d}{d\tau}\rho^i &= 2\epsilon^{ijk}M^j\rho^k - \Gamma^i\rho^0 - \Gamma^0\rho^i - h^{ij}\rho^j , \end{aligned} \tag{3.9}$$

where, from here on, $i, j, k = 1, 2, 3$ only. Notice that the antisymmetric part of $h^{ij}$ can be absorbed into $M^j$, so that we may assume from here on that $h$ is symmetric.

EHNS simplify this formalism by imposing one further assumption. If the action of quantum gravity is universal among flavors, the new term cannot change strangeness. Alternatively, if the basis chosen by the quantum gravity interactions is close to the basis of quark mass eigenstates, the strangeness nonconservation will be proportional to the square of the rotation angle between these two bases. If this angle is of order the Cabibbo angle, strangeness violation will be a higher-order effect. Under either assumption, we may concentrate on the part of $h$ which conserves strangeness. Since strangeness is measured by the operator $\mathcal{O}_S = -\sigma^1$ in the basis of $CP$ eigenstates, the restricted $\delta H$ must satisfy

$$\mathrm{tr}[\sigma^1 h^{ij}\sigma^i\rho^j] = 0 \ , \qquad i.e., \ h^{1j} = 0. \tag{3.10}$$

This leaves

$$h = 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \beta & \gamma \end{pmatrix} , \tag{3.11}$$

where the EHNS parameters $\alpha$, $\beta$, $\gamma$ satisfy

$$\alpha, \gamma > 0 \ , \qquad \alpha\gamma > \beta^2. \tag{3.12}$$

The equations (3.9) for the density matrix components are linear equations which can be solved by diagonalizing a $4 \times 4$ matrix. Since $CP$ violation is a small effect, of order $10^{-3}$ of $\Delta m$ and $\Delta\Gamma$, a perturbative solution is quite appropriate. To obtain this solution, we first rewrite the equations (3.9) in the basis of matrix elements of the density matrix:

$$\rho = \begin{pmatrix} \rho_1 & \rho \\ \bar{\rho} & \rho_2 \end{pmatrix} \tag{3.13}$$

It is useful to first write the purely quantum-mechanical equation (3.7) in this basis:

$$\frac{d}{d\tau}\begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho \\ \bar{\rho} \end{pmatrix} = \left[ -\bar{\Gamma} + \begin{pmatrix} -\Delta\Gamma/2 & 0 & +i\epsilon_L^* d^* & -i\epsilon_L d \\ 0 & \Delta\Gamma/2 & +i\epsilon_S d & -i\epsilon_S^* d^* \\ -i\epsilon_S^* d^* & -i\epsilon_L d & +i\Delta m & 0 \\ +i\epsilon_S d & +i\epsilon_L^* d^* & 0 & -i\Delta m \end{pmatrix} \right] \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho \\ \bar{\rho} \end{pmatrix}, \tag{3.14}$$

with

$$d = \Delta m + \frac{i}{2}\Delta\Gamma. \tag{3.15}$$

It is straightforward to check that the parameters are chosen so that eigenvectors of the matrix reproduce (2.9), with the correct eigenvalues. The perturbation (3.11) adds to the quantity in brackets the matrix

$$\begin{pmatrix} -\gamma & \gamma & -i\beta & +i\beta \\ \gamma & -\gamma & +i\beta & -i\beta \\ +i\beta & -i\beta & -\alpha & \alpha \\ -i\beta & +i\beta & \alpha & -\alpha \end{pmatrix} \tag{3.16}$$

Treating this addition to the equations in perturbation theory, and working to the order of the leading contribution to each matrix element, we can work out the new eigenmodes and eigenvalues. The eigenmodes are:

$$\rho_L = \begin{pmatrix} |\epsilon_L|^2 + \gamma/\Delta\Gamma + 4\beta(\Delta m/\Delta\Gamma)\mathrm{Im}[\epsilon_L/d^*] - \beta^2/|d|^2 & \epsilon_L + \beta/d \\ \epsilon_L^* + \beta/d^* & 1 \end{pmatrix}$$
$$\rho_S = \begin{pmatrix} 1 & \epsilon_S^* - \beta/d^* \\ \epsilon_S - \beta/d & |\epsilon_S|^2 - \gamma/\Delta\Gamma - 4\beta(\Delta m/\Delta\Gamma)\mathrm{Im}[\epsilon_S/d^*] - \beta^2/|d|^2 \end{pmatrix} .$$
$$\rho_I = \begin{pmatrix} \epsilon_L^* - \beta/d^* & 1 \\ -i\alpha/2\Delta m & \epsilon_S + \beta/d \end{pmatrix}$$
$$\rho_{\bar{I}} = \begin{pmatrix} \epsilon_L - \beta/d & i\alpha/2\Delta m \\ 1 & \epsilon_S^* + \beta/d^* \end{pmatrix} \tag{3.17}$$

The corresponding eigenvalues are corrected by the shifts

$$\begin{aligned} \Gamma_L \to \Gamma_L + \gamma \ , \qquad & \Gamma_S \to \Gamma_S + \gamma \ , \\ \bar{\Gamma} \to \bar{\Gamma} + \alpha \ , \qquad & \Delta m \to \Delta m \cdot (1 - \tfrac{1}{2}(\beta/\Delta\Gamma)^2) \ . \end{aligned} \tag{3.18}$$

The shifts of $\Gamma_L$, $\Gamma_S$, and $\Delta m$ can be absorbed by redefinition of these parameters. The shift of $\Delta m$ is of relative size $10^{-6}$ and so is negligible in any event. If we redefine $\bar{\Gamma}$ to be the average of the new $\Gamma_S$ and $\Gamma_L$, then the interference terms $\rho_I$ and $\rho_{\bar{I}}$ fall off at the rate

$$\bar{\Gamma} + (\alpha - \gamma) \ . \tag{3.19}$$

This correction is not relevant to current experiments unless $\alpha$ is as large as $10^{-2}\bar{\Gamma}$; in that case $\alpha$ would be 10 times larger than the familiar $CP$-violating parameters. We will retain this shift, for completeness, in our formulae below, but we will ignore it in our analysis of the present experimental situation.

To summarize, under the influence of quantum mechanics violation as described by the formalism of EHNS, the most general initial density matrix evolves according to

$$\rho_K(\tau) \;=\; A_L\,\rho_L\,e^{-\Gamma_L\tau} \;+\; A_S\,\rho_S\,e^{-\Gamma_S\tau} \\ +\,A_I\,\rho_I\,e^{-(\bar{\Gamma}+\alpha-\gamma)\tau}\,e^{-i\Delta m\tau} \;+\; A_{\bar{I}}\,\rho_{\bar{I}}\,e^{-(\bar{\Gamma}+\alpha-\gamma)\tau}\,e^{+i\Delta m\tau}\,, \tag{3.20}$$

where now the eigenmodes are given by (3.17).

To understand the changes that have been induced in the evolution of density matrices, it is useful to rewrite the eigenmodes slightly in order to emphasize their similarity to (2.9). It is very convenient to introduce the split $\epsilon$ parameters:

$$\epsilon_L^{\pm} = \epsilon_L \pm \beta/d\;,\epsilon_S^{\pm} = \epsilon_S \pm \beta/d\;. \tag{3.21}$$

Then the expressions for the eigenmodes can be rewritten as follows:

$$\begin{aligned} \rho_L &= \begin{pmatrix} |\epsilon_L^-|^2 + \gamma/\Delta\Gamma + 4(\beta/\Delta\Gamma)\mathrm{Im}[\epsilon_L^- d/d^*] & \epsilon_L^+ \\ \epsilon_L^{+*} & 1 \end{pmatrix} \\ \rho_S &= \begin{pmatrix} 1 & \epsilon_S^{-*} \\ \epsilon_S^- & |\epsilon_S^+|^2 - \gamma/\Delta\Gamma - 4(\beta/\Delta\Gamma)\mathrm{Im}[\epsilon_S^+ d/d^*] \end{pmatrix} \\ \rho_I &= \begin{pmatrix} \epsilon_L^{-*} & 1 \\ -i\alpha/2\Delta m & \epsilon_S^+ \end{pmatrix} \\ \rho_{\bar{I}} &= \begin{pmatrix} \epsilon_L^- & i\alpha/2\Delta m \\ 1 & \epsilon_S^{+*}. \end{pmatrix} \end{aligned} \tag{3.22}$$

The eigenmodes are written here to sufficient accuracy for the analysis in the rest of this paper. The complete expressions for these eigenmodes to second order in small parameters are given in the Appendix.

Except for their smallest matrix elements, of which only $\rho_{L1}$ and $\rho_{S2}$ are important in practice, the expressions (3.22) take precisely the form of the density matrices of pure states given in (2.9). However, the various modes contain differently shifted $\epsilon$ parameters. We will see in the next section that different neutral kaon experiments are sensitive to different choices of these parameters, in such a way that the various sources of $CPT$ violation can be distinguished.

## 4. Experimental determination of $\beta$ and $\gamma$

In this section, we will show how present experimental data constrains the EHNS parameters. We will show how to combine the very accurate measurements of the $K^0$ parameters from the experiments of the early 1970's with new data on time-dependent $K^0$ evolution from the CPLEAR experiment. This comparison will give stringent constraints on $\beta$ and $\gamma$ which limit their effects to be at most about 10% of the observed $CP$ violation in the neutral kaon system. This rules out the possibility, suggested in ref. 16, that a deviation from quantum mechanics of the nature considered here is the major source of observed $CP$ violation. We will also recommend a parametrization of the future, more accurate CPLEAR data which will facilitate comparison of this data with the EHNS formalism.

In this section, we will consider specifically the following two observables of the neutral kaon system:

$$\begin{aligned} R_{+-}(\tau) &= \frac{N(K(\tau)\to\pi^+\pi^-)}{N(K(\tau=0)\to\pi^+\pi^-)} \\ \delta(\tau) &= \frac{N(K(\tau)\to\pi^-\ell^+\nu) - N(K(\tau)\to\pi^+\ell^-\bar\nu)}{N(K(\tau)\to\pi^-\ell^+\nu) + N(K(\tau)\to\pi^+\ell^-\bar\nu)} \end{aligned} \tag{4.1}$$

These quantities are given in terms of density matrices and the decay operators defined in Section 2 by

$$R_{+-}(\tau) = \frac{\mathrm{tr}\,\rho_K(\tau)\mathcal{O}_{+-}}{\mathrm{tr}\,\rho_K(0)\mathcal{O}_{+-}}\ , \quad \delta(\tau) = \frac{\mathrm{tr}\,\rho_K(\tau)\big(\mathcal{O}_{\ell^+}-\mathcal{O}_{\ell^-}\big)}{\mathrm{tr}\,\rho_K(\tau)\big(\mathcal{O}_{\ell^+}+\mathcal{O}_{\ell^-}\big)} \tag{4.2}$$

In our analysis, we may ignore the effects of $\mathrm{Im}A_2$ in $\mathcal{O}_{+-}$, since $\epsilon'/\epsilon \sim 10^{-4}$. Thus, the formulae we present for the charged pion decay apply equally well to the neutral pion decay. It is easy to restore the effect of $\epsilon'$ by evaluating (4.2) more exactly.

In a $K^0$ beam which has evolved to large $\tau$, the quantities $\delta$ and $R_{+-}$ reflect the pure $K_L$ eigenstate. In this case, we can evaluate (4.2) with the density matrix $\rho_L$ and obtain the following results:

$$\begin{aligned} \delta_L &= 2\mathrm{Re}\,\epsilon_L^+ \\ R_L &= \rho_{L1} = |\epsilon_L^-|^2 + \gamma/\Delta\Gamma + 4(\beta/\Delta\Gamma)\mathrm{Im}[\epsilon_L^- d/d^*] \end{aligned} \tag{4.3}$$

Alternatively, we could consider the evolution of a pure $K^0$ or $\bar{K}^0$ state created in a strong interaction process. For state which is initially pure $K^0$, the evolving density matrix is given by (3.20) with $A_L = A_S = A_I = A_{\bar{I}} = \frac{1}{2}$, up to corrections of order $\epsilon, \alpha$. (For an initial $\bar{K}^0$, reverse the sign of the interference terms.) Then the time-dependent quantities (4.1) are given by$^\star$

$$\begin{aligned} \delta(\tau) &= \frac{2\cos(\Delta m\tau)e^{-(\bar{\Gamma}+\alpha-\gamma)\tau} + 2\mathrm{Re}\,\epsilon_S^- e^{-\Gamma_S\tau} + 2\mathrm{Re}\,\epsilon_L^+ e^{-\Gamma_L\tau}}{e^{-\Gamma_S\tau} + e^{-\Gamma_L\tau}} \\ R_{+-}(\tau) &= e^{-\Gamma_S\tau} + R_L e^{-\Gamma_L\tau} + 2|\bar{\eta}_{+-}|\cos(\Delta m\tau + \phi_{+-})e^{-(\bar{\Gamma}+\alpha-\gamma)\tau} \end{aligned} \tag{4.4}$$

with $R_L$ as in (4.3), and

$$|\bar{\eta}_{+-}|e^{i\phi_{+-}} = \epsilon_L^- \ . \tag{4.5}$$

Notice that the interference term in the time-dependent asymmetry depends on $\epsilon_L^-$, while the $K_L$ lepton asymmetry depends on $\epsilon_L^+$. Thus, by comparing time-dependent and long-time measurements, we can find a constraint on $\beta$ which is independent of other sources of $CPT$ violation. By comparing the value of $|\epsilon_L^-|^2$ to a determination of $R_L$, and using this bound on $\beta$, we can also obtain a bound on $\gamma$.

The difficulty in implementing this program is that experiments on the time-dependent evolution of neutral kaon beams typically report fits of their data to the conventional $CPT$-conserving theory, in which $\epsilon_L^-$, $\epsilon_L^+$, and the square root of

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$\star$ This formula includes all corrections linear in $\alpha$.

$\rho_{L1}$ are not distinguished. Thus, it is important to understand which particular parameters are actually constrained by each given experiment.

The value of $\delta_L$ poses no difficulty. The current world average[9] gives

$$\delta_L = 2\mathrm{Re}\,\epsilon_L^+ = (3.27 \pm 0.12) \times 10^{-3} \ . \tag{4.6}$$

In the early 1970's, the CERN-Heidelberg collaboration carried out beautiful studies of the time-dependence of $K_L$–$K_S$ interference.[17–19] These studies confirmed that the conventional, $CPT$-conserving parametrization of the neutral kaon system indeed gives a good description of its time-dependent phenomena. However, it is very difficult to reconstruct the constraints that these experiments put on more general models of time-dependence. The experiments involved kaon production from a platinum target; in this situation, the proportion of $K^0$ versus $\bar{K}^0$ initial states is not known *a priori* and is also momentum-dependent. The particle-antiparticle asymmetry must be described by an unknown function

$$A(p) = \frac{N(p) - \bar{N}(p)}{N(p) + \bar{N}(p)} \tag{4.7}$$

In ref. 18, data on neutral kaon decays to $\pi^+\pi^-$ was binned in momentum and decay time and these distributions were used to fit for the parameters $|\eta_{+-}|$, $\phi_{+-}$, and $A(p)$ (in each bin) from the assumed relation

$$R_{+-}(\tau) = e^{-\Gamma_S\tau} + |\eta_{+-}|^2 e^{-\Gamma_L\tau} + 2A(p)|\eta_{+-}|\cos(\Delta m\tau + \phi_{+-})e^{-\bar{\Gamma}\tau}. \tag{4.8}$$

Unfortunately, the absolute magnitude of the interference term can be absorbed into the parameters $A(p)$, so it is not possible to determine the coefficients $R_L$ and $|\bar{\eta}_{+-}|$ separately. Since the strongest constraint on $|\eta_{+-}|$ from this data set comes from the long-time tail of the decay distribution, we will consider the CERN-Heidelberg determination of $|\eta_{+-}|$ to be a measurement of $R_L$:

$$\sqrt{R_L} = (2.30 \pm 0.035) \times 10^{-3}. \tag{4.9}$$

From Fig. 4 of ref. 18, it is clear that quantum mechanics does correctly describe the interference region to few-percent accuracy after adjustment of the $A(p)$, but we will not make use that information in the discussion below.

Shortly afterward, the experimental group of Carithers *et al.*, working at the LBL Bevatron, used their data on $K \to \pi\pi$ decays in a regenerated $K_L$ beam to directly measure the magnitude of the interference term in neutral kaon evolution.[12] In the notation of eq. (4.4), they obtained the result

$$\frac{|\bar{\eta}_{+-}|}{\sqrt{R_L}} = 0.972 \pm 0.021 \ . \tag{4.10}$$

Very recently, the CPLEAR collaboration has published as its first result a determination of $|\bar{\eta}_{+-}|$ using hadronically produced neutral kaons tagged by an accompanying charged kaon.[3] Their determination is dominated by the interference region, and so we may interpret their result as a measurement of $|\bar{\eta}_{+-}|$. This gives

$$|\bar{\eta}_{+-}| = (2.32 \pm 0.14) \times 10^{-3} \ . \tag{4.11}$$

Averaging these measurements using the CERN-Heidelberg value for $R_L$, we find

$$|\bar{\eta}_{+-}| = (2.249 \pm 0.054) \times 10^{-3} \ . \tag{4.12}$$

Finally, we see no difficulty in accepting the world average of measurements of the interference phase $\phi_{+-}$ as a determination of the phase of $\epsilon_L^-$:

$$\phi_{+-} = (46.5 \pm 1.2)^\circ \tag{4.13}$$

The comparison of these four numbers allows us to constrain $\beta$ and $\gamma$. We can find $\beta$ from the relation

$$\mathrm{Re}\frac{2\beta}{d} = \mathrm{Re}\,\epsilon_L^+ - \mathrm{Re}\,\epsilon_L^- \ . \tag{4.14}$$

To analyze this relation, we use the fact that $\beta$ is real and $d$ is very well known:[9]

$$d = \Delta m + \frac{i}{2}\Delta\Gamma = \big((3.522 \pm 0.016) + i(3.682 \pm 0.008)\big) \times 10^{-15}\mathrm{GeV} \qquad (4.15)$$

It is convenient to parametrize

$$d = |d| e^{i(\pi/2 - \phi_{SW})} \ , \qquad (4.16)$$

where $\phi_{SW} = (43.73 \pm 0.15)^\circ$ is the *superweak phase* of Maiani's conventions.[20] With this definition

$$\beta = \frac{|d|}{2\sin\phi_{SW}}\big(\frac{\delta_L}{2} - |\bar\eta_{+-}|\cos\phi_{+-}\big) \qquad (4.17)$$

This yields

$$\beta = (3.2 \pm 2.9) \times 10^{-19} \ \mathrm{GeV} \ . \qquad (4.18)$$

With this determination of $\beta$, we can evaluate $\gamma$ by comparing $R_L$ to $\epsilon_L^-$:

$$\gamma = \Delta\Gamma\big[R_L - |\bar\eta_{+-}|^2 - 4(\beta/\Delta\Gamma)|\bar\eta_{+-}|\sin(2\phi_{SW} - \phi_{+-})\big] \ . \qquad (4.19)$$

which gives

$$\gamma = (-0.2 \pm 2.2) \times 10^{-21} \ \mathrm{GeV} \ . \qquad (4.20)$$

These constraints are similar to those obtained by Ellis, Mavromatos, and Nanopoulos[15,16] in one-parameter fits for $\beta$ and $\gamma$. However, our constraints hold in the general three-parameter EHNS model; they exclude the possibility that $\beta$ and $\gamma$ could give large individual contributions which cancel in $R_L$.

The geometry of these constraints is shown in Fig. 1. In Fig. 1(a), we show the systematics of the various values of $\epsilon$. This figure should be compared to Fig. 1 of ref. 15. The parameter $\epsilon_L^-$, considered as a vector in the complex plane, is directly determined by the interference measurement. From the endpoint of this vector, $\epsilon_L^+$

is found by making an excursion downward at 45° by a displacement proportional to $\beta$. Similarly, $R_L$ is found by moving outward a distance proportional to $\gamma$, after correcting for a $\beta$ effect. In Fig. 1(b), we show the constraints on these excursions given by the experimental values (4.6), (4.9), (4.12), (4.13). The various constraints overlap in the $\epsilon$ plane in such a way as to constraint $\beta$ and $\gamma$ to contribute only a very small part of the $CP$ violation phenomenology.

It is important to note that the constraints on $CPT$ violation outside of quantum mechanics that we have considered here are quite independent of the possibility of $CPT$ violation within quantum mechanics. Such a source of $CPT$ violation can be constrained, just as in the analysis without quantum mechanics violation,[20] by verifying the extend to which $\epsilon_L = \epsilon_L^- + \beta/d$ is parallel to $d$ in the complex plane. We will give the precise argument in the next section. Since we have not used any information on the absolute phase of $\epsilon_L$ in order to bound $\beta$ and $\gamma$, the standard constraints on $CPT$ violation within quantum mechanics are not significantly weakened when we allow for the presence of the EHNS parameters. In the future, the measurement of $\epsilon_S^\pm$ will allow a stronger constraint on this type of $CPT$ violation.

To derive further constraints on the violation of quantum mechanics, we strongly recommend that time dependent decay distributions be fit directly to the formulae (4.4), to determine the five independent parameters $R_L$, $\delta_L$, $|\bar{\eta}_{+-}|$, $\phi_{+-}$, $\alpha$. This will make it possible to check, without the ambiguities of our analysis here, whether the rapport among these parameters definitively excludes the presence of new terms in the density matrix evolution equation.

## 5. Effects of $CPT$ violation in decay amplitudes

At the end of Section 2, we argued that, whether $CPT$ violation arises from the breakdown of quantum mechanics or from the breakdown of local quantum field theory within quantum mechanics, the effects of direct $CPT$ violation in decay amplitudes should be small compared to those in neutral kaon propagation. Nevertheless, much emphasis is given in the literature on $CPT$ violation to disentangling effects of $CPT$ violation in decay vertices from $CPT$ violation in the kaon mass matrix. In this section, we will show that this separation can still be made, and constraints on $\beta$ and $\gamma$ deduced, if one allows for $CPT$-violating additions to the kaon decay vertices. However, this analysis will require additional assumptions which, though not unreasonable, are not airtight.

To begin the analysis, we need the more complicated forms of the decay operators $\mathcal{O}_{\mathcal{P}}$ which allow for $CPT$ violation in the decay amplitudes. The parametrization given in Maiani's review article[20] leads to the following expressions, which replace (2.11), (2.12), (2.15), and (2.18): for the leptonic decay amplitudes,

$$\mathcal{O}_{\ell^+} = \frac{|a+b|^2}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \;, \qquad \mathcal{O}_{\ell^-} = \frac{|a-b|^2}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \;; \tag{5.1}$$

for the $\pi\pi$ decay amplitudes,

$$\mathcal{O}_{+-} = |X_{+-}|^2\begin{pmatrix} 1 & Y_{+-} \\ Y^*_{+-} & |Y_{+-}|^2 \end{pmatrix} \;, \qquad \mathcal{O}_{00} = |X_{00}|^2\begin{pmatrix} 1 & Y_{00} \\ Y^*_{00} & |Y_{00}|^2 \end{pmatrix}, \tag{5.2}$$

where

$$X = \left\langle \pi\pi \,\middle|\, K_1 \right\rangle \;, \qquad Y = \frac{\left\langle \pi\pi \mid K_2 \right\rangle}{\left\langle \pi\pi \mid K_1 \right\rangle}. \tag{5.3}$$

More explicitly,

$$\begin{aligned} Y_{+-} &= \left(\frac{\mathrm{Re}B_0}{A_0}\right) + \epsilon' \\ Y_{00} &= \left(\frac{\mathrm{Re}B_0}{A_0}\right) - 2\epsilon' \end{aligned} \tag{5.4}$$

The quantities $\mathrm{Re}(B_0/A_0)$ and $\mathrm{Re}(b/a)$ parametrize $CPT$-violating decay amplitudes; $\epsilon'$ has the $CPT$ conserving value (2.21) shifted by the amount $e^{i\delta}(\mathrm{Re}B_0/A_0 - \mathrm{Re}B_2/A_2)/\sqrt{2}$ which accounts for a different degree of $CPT$ violation in the isospin $I = 0$ and $I = 2$ pion-decay channels.[20] With this generalization, the value of $\delta_L$ becomes

$$\delta_L = 2\mathrm{Re}\big[\epsilon_L^+ + \frac{b}{a}\big] \tag{5.5}$$

and the parameters $R_L$, $|\bar{\eta}_{+-}|$, and $\phi_{+-}$ in (4.4) are shifted to

$$\begin{aligned} R_L &= \gamma/\Delta\Gamma + |\bar{\eta}_{+-}|^2 + 4(\beta/\Delta\Gamma)\mathrm{Im}\big[\bar{\eta}_{+-}d/d^* - Y_{+-}\big] \\ |\bar{\eta}_{+-}|e^{i\phi_{+-}} &= \epsilon_L^- + Y_{+-} \ . \end{aligned} \tag{5.6}$$

When we include these corrections into the relation between $\epsilon_L^-$ and $\epsilon_L^+$, we find, instead of (4.17), the following expression for $\beta$:

$$\beta + \frac{|d|}{2\sin\phi_{SW}}\mathrm{Re}\big(\frac{b}{a} - \frac{B_0}{A_0}\big) = \frac{|d|}{2\sin\phi_{SW}}\big(\frac{\delta_L}{2} - |\bar{\eta}_{+-}|\cos\phi_{+-}\big) \ . \tag{5.7}$$

The relation for $\gamma$ remains

$$\gamma = \Delta\Gamma\big[R_L - |\bar{\eta}_{+-}|^2 - 4(\beta/\Delta\Gamma)|\bar{\eta}_{+-}|\sin(2\phi_{SW} - \phi_{+-})\big] \tag{5.8}$$

if we ignore a very small correction proportional to $\beta \cdot \mathrm{Im}\, Y_{+-} = \beta \cdot \mathrm{Im}\, \epsilon'$. Thus, our previous constraints on $\beta$ and $\gamma$ now appear as constraints on combinations of $CPT$-violating parameters.

For completeness, we should add one further constraint on a combination of parameters characterizing $CPT$ violation within and outside of quantum mechanics. This constraint, which is reviewed in ref. 20, uses unitarity and the dominance of the isospin 0 $\pi\pi$ decay channel to determine the phases of the $CPT$-conserving and $CPT$-violating mixing parameters. If we denote these components of the kaon mixing parameters as $\epsilon_M$ and $\Delta$, as in eq. (2.4), then this constraint implies that the phase of $\epsilon_M$ is $\phi_{SW}$ of eq. (4.16), and the phase of the combination

$$\Delta - \frac{\mathrm{Re}B_0}{A_0} \tag{5.9}$$

is $\phi_{SW} \pm \pi/2$.

To apply this relation, rewrite the second of eqs. (5.6) as:

$$|\bar{\eta}_{+-}|e^{i\phi_{+-}} = \epsilon_M - \big(\Delta - \frac{\mathrm{Re}B_0}{A_0}\big) - \frac{\beta}{d} \ . \tag{5.10}$$

Notice that, because $d$ points at 45°, the last term has a phase of (−45°), so that it is also perpendicular to $\epsilon_M$. On the other hand, the experimental value of $\phi_{+-}$ is very close to that of $\phi_{SW}$, though there is a small discrepancy:

$$\phi_{+-} - \phi_{SW} = (2.8 \pm 1.2)^\circ \ . \tag{5.11}$$

Thus, by comparing the components of the right and left hand sides of (5.10) perperdicular to $\epsilon_M$, we obtain a third constraint on the parameters of $CPT$ violation.

This additional constraint reads:

$$\beta \pm |d| \left|\Delta - \frac{\mathrm{Re}B_0}{A_0}\right| = (-5.6 \pm 2.5) \times 10^{-19} \ \mathrm{GeV} \ . \tag{5.12}$$

This must be combined with the results of eqs. (5.7) and (5.8):

$$\begin{aligned} \beta + \frac{|d|}{2\sin\phi_{SW}}\mathrm{Re}\big(\frac{b}{a} - \frac{B_0}{A_0}\big) &= (3.2 \pm 2.9) \times 10^{-19} \ \mathrm{GeV} \\ \gamma - 2|\bar{\eta}_{+-}d|\mathrm{Re}\big(\frac{b}{a} - \frac{B_0}{A_0}\big) &= (-0.2 \pm 2.2) \times 10^{-21} \ \mathrm{GeV} \ . \end{aligned} \tag{5.13}$$

In the last relation, we have used the approximation $\phi_{SW} \approx \phi_{+-}$. These three equations provide three constraints on four parameters and so cannot disprove the existence of $CPT$ violation. However, they imply that, unless there are unnatural cancellations among these parameters, the magnitude of $CPT$ violation should be at most about a tenth that of $CP$ violation.

## 6. Tests of quantum mechanics at a $\phi$ factory: formalism

A high-luminosity $\phi$ factory has been recognized as a facility which gives particularly incisive tests of $CPT$ violation.[5] In models in which $CPT$ violation arises within quantum mechanics, Peccei and collaborators have shown how, by studying the full set of possible time-dependent asymmetries observable at a $\phi$ factory, one may disentangle $CPT$-violating terms in the neutral kaon mass matrix from $CPT$-violating contributions to kaon decay amplitudes.[4,21] This analysis is made possible by the very simple time-dependence predicted for the $K_0$–$\bar{K}_0$ state which evolves from the decay of the $\phi$. Since the $\phi$ has spin 1, its decay to two spinless bosons produces an antisymmetric spatial wavefunction. This means that, when the $\phi$ decays to two neutral kaons, those particles must remain in opposite mass eigenstates until one decays. This strong constraint from quantum-mechanical coherence governs the whole phenomenology of $\phi$ factory measurements.

Because of the importance of quantum-mechanical coherence in $\phi$ decays, a $\phi$ factory is also an ideal place to search for terms in the kaon evolution which violate quantum coherence. In this section, we will explain how the standard formulae for kaon correlations in $\phi$ decay are changed by the introduction of quantum mechanics violation according to the model of EHNS, and we will point out particularly incisive measurements for determining or constraining the EHNS parameters.

The key to our analysis will be the form of the density matrix for the two-particle system which results from $\phi$ decay. We will first review the form of this density matrix in the case in which $CPT$ is violated only by corrections within quantum mechanics. Then we will show how this result is generalized in the EHNS model.

In the following discussion, we will assume that the $\phi$ resonance is pure spin 1, with no quantum mechanics violation in its decay amplitudes. Throughout our analysis, we will ignore background processes which produce $K_0\bar{K}_0$ in a spin 0 combination, and also effects of finite detector size. The influence of these effects in conventional $\phi$ factory analyses are described in ref. 5.

A spin 1 $\phi$ meson decays to an antisymmetric state of two kaons. If the kaons are neutral, we can describe the resulting wavefunction, in the basis of $CP$ eigenstates $|K_1\rangle$, $|K_2\rangle$, as

$$\phi \to \frac{1}{\sqrt{2}}\Big(|K_1, p> \otimes |K_2, -p> - |K_2, p> \otimes |K_1, -p>\Big) \ . \tag{6.1}$$

The two-kaon density matrix resulting from this decay is a $4 \times 4$ matrix. We can express this matrix concisely by introducing the set of $2 \times 2$ matrices appropriate to the $|K_1\rangle$, $|K_2\rangle$ basis:

$$\hat{\rho}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ , \quad \hat{\rho}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ , \quad \hat{\rho}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \quad \hat{\rho}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ . \tag{6.2}$$

Then the state (6.1) corresponds to the density matrix

$$P = \frac{1}{2}\Big(\hat{\rho}_1 \otimes \hat{\rho}_2 + \hat{\rho}_2 \otimes \hat{\rho}_1 - \hat{\rho}_+ \otimes \hat{\rho}_- - \hat{\rho}_- \otimes \hat{\rho}_+\Big) \ . \tag{6.3}$$

To work with the density matrix (6.3), we must express its components $\rho_i$ in terms of eigenstates of the single-particle density matrix evolution equation. Then we can assign each eigenstate its natural time-dependence. This procedure accounts correctly the full time-dependence of correlations.

To see how this works, we will first express $P$ in terms of eigenmodes in the quantum-mechanical case. It is straightforward to express the components (6.2) in terms of the four matrices (2.9) and then to express the combination (6.3) in terms of these elements. One finds, to no great surprise, an antisymmetric combination of $|K_L\rangle$ and $|K_S\rangle$. Here and for the rest of this section, we will denote results derived assuming quantum mechanical evolution (but not $CPT$ symmetry) with a superscript diamond. Thus,

$$P^{(\diamond)} = \frac{1 + 2\mathrm{Re}(\epsilon_S \epsilon_L)}{2}\Big(\rho_S \otimes \rho_L + \rho_L \otimes \rho_S - \rho_I \otimes \rho_{\bar{I}} - \rho_{\bar{I}} \otimes \rho_I\Big) \ . \tag{6.4}$$

The prefactor, which we have written to order $\epsilon^2$, corrects for the fact that $|K_L\rangle$ and $|K_S\rangle$ are not orthogonal. Supplying the proper time-dependence, we find the density matrix for processes in which the first kaon decays at proper time $\tau_1$ and the second at proper time $\tau_2$. We find

$$P^{(\diamondsuit)}(\tau_1,\tau_2) = \frac{1+2\mathrm{Re}(\epsilon_S\epsilon_L)}{2}\Big(\rho_S\otimes\rho_L e^{-\Gamma_S\tau_1}e^{-\Gamma_L\tau_2} + \rho_L\otimes\rho_S e^{-\Gamma_L\tau_1}e^{-\Gamma_S\tau_2} - \rho_I\otimes\rho_{\bar I} e^{-i\Delta m(\tau_1-\tau_2)}e^{-\bar\Gamma(\tau_1+\tau_2)} - \rho_{\bar I}\otimes\rho_I e^{+i\Delta m(\tau_1-\tau_2)}e^{-\bar\Gamma(\tau_1+\tau_2)}\Big)\ . \tag{6.5}$$

It is equally straightforward to perform this computation when quantum mechanics violation is included. One must first work out expressions for the components (6.2) in terms of the eigenmodes (3.22). To first order in small quantities, we find

$$\begin{aligned}
\hat\rho_1 &= \rho_S + (\gamma/\Delta\Gamma)\rho_L - \epsilon_S^{-*}\rho_I - \epsilon_S^{-}\rho_{\bar I}\\
\hat\rho_2 &= -(\gamma/\Delta\Gamma)\rho_S + \rho_L - \epsilon_L^{+}\rho_I - \epsilon_L^{+*}\rho_{\bar I}\\
\hat\rho_+ &= -\epsilon_L^{-*}\rho_S - \epsilon_S^{+}\rho_L + \rho_I + i(\alpha/2\Delta m)\rho_{\bar I}\\
\hat\rho_- &= -\epsilon_L^{-}\rho_S - \epsilon_S^{+*}\rho_L - i(\alpha/2\Delta m)\rho_I + \rho_{\bar I}\ .
\end{aligned} \tag{6.6}$$

Inserting these expressions into (6.4), we find

$$\begin{aligned}
P = \frac{1}{2}\Big[&\rho_S\otimes\rho_L + \rho_L\otimes\rho_S - \rho_I\otimes\rho_{\bar I} - \rho_{\bar I}\otimes\rho_I\\
&- 2\frac{\beta}{d}(\rho_S\otimes\rho_I + \rho_I\otimes\rho_S) - 2\frac{\beta}{d^*}(\rho_S\otimes\rho_{\bar I} + \rho_{\bar I}\otimes\rho_S)\\
&+ 2\frac{\beta}{d^*}(\rho_L\otimes\rho_I + \rho_I\otimes\rho_L) + 2\frac{\beta}{d}(\rho_L\otimes\rho_{\bar I} + \rho_{\bar I}\otimes\rho_L)\\
&+ i\frac{\alpha}{\Delta m}\big(\rho_I\otimes\rho_I - \rho_{\bar I}\otimes\rho_{\bar I}\big) + \frac{2\gamma}{\Delta\Gamma}\big(\rho_L\otimes\rho_L - \rho_S\otimes\rho_S\big)\Big]\ .
\end{aligned} \tag{6.7}$$

In eq. (6.7), each coefficient is given to first order in small quantities. We will carry out our analysis to this order. However, if some of the EHNS parameters are more severely constrained than others, one might need the expressions of second order in the less constrained parameters to determine the correct limits for the more constrained ones. For example, we saw in Section 4 that the terms quadratic in $\beta$ affect the limits on $\gamma$. In case a similar situation arises in $\phi$ factory experiments, we have given in the Appendix the complete formula for $P$ correct to second order in small quantities.

When we study time-dependent correlations, each term of (6.7) will lead to a characteristic exponential behavior. The first line of (6.7) is identical (within the approximation given) to (6.4) and thus contains only time-dependences allowed within quantum mechanics. The next three lines of (6.7), however, contain completely new structures. The second line of (6.7) leads to terms in the decay distribution which behave as

$$\cos(\Delta m \tau_1 - \phi) e^{-(\bar{\Gamma}+\alpha-\gamma)\tau_1} e^{-\Gamma_S \tau_2} \ , \tag{6.8}$$

and, in the same way, with 1 and 2 interchanged. The third line leads to similar expressions with the decay rate $\Gamma_L$. The first term in the fourth line leads to the even more bizarre time dependence

$$\sin(\Delta m(\tau_1 + \tau_2) - \phi') e^{-(\bar{\Gamma}+\alpha-\gamma)(\tau_1+\tau_2)} \ . \tag{6.9}$$

If we interpret frequencies as energies, both of the forms (6.8) and (6.9) signal a switch from positive to negative values of the energy. This is a rather subtle breakdown of energy conservation, which should be expected in the framework of density matrix evolution equations according to the analysis of ref. 10. This subtle effect does not obviously lead to macroscopic violations of energy conservation. However, in the $\phi$ factory experiments, one does not need to wait for the problems of energy conservation to built up to a macroscopic violation; one can instead track these violations directly in the frequency dependence of corrections. Finally, the second term in the fourth line contains the time-dependences

$$e^{-\Gamma_S(\tau_1+\tau_2)} \ , \qquad e^{-\Gamma_L(\tau_1+\tau_2)} \ . \tag{6.10}$$

Both terms in this line signal a breakdown of the antisymmetry of the final state wave function, which corresponds to a subtle breakdown of angular momentum conservation.

The basic observables computed from $P$ are double differential decay rates, the probabilities that the kaon with momentum $p$ decays into the final state $f_1$ at proper time $\tau_1$ while the kaon with momentum $(-p)$ decays to the final state $f_2$ at proper time $\tau_2$. We denote this quantity as $\mathcal{P}(f_1, \tau_1; f_2, \tau_2)$. If we denote the expression (6.7) schematically as

$$P = \sum_{i,j} A_{ij}\, \rho_i \otimes \rho_j \ , \tag{6.11}$$

where $i, j$ run over $S, L, I, \bar{I}$, and write the corresponding eigenvalues as $\lambda_i$, then the double decay rate is given by

$$\mathcal{P}(f_1, \tau_1; f_2, \tau_2) = \sum_{i,j} A_{ij}\ \mathrm{tr}[\rho_i \mathcal{O}_{f_1}]\, \mathrm{tr}[\rho_j \mathcal{O}_{f_2}] e^{-\lambda_i \tau_1 - \lambda_j \tau_2} \ . \tag{6.12}$$

Since it is easier to understand a distribution in one variable, much of the analysis of $\phi$ factory experiments has made use of the integrated distribution at fixed time interval $\Delta\tau = \tau_1 - \tau_2$. We will assume, in working with this quantity, that $\Delta\tau > 0$. Then this time interval distribution is defined as

$$\bar{\mathcal{P}}(f_1; f_2; \Delta\tau) = \int_{\Delta\tau}^{\infty} d(\tau_1 + \tau_2)\, \mathcal{P}(f_1, \tau_1; f_2, \tau_2) \ . \tag{6.13}$$

This time interval distribution is very useful for obtaining the standard $CP$ violation parameters of the neutral kaon system. However, since (6.13) integrates out one of the exponentials in (6.12), this integral does not possess the strange time dependences which signal quantum mechanics violation. A different quantity which shows these unusual effects more clearly is the double decay rate interpolated to equal times:

$$\mathcal{Q}(f_1; f_2; \tau) = \mathcal{P}(f_1, \tau; f_2, \tau) \ . \tag{6.14}$$

In the case in which quantum mechanics is exact and the density matrix is given by (6.5), this expression is a linear combination of decreasing exponentials, with no oscillatory terms. However, when we introduce the EHNS parameters, $\mathcal{Q}(\tau)$ can acquire terms with the oscillatory dependences $\cos(\Delta m \Delta\tau - \phi)$ and $\cos(2\Delta m \Delta\tau - \phi')$.

The most striking example of this modification of the quantum mechanical prediction appears in the case of decay to identical final states $f_1 = f_2 = f$. In this case, the quantum mechanical prediction for the double decay rate is especially simple.[4] Using

$$\mathrm{tr}[\rho_S \mathcal{O}_f]\,\mathrm{tr}[\rho_L \mathcal{O}_f] = \mathrm{tr}[\rho_I \mathcal{O}_f]\,\mathrm{tr}[\rho_{\bar{I}} \mathcal{O}_f] = \left|\langle f \mid K_S\rangle\right|^2 \left|\langle f \mid K_L\rangle\right|^2 , \tag{6.15}$$

we can rewrite the expression for the double decay rate as

$$\mathcal{P}^{(\diamondsuit)}(f,\tau_1;f,\tau_2) = C \times \left[e^{-\Gamma_S\tau_1 - \Gamma_L\tau_2} + e^{-\Gamma_L\tau_1 - \Gamma_S\tau_2} - 2\cos(\Delta m(\tau_1-\tau_2))e^{-\bar{\Gamma}(\tau_1+\tau_2)}\right] . \tag{6.16}$$

This quantity depends on the two times in a manner completely fixed by quantum mechanics irrespective of the properties of the decay amplitudes. In particular, at $\tau_1 = \tau_2$, the double distribution *vanishes*, as a consequence of the antisymmetry of the final state wavefunction. All of these conclusions hold whether or not $CPT$ symmetry is preserved.

On the other hand, when quantum mechanics is violated, decays to identical final states can have a much less constrained structure which includes all of the time dependences found in the most general case. We will see examples of this in the next section.

## 7. Tests of quantum mechanics at a $\phi$ factory: identical final states

Now that we have clarified the general form of the effects of quantum mechanics violation in the EHNS model, we will present expressions for the dependence of particular observables on the EHNS parameters. In particular, we will show explicitly how the terms of the equal time distributions $\mathcal{Q}(f;f;\tau)$, defined in eq. (6.14), can be used to constrain these parameters. In this section, we will return to the framework of Section 4, in which $CPT$ violation in decay vertices and $\epsilon'$ are neglected. However, in contrast to the case of experiments on single kaons, the constraints on the EHNS parameters from $\phi$ factory experiments are not essentially affected by the inclusion of $CPT$ violation in decay vertices. In the next section, we will explain this point and also discuss some aspects of the measurement of these decay parameters.

Consider first the case in which both kaons decay semileptonically. The cases in which both kaons decay to $\pi^-\ell^+\nu$ or to $\pi^+\ell^-\bar{\nu}$ are examples of decays to identical final states whose special properties were discussed at the end of the previous section. It is straightforward to work out the double time distribution for these cases by using the expression (6.7), the explicit forms of the density matrix eigenmodes (3.22), and the decay operators (2.11), (2.12). We find, to first order in small parameters

$$
\begin{aligned}
&\mathcal{P}(\ell^\pm,\tau_1;\ell^\pm,\tau_2) = \frac{|a|^4}{8}\\
&\times\Big\{(1\pm 4\mathrm{Re}\,\epsilon_M)\big[e^{-\Gamma_S\tau_1-\Gamma_L\tau_2}+e^{-\Gamma_L\tau_1-\Gamma_S\tau_2}-2\cos(\Delta m(\tau_1-\tau_2)e^{-(\bar{\Gamma}+\alpha-\gamma)(\tau_1+\tau_2)}\big]\\
&\quad\pm 4\frac{\beta}{|d|}\sin(\Delta m\tau_1-\phi_{SW})e^{-(\bar{\Gamma}+\alpha-\gamma)\tau_1}e^{-\Gamma_S\tau_2}+(1\leftrightarrow 2)\\
&\quad\pm 4\frac{\beta}{|d|}\sin(\Delta m\tau_1+\phi_{SW})e^{-(\bar{\Gamma}+\alpha-\gamma)\tau_1}e^{-\Gamma_L\tau_2}+(1\leftrightarrow 2)\\
&\quad+2\frac{\alpha}{\Delta m}\sin\Delta m(\tau_1+\tau_2)\,e^{-(\bar{\Gamma}+\alpha-\gamma)(\tau_1+\tau_2)}+2\frac{\gamma}{\Delta\Gamma}\big[e^{-\Gamma_L(\tau_1+\tau_2)}-e^{-\Gamma_S(\tau_1+\tau_2)}\big]\Big\}.
\end{aligned}
\tag{7.1}
$$

Notice that the first term in the brackets has a form quite close to the canonical form (6.16) predicted by quantum mechanics, while the remaining terms give systematic corrections to this result. For comparison,

$$\begin{aligned}\mathcal{P}(\ell^{\pm},\tau_1;\ell^{\mp},\tau_2) = {}& \frac{|a|^4}{8} \\ & \times\Big\{ \big[e^{-\Gamma_S\tau_1-\Gamma_L\tau_2} + e^{-\Gamma_L\tau_1-\Gamma_S\tau_2} + 2\cos(\Delta m(\tau_1-\tau_2))e^{-\bar{\Gamma}(\tau_1+\tau_2)}\big] \\ & \quad + \mathcal{O}(\epsilon,\alpha,\beta,\gamma)\Big\}.\end{aligned} \tag{7.2}$$

The complete expression is given in Appendix B.

The form of the corrections to quantum mechanics are easiest to see by interpolating to the line $\tau_1 = \tau_2$. On this line, (7.2) becomes

$$\mathcal{P}(\ell^{\pm},\tau;\ell^{\mp},\tau) = \frac{|a|^4}{8}\cdot 4e^{-2\bar{\Gamma}\tau}\ , \tag{7.3}$$

and (7.1) has a similar, though less dramatic, simplification. Then one finds

$$\begin{aligned}\mathcal{Q}(\ell^{\pm};\ell^{\pm};\tau)/\mathcal{Q}(\ell^{\pm};\ell^{\mp};\tau) = {}& \\ & \frac{1}{2}\big[1-e^{-2(\alpha-\gamma)\tau}\big(1-\frac{\alpha}{\Delta m}\sin 2\Delta m\tau\big)\big] \\ & +\frac{1}{2}\frac{\gamma}{\Delta\Gamma}\big[e^{+\Delta\Gamma\tau}-e^{-\Delta\Gamma\tau}\big] \\ & \pm 2\frac{\beta}{|d|}\big[\sin(\Delta m\tau-\phi_{SW})e^{-\Delta\Gamma\tau/2}+\sin(\Delta m\tau+\phi_{SW})e^{+\Delta\Gamma\tau/2}\big]\end{aligned} \tag{7.4}$$

The three coefficients $\alpha$, $\beta$, and $\gamma$ are selected by terms which are monotonic in $\tau$, oscillatory with frequency $\Delta m$,and oscillatory with frequency $2\Delta m$.

It is amusing to note that the positivity of the expression (7.4) under the conditions (3.12) is maintained by a delicate interplay of the correction terms. This is most easily seem by examining the limits $\alpha \gg \gamma > 0$ and $\gamma \gg \alpha > 0$.

Similar information is provided by the decay distribution to $\pi\pi$ final states on both sides of the $\phi$ decay process. Since we ignore $\epsilon'$ effects, the decay distributions to $\pi^+\pi^-$ and $\pi^0\pi^0$ final states are identical up to the overall normalization. For the specific case of $\pi^+\pi^-$ decays on both sides, we find, to second order in small parameters

$$
\begin{aligned}
\mathcal{P}(\pi^+\pi^-,\tau_1;\pi^+\pi^-,\tau_2) &= 2|A_0|^4 \\
&\times\Big\{ R_L\big[e^{-\Gamma_S\tau_1-\Gamma_L\tau_2}+e^{-\Gamma_L\tau_1-\Gamma_S\tau_2}\big] \\
&\quad - 2|\bar\eta_{+-}|^2\cos(\Delta m(\tau_1-\tau_2))e^{-(\bar\Gamma+\alpha-\gamma)(\tau_1+\tau_2)} \\
&\quad + 4\frac{\beta}{|d|}|\bar\eta_{+-}|\sin(\Delta m\tau_1+\phi_{+-}-\phi_{SW})e^{-(\bar\Gamma+\alpha-\gamma)\tau_1}e^{-\Gamma_S\tau_2}+(1\leftrightarrow 2) \\
&\quad - 2\Big[\frac{\gamma}{\Delta\Gamma}+2\frac{\beta}{|d|}|\bar\eta_{+-}|\frac{\sin\phi_{+-}}{\cos\phi_{SW}}\Big]e^{-\Gamma_S(\tau_1+\tau_2)}\Big\}.
\end{aligned}
\tag{7.5}
$$

where $|\bar\eta_{+-}|e^{i\phi_{+-}} = \epsilon_L^-$ and $R_L$ is defined as in (4.3). Specializing to the line $\tau_1=\tau_2=\tau$, we find

$$
\begin{aligned}
\mathcal{Q}(\pi^+\pi^-;\pi^+\pi^-;\tau) &\propto e^{-2\bar\Gamma\tau} \\
&\times\Big\{|\bar\eta_{+-}|^2\big[1-e^{-2(\alpha-\gamma)\tau}\big]-2\frac{\beta}{|d|}|\bar\eta_{+-}|\frac{\sin(\phi_{+-}-2\phi_{SW})+e^{-\Delta\Gamma\tau}\sin\phi_{+-}}{\cos\phi_{SW}} \\
&\quad+\frac{\gamma}{\Delta\Gamma}\big[1-e^{-\Delta\Gamma\tau}\big] \\
&\quad+4\frac{\beta}{|d|}|\bar\eta_{+-}|\sin(\Delta m\tau+\phi_{+-}-\phi_{SW})e^{-\Delta\Gamma\tau/2}\Big\}+\mathcal{O}^3(\alpha,\beta,\gamma,\epsilon_M,\Delta).
\end{aligned}
\tag{7.6}
$$

This distribution is less sensitive to $\alpha$; its leading $\alpha$ effect is of order $\alpha\cdot|\bar\eta_{+-}|^2$. However, the measurement of this distribution does allow one to put independent constraints on $\beta$ and $\gamma$.

The corresponding distributions for the three pion decay channels can be obtained by tracing the density matrix $P$ with the operator $\mathcal{O}_{3\pi}\otimes\mathcal{O}_{3\pi}$ as in (6.12). $\mathcal{O}_{3\pi}$ is expressed in terms of the three pion decay amplitudes as

$$
\mathcal{O}_{3\pi} = |X_{3\pi}|^2\begin{pmatrix} |Y_{3\pi}|^2 & Y_{3\pi}^* \\ Y_{3\pi} & 1\end{pmatrix}\,, \tag{7.7}
$$

where

$$X_{3\pi} = \Big\langle 3\pi \Big| K_2 \Big\rangle \ , \qquad Y_{3\pi} = \frac{\langle 3\pi \mid K_1 \rangle}{\langle 3\pi \mid K_2 \rangle}. \tag{7.8}$$

In this section we ignore $CP$ and $CPT$ violation in the decay amplitudes, so we set $Y_{3\pi}$ to zero; then

$$\begin{aligned}
\mathcal{P}(3\pi,\tau_1;3\pi,\tau_2) = & \frac{|X_{3\pi}|^4}{2} \\
& \times\Big\{ R_S\big[e^{-\Gamma_S\tau_1-\Gamma_L\tau_2} + e^{-\Gamma_L\tau_1-\Gamma_S\tau_2}\big] \\
& \quad - 2|\bar\eta_{3\pi}|^2\cos(\Delta m(\tau_1-\tau_2))e^{-(\bar\Gamma+\alpha-\gamma)(\tau_1+\tau_2)} \\
& \quad + 4\frac{\beta}{|d|}|\bar\eta_{3\pi}|\sin(\Delta m\tau_1 - \phi_{3\pi} + \phi_{SW})e^{-(\bar\Gamma+\alpha-\gamma)\tau_1}e^{-\Gamma_L\tau_2} + (1\leftrightarrow 2) \\
& \quad + 2\left[\frac{\gamma}{\Delta\Gamma} + 2\frac{\beta}{|d|}|\bar\eta_{3\pi}|\frac{\sin\phi_{3\pi}}{\cos\phi_{SW}}\right]e^{-\Gamma_L(\tau_1+\tau_2)}\Big\} + \mathcal{O}^3(\alpha,\beta,\gamma,\epsilon_M,\Delta).
\end{aligned} \tag{7.9}$$

where $|\bar\eta_{3\pi}|e^{i\phi_{3\pi}} = \epsilon_S^+$ and $R_S = -\gamma/\Delta\Gamma + |\bar\eta_{3\pi}|^2 - 4(\beta/\Delta\Gamma)\mathrm{Im}\big[\bar\eta_{3\pi}d/d^*\big]$.

This distribution reduces, on the line $\tau_1=\tau_2=\tau$, to

$$\begin{aligned}
& \mathcal{Q}(3\pi;3\pi;\tau) \propto e^{-2\bar\Gamma\tau} \\
& \quad \times\Big\{ |\bar\eta_{3\pi}|^2\big[1-e^{-2(\alpha-\gamma)\tau}\big] - 2\frac{\beta}{|d|}|\bar\eta_{3\pi}|\frac{\sin(2\phi_{SW}-\phi_{3\pi}) - e^{\Delta\Gamma\tau}\sin\phi_{3\pi}}{\cos\phi_{SW}} \\
& -\frac{\gamma}{\Delta\Gamma}\big[1-e^{\Delta\Gamma\tau}\big] \\
& +4\frac{\beta}{|d|}|\bar\eta_{3\pi}|\sin(\Delta m\tau - \phi_{3\pi}+\phi_{SW})e^{\Delta\Gamma\tau/2}\Big\} + \mathcal{O}^3(\alpha,\beta,\gamma,\epsilon_M,\Delta).
\end{aligned} \tag{7.10}$$

This distribution has a sensitivity to $\gamma$ which survives at large time ($\Gamma_S\tau \gg 1$) as

$$\lim_{\Gamma_S\tau\gg 1} \ \mathcal{Q}(3\pi;3\pi;\tau) \propto \frac{\gamma}{\Delta\Gamma}e^{-2\Gamma_L\tau}\,. \tag{7.11}$$

Its origin can be traced to the $\rho_L\otimes\rho_L$ propagating mode of the density matrix (6.7). This behavior contrasts with the behavior of the equal-time distribution $\mathcal{Q}(\pi^+\pi^-;\pi^+\pi^-;\tau)$; the latter vanishes, at large time, as $\sim e^{-\Gamma_S\tau}$.

# 8. Tests of quantum mechanics at a $\phi$ factory: general final states

In the previous section, we showed how to constrain the EHNS parameters independently of one another by considering the most straightforward experiments on $\phi$ decay, analyzed with the simplest theory. In this section, we will generalize our analysis to include direct $CPT$ violation and the effects of $\epsilon'$. in the $K^0$ decay matrix elements, as we have discussed for single kaon experiments in Section 5. We will first re-examine the determination of the EHNS parameters. We will show that these complications have virtually no effect on the method, or even the formulae, given in the previous section for the determination of $\alpha$, $\beta$, and $\gamma$. Then we will consider the reciprocal problem of the effect of quantum mechanics violation on the experimental determination of the kaon decay matrix elements. We will show that the measurements of these decay matrix elements can be affected if $\beta$ and $\gamma$ are nonzero. The measurements most sensitive to the modifications of kaon decay within quantum mechanics are asymmetries of the integrated time distributions (6.13).[4] We will present formulae which show how these asymmetries are shifted by quantum mechanics violation.

The modification of the formulae for the $\mathcal{Q}(f;f;\tau)$ in the presence of the effects discussed in Section 5 is quite minor. In the formulae for leptonic double decay distributions, $|a|^2$ is changed to $|a+b|^2$ for decays to $\ell^+$ and to $|a-b|^2$ for decays to $\ell^-$. This has no effect on the functional form of the double time distribution except for the simple replacement

$$\epsilon_M \to \epsilon_M + \frac{b}{a}\,. \tag{8.1}$$

In particular, the formula (7.4) is still valid in this more general context. Similarly, the inclusion of more general effects in the kaon decay vertices changes the relative normalization of $\pi^+\pi^-$ and $\pi^0\pi^0$ decay rates. However, the formula (7.6) remains valid with the replacement

$$|\bar{\eta}_{+-}|e^{i\phi_{+-}} = \epsilon_L^- + Y_{+-}\ , \tag{8.2}$$

as in eq. (5.6). This is equally true for the case of the three pion decay, for which formula (7.10) remains valid with the substitution*

$$|\bar{\eta}_{3\pi}|e^{i\phi_{3\pi}} = \epsilon_S^+ + Y_{3\pi} \ . \tag{8.3}$$

Thus, there is no difficulty in constraining $CPT$ violation from outside quantum mechanics in $\phi$ factory experiments even in this more general context. For reference, the complete expressions for the double time distributions to pion or leptonic final states are given in Appendix B.

It is also interesting to ask whether the converse of this statement is true, whether quantum mechanics violation can interfere with the measurement of $\epsilon'$ and other more standard asymmetries which should appear at a $\phi$ factory. In the following, we briefly discuss how the determination of certain of these quantities might be affected.

One would not immediately expect the measured value of $\epsilon'/\epsilon$ to be significantly affected by quantum mechanics violation, since $\epsilon'$ is a property of the difference between the decay rates into $\pi^+\pi^-$ and $\pi^0\pi^0$ while the parameters $\beta$ and $\gamma$ affect the time evolution of the kaon system prior its decay. From this argument, one would expect $\epsilon'/\epsilon$ to be at most corrected by a factor $1+\mathcal{O}(\,\beta/d\epsilon\,,\,\gamma/|d||\epsilon|^2)$. Using the bounds on $\beta$ and $\gamma$ derived in Section 4, we estimate the terms in parenthesis to be at most corrections of the order of 5% and 25% respectively. These corrections are mild and can be reduced further if no evidence for nonzero $\beta$ and $\gamma$ is found at a $\phi$ factory.

However, if $\beta$ and $\gamma$ prove to be nonzero, they will have two important effects in the measurement of $\epsilon'/\epsilon$. First, any time-dependent method of determining $\epsilon'/\epsilon$ would be complicated by the new time dependences introduced by quantum-mechanics violating terms in the density matrix evolution. As we have already

* We also assume here that we can neglect a term proportional to $\mathrm{Im}Y_{3\pi}$; we expect this term to be no bigger than $\mathrm{Im}Y_{+-} \sim \mathrm{Im}\epsilon'$.

pointed out, this effect can be minimized by considering the integrated distributions (6.13) at fixed time interval $\Delta\tau = \tau_1 - \tau_2$. Second, because $\beta/d$ and $\epsilon$ are close to being orthogonal in the complex plane, effects of nonzero $\beta$ can mix the real and imaginary parts of $\epsilon'/\epsilon$.

To demonstrate these points quantitatively, we consider the determination[24] of $\epsilon'/\epsilon$ from the measurement of the quantity

$$\mathcal{A}(\pi^+\pi^-;\pi^0\pi^0;\Delta\tau) = \frac{\bar{\mathcal{P}}(\pi^+\pi^-;\pi^0\pi^0;\Delta\tau) - \bar{\mathcal{P}}(\pi^0\pi^0;\pi^+\pi^-;\Delta\tau)}{\bar{\mathcal{P}}(\pi^+\pi^-;\pi^0\pi^0;\Delta\tau) + \bar{\mathcal{P}}(\pi^0\pi^0;\pi^+\pi^-;\Delta\tau)} \,. \tag{8.4}$$

This asymmetry can be straightforwardly computed using the formulae of Appendix B. We find

$$\mathcal{A}(\pi^+\pi^-;\pi^0\pi^0;\Delta\tau) = 3\mathrm{Re}\,\frac{\epsilon'}{\epsilon} \times \frac{\mathcal{N}_R}{\mathcal{D}} - 3\,\mathrm{Im}\,\frac{\epsilon'}{\epsilon} \times \frac{\mathcal{N}_I}{\mathcal{D}}\,, \tag{8.5}$$

The coefficients $\mathcal{N}_{R,I}$ and $\mathcal{D}$ are functions of $\Delta\tau$, $\beta/d$, $\gamma$, and $|\bar{\eta}_{+-}|$ which are given in Appendix C.

In the context of pure quantum mechanics, the quantities $\mathcal{N}_{R,I}$ and $\mathcal{D}$ have a simple functional form, and the quantities $\mathrm{Re}\,\epsilon'/\epsilon$, $\mathrm{Im}\,\epsilon'/\epsilon$ can be extracted from $\mathcal{A}^{(\diamondsuit)}(\pi^+\pi^-;\pi^0\pi^0;\Delta\tau)$ by a two-parameter fit. In presence of quantum mechanics violation, this is no longer true. For example, in the limit $\Gamma_S\Delta\tau \gg 1$,

$$\mathcal{A}^{(\diamondsuit)}(\pi^+\pi^-;\pi^0\pi^0;\Delta\tau) \to 3\mathrm{Re}\,\epsilon'/\epsilon \ . \tag{8.6}$$

In the same limit, (8.5) becomes

$$\begin{aligned}
&\mathcal{A}(\pi^+\pi^-;\pi^0\pi^0;\Delta\tau) \to \\
&\qquad 3\mathrm{Re}\,\epsilon'/\epsilon\ \left[\frac{1 + (2\beta/|d||\bar{\eta}_{+-}|)\sin(\phi_{SW} - \phi_{+-})}{1 + (\gamma/\Delta\Gamma|\bar{\eta}_{+-}|^2) + (2\beta/|d||\bar{\eta}_{+-}|)(\sin(2\phi_{SW} - \phi_{+-})/\cos\phi_{SW})}\right] \,. \\
&\qquad -3\mathrm{Im}\,\epsilon'/\epsilon\ \left[(2\beta/|d||\bar{\eta}_{+-}|)\,\cos(\phi_{SW} - \phi_{+-})\right]
\end{aligned} \tag{8.7}$$

To understand the role of the near-orthogonality of $\beta/d$ and $\bar{\eta}_{+-}$, we may approximate $\phi_{+-} \approx \phi_{SW} \approx 45°$; then (8.7) simplifies to

$$\mathcal{A}(\pi^+\pi^-;\pi^0\pi^0;\Delta\tau) \rightarrow \\ 3\mathrm{Re}\,\epsilon'/\epsilon\left[1-\frac{\gamma}{\sqrt{2}|d||\bar{\eta}_{+-}|^2}-2\frac{\beta}{|d||\bar{\eta}_{+-}|}\right]-3\,\mathrm{Im}\,\epsilon'/\epsilon\left[2\frac{\beta}{|d||\bar{\eta}_{+-}|}\right]. \quad (8.8)$$

This makes clear that, if one opens the possibility of quantum mechanics violation in the neutral kaon system at the level $\beta \sim m_K^2/m_{\mathrm{Pl}}$, one must constrain or determine $\beta$ in order to measure $\mathrm{Re}\,\epsilon'$.

Similar effects are present in asymmetries involving the semileptonic decay distributions $\bar{\mathcal{P}}(\ell^\pm;\ell^\pm;\Delta\tau)$. For simplicity, we again consider only the large-time limit $\Gamma_S\Delta\tau \gg 1$. Three particularly informative asymmetries are

$$\begin{aligned}
&\frac{\bar{\mathcal{P}}(\ell^+;\ell^+;\Delta\tau)-\bar{\mathcal{P}}(\ell^-;\ell^+;\Delta\tau)}{\bar{\mathcal{P}}(\ell^+;\ell^+;\Delta\tau)+\bar{\mathcal{P}}(\ell^-;\ell^-;\Delta\tau)} \rightarrow \delta_L \\
&\frac{\bar{\mathcal{P}}(\ell^+;\ell^+;\Delta\tau)-\bar{\mathcal{P}}(\ell^-;\ell^-;\Delta\tau)}{\bar{\mathcal{P}}(\ell^+;\ell^+;\Delta\tau)+\bar{\mathcal{P}}(\ell^-;\ell^-;\Delta\tau)} \rightarrow 4\mathrm{Re}\,(\epsilon_M+b/a)\ +\ 8\frac{\beta}{|d|}\sin 2\phi_{SW}\ \cos\phi_{SW} \\
&\frac{\bar{\mathcal{P}}(\ell^+;\ell^-;\Delta\tau)-\bar{\mathcal{P}}(\ell^-;\ell^+;\Delta\tau)}{\bar{\mathcal{P}}(\ell^+;\ell^+;\Delta\tau)\ +\ \bar{\mathcal{P}}(\ell^-;\ell^-;\Delta\tau)} \rightarrow -4\mathrm{Re}\,\Delta+4\frac{\beta}{|d|}\sin\phi_{SW}\left[1-4\cos^2\phi_{SW}\right].
\end{aligned} \quad (8.9)$$

The first of the above formulae yields a direct determination of $\delta_L$, even in the presence of quantum mechanics violation. However, the other two formulae are more complicated. If quantum mechanics is assumed to be valid, these two limits give simple constraints on the $CPT$ violating parameters $\Delta$ and $\mathrm{Re}(b/a)$. However, in the more general context of quantum mechanics violation, these parameters are constrained only to the extent that $\beta$ is known from the experiments described in Section 7.

It is remarkable how sensitively the parameters of quantum mechanics violation affect the various observable quantities of the $K^0$–$\bar{K}^0$ system as observed at the $\phi$. Using the strategies we have discussed, it is likely that all three of the EHNS parameters of quantum mechanics violation can be bounded, or measured, at a level well below $(1/m_{Pl})$. Perhaps there are still more surprises waiting for us in neutral kaon physics.

# APPENDIX A: Eigenmodes of density matrix evolution

In this appendix, we give more exact formulae for the eigenmodes of the density matrix evolution and for the components of the density matrix for neutral kaon pairs resulting from $\phi$ decay. The expressions below are complete through second order in $CP$-violating parameters.

First, we present the density matrix eigenmodes. These are given as column vectors

$$\begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_I \\ \rho_{\bar{I}} \end{pmatrix} , \tag{A.1}$$

where these elements are defined by eq. (3.13). The expressions (3.22) should be replaced by:

$$\rho_S = \begin{pmatrix} 1 \\ -\gamma/\Delta\Gamma + 2\mathrm{Im}[\epsilon_S^+ \epsilon_S^{-*} d]/\Delta\Gamma \\ \epsilon_S^{-*} - (d\epsilon_L^+ \gamma/d^* \Delta\Gamma) + i(\epsilon_S^{-*}\gamma/d^*) - (2\alpha \mathrm{Im}\, \epsilon_S^-]/d^*) \\ \epsilon_S^- - (d^* \epsilon_L^{+*} \gamma/d\Delta\Gamma) - i(\epsilon_S^- \gamma/d) - (2\alpha \mathrm{Im}\, \epsilon_S^-]/d) \end{pmatrix} , \tag{A.2}$$

$$\rho_L = \begin{pmatrix} \gamma/\Delta\Gamma - 2\mathrm{Im}[\epsilon_L^+ \epsilon_L^{-*} d^*]/\Delta\Gamma \\ 1 \\ \epsilon_L^+ + (d^* \epsilon_S^{-*} \gamma/d\Delta\Gamma) + i(\epsilon_L^+ \gamma/d) + (2\alpha \mathrm{Im}\, \epsilon_L^+]/d) \\ \epsilon_L^{+*} + (d\epsilon_S^- \gamma/d^* \Delta\Gamma) - i(\epsilon_L^{+*}\gamma/d^*) + (2\alpha \mathrm{Im}\, \epsilon_L^+]/d^*) \end{pmatrix} , \tag{A.3}$$

$$\rho_I = \begin{pmatrix} \epsilon_L^{-*}(1 + i\gamma/d^* - i\alpha/d^*) - i\epsilon_S^+\gamma/d^* + i(d\epsilon_L^-\alpha/2d^*\Delta m) \\ \epsilon_S^+(1 + i\gamma/d - i\alpha/d) - i\epsilon_L^{-*}\gamma/d + i(d^*\epsilon_S^{+*}\alpha/2d\Delta m) \\ 1 \\ -i(\alpha/2\Delta m) + (\epsilon_S^-\epsilon_L^{-*}d + \epsilon_L^{+*}\epsilon_S^{+*}d^*)/2\Delta m \end{pmatrix}, \tag{A.4}$$

$$\rho_{\bar{I}} = \begin{pmatrix} \epsilon_L^-(1 - i\gamma/d + i\alpha/d) + i\epsilon_S^{+*}\gamma/d^* - i(d^*\epsilon_L^{-*}\alpha/2d\Delta m) \\ \epsilon_S^{+*}(1 - i\gamma/d^* + i\alpha/d^*) + i\epsilon_L^-\gamma/d^* - i(d\epsilon_S^+\alpha/2d^*\Delta m) \\ i(\alpha/2\Delta m) + (\epsilon_S^{-*}\epsilon_L^- d^* + \epsilon_L^+\epsilon_S^{+*}d)/2\Delta m \\ 1 \end{pmatrix}. \tag{A.5}$$

Note that the expressions for $\rho_{L1}$ and $\rho_{S2}$ given in (3.22) agree with the expressions above.

Next, we present a more precise form for the density matrix $P$ by quoting the matrix elements of $A_{ij}$, defined by eq. (6.11), to second order in small parameters. We find:

$$
\begin{aligned}
A_{SL} &= A_{LS} = \frac{1}{2}(1+2\mathrm{Re}(\epsilon_S\epsilon_L)) + \beta\mathrm{Re}\frac{\epsilon_S-\epsilon_L}{d} + \frac{3}{2}\beta^2\ \frac{d^2+d^{*2}}{|d|^4} - \frac{3}{2}\ \frac{\gamma^2}{(\Delta\Gamma)^2} \\
A_{SS} &= -\frac{\gamma}{\Delta\Gamma} - 2\ \frac{\beta^2}{|d|^2} - 4\,\frac{\beta}{\Delta\Gamma}\,\mathrm{Im}\epsilon_L \\
A_{LL} &= \frac{\gamma}{\Delta\Gamma} - 2\,\frac{\beta^2}{|d|^2} + 4\,\frac{\beta}{\Delta\Gamma}\,\mathrm{Im}\epsilon_S \\
A_{SI} &= A_{IS} = (A_{S\overline{I}})^* = (A_{\overline{I}S})^* \\
&= -\frac{\beta}{d} - 2\,\frac{\alpha}{d}\ \mathrm{Im}\epsilon_L - i\,\frac{\gamma}{d}\,(\epsilon_L-\epsilon_S^*) + i\,\frac{\alpha\beta}{2\Delta m\, d^*} - \frac{\gamma\beta}{\Delta\Gamma\, d^*} \\
A_{LI} &= A_{IL} = (A_{L\overline{I}})^* = (A_{\overline{I}L})^* \\
&= \frac{\beta}{d^*} + 2\,\frac{\alpha}{d^*}\,\mathrm{Im}\epsilon_S + i\,\frac{\gamma}{d^*}\,(\epsilon_L-\epsilon_S^*) - i\,\frac{\alpha\beta}{2\Delta m d} - \frac{\beta\gamma}{\Delta\Gamma d} \\
A_{I\bar{I}} &= A_{\bar{I}I} = -\frac{1}{2}\,(1+2\mathrm{Re}\epsilon_L\epsilon_S) + \beta\,\mathrm{Re}\left(\frac{\epsilon_S-\epsilon_L}{d}\right) \\
&\qquad + \frac{3}{2}\,\beta^2\,\frac{d^2+(d^*)^2}{|d|^4} - \frac{3}{8}\ \frac{\alpha^2}{(\Delta m)^2} \\
A_{II} &= (A_{\overline{I}\overline{I}})^* = \frac{i\alpha}{2\Delta m} - 2\,\frac{\beta^2}{|d|^2} - \frac{\beta}{\Delta m}\,(\epsilon_L-\epsilon_S^*)
\end{aligned}
\tag{A.6}
$$

## APPENDIX B: Double time distributions

In this appendix, we give the complete formulae for the double time distributions $\mathcal{P}(f_1,\tau_1;f_2,\tau_2)$ for $\phi$ decay to the various final states discussed in the text. The expressions below are complete through first order in $CP$-violating parameters unless it is specified otherwise.

$$
\begin{aligned}
\mathcal{P}(\ell^{\pm},\tau_1;\ell^{\pm},\tau_2) = \frac{|a|^4}{8}& \\
\times\Bigg\{ \big[(1 \pm 4\mathrm{Re}\,(\epsilon_M + \frac{b}{a})\big] &\big[e^{-\Gamma_S\tau_1-\Gamma_L\tau_2} + e^{-\Gamma_L\tau_1-\Gamma_S\tau_2} - 2\cos(\Delta m(\tau_1-\tau_2))e^{-(\bar\Gamma+\alpha-\gamma)(\tau_1+\tau_2)}\big] \\
&\pm 4\frac{\beta}{|d|}\sin(\Delta m\tau_1 - \phi_{SW})e^{-(\bar\Gamma+\alpha-\gamma)\tau_1}e^{-\Gamma_S\tau_2} + (1\leftrightarrow 2) \\
&\pm 4\frac{\beta}{|d|}\sin(\Delta m\tau_1 + \phi_{SW})e^{-(\bar\Gamma+\alpha-\gamma)\tau_1}e^{-\Gamma_L\tau_2} + (1\leftrightarrow 2) \\
&+ 2\frac{\alpha}{\Delta m}\sin\Delta m(\tau_1+\tau_2)\,e^{-(\bar\Gamma+\alpha-\gamma)(\tau_1+\tau_2)} + 2\frac{\gamma}{\Delta\Gamma}\big[e^{-\Gamma_L(\tau_1+\tau_2)} - e^{-\Gamma_S(\tau_1+\tau_2)}\big]\Bigg\}.
\end{aligned} \tag{B.1}
$$

$$
\begin{aligned}
\mathcal{P}(\ell^{+},\tau_1;\ell^{-},\tau_2) = \frac{|a|^4}{8}& \\
\times\Bigg\{&(1+4\mathrm{Re}\,(\Delta-\beta/d))e^{-\Gamma_S\tau_1-\Gamma_L\tau_2} + (1-4\mathrm{Re}\,(\Delta-\beta/d))e^{-\Gamma_L\tau_1-\Gamma_S\tau_2} \\
&+ 2\cos\big(\Delta m(\tau_1-\tau_2) - 4\mathrm{Im}(\Delta+\beta/d)\big)\,e^{-(\bar\Gamma+\alpha-\gamma)(\tau_1+\tau_2)} \\
&+ 4\frac{\beta}{|d|}\sin(\Delta m\tau_1 - \phi_{SW})e^{-(\bar\Gamma+\alpha-\gamma)\tau_1}e^{-\Gamma_S\tau_2} - (1\leftrightarrow 2) \\
&+ 4\frac{\beta}{|d|}\sin(\Delta m\tau_1 + \phi_{SW})e^{-(\bar\Gamma+\alpha-\gamma)\tau_1}e^{-\Gamma_L\tau_2} - (1\leftrightarrow 2) \\
&- 2\frac{\alpha}{\Delta m}\sin\Delta m(\tau_1+\tau_2)\,e^{-(\bar\Gamma+\alpha-\gamma)(\tau_1+\tau_2)} + 2\frac{\gamma}{\Delta\Gamma}\big[e^{-\Gamma_L(\tau_1+\tau_2)} - e^{-\Gamma_S(\tau_1+\tau_2)}\big]\Bigg\}.
\end{aligned} \tag{B.2}
$$

$$
\begin{aligned}
\mathcal{P}(\pi^+\pi^-,\tau_1;\pi^+\pi^-,\tau_2) = \frac{|X_{+-}|^4}{2}& \\
\times\Bigg\{R_L\big[e^{-\Gamma_S\tau_1-\Gamma_L\tau_2} + e^{-\Gamma_L\tau_1-\Gamma_S\tau_2}\big] &- 2|\bar\eta_{+-}|^2\cos(\Delta m(\tau_1-\tau_2))\,e^{-(\bar\Gamma+\alpha-\gamma)(\tau_1+\tau_2)} \\
&+ 4\frac{\beta}{|d|}|\bar\eta_{+-}|\sin(\Delta m\tau_1 + \phi_{+-} - \phi_{SW})e^{-(\bar\Gamma+\alpha-\gamma)\tau_1}e^{-\Gamma_S\tau_2} + (1\leftrightarrow 2) \\
&- 2\bigg(\frac{\gamma}{\Delta\Gamma} + 4\frac{\beta}{\Delta\Gamma}\mathrm{Im}\big[\bar\eta_{+-} - Y_{+-}\big]\bigg)e^{-\Gamma_S(\tau_1+\tau_2)}\Bigg\} \;+\; \mathcal{O}^3(\alpha,\beta,\gamma,\epsilon_{S,L},Y_{+-})\;,
\end{aligned} \tag{B.3}
$$

using $\bar{\eta}_{+-}e^{i\phi_{+-}} = \epsilon_L^- + Y^{+-}$ and

$$R_L = \gamma/\Delta\Gamma + |\bar{\eta}_{+-}|^2 + 4(\beta/\Delta\Gamma)\mathrm{Im}\big[\bar{\eta}_{+-}d/d^* - Y_{+-}\big]\ . \tag{B.4}$$

$$\begin{aligned}
\mathcal{P}(\pi^+\pi^-,\tau_1;\pi^0\pi^0,\tau_2) &= \frac{|X_{+-}|^2|X_{00}|^2}{2} \\
&\times\Bigg\{ R_L^{00}e^{-\Gamma_S\tau_1-\Gamma_L\tau_2} + R_L^{+-}e^{-\Gamma_L\tau_1-\Gamma_S\tau_2} \\
&\quad - 2|\bar{\eta}_{+-}||\bar{\eta}_{00}|\cos(\Delta m(\tau_1-\tau_2)+\phi_{+-}-\phi_{00})\,e^{-(\bar{\Gamma}+\alpha-\gamma)(\tau_1+\tau_2)} \\
&\quad + 4\frac{\beta}{|d|}|\bar{\eta}_{+-}|\sin(\Delta m\tau_1 - \phi_{SW} + \phi_{+-})e^{-(\bar{\Gamma}+\alpha-\gamma)\tau_1}e^{-\Gamma_S\tau_2} \\
&\quad + 4\frac{\beta}{|d|}|\bar{\eta}_{00}|\sin(\Delta m\tau_2 - \phi_{SW} + \phi_{00})e^{-(\bar{\Gamma}+\alpha-\gamma)\tau_2}e^{-\Gamma_S\tau_1} \\
&\quad - 2\left(\frac{\gamma}{\Delta\Gamma} + 2\frac{\beta}{\Delta\Gamma}\mathrm{Im}\big[\bar{\eta}_{+-} - Y_{+-} + \bar{\eta}_{00} - Y_{00}\big]\right)e^{-\Gamma_S(\tau_1+\tau_2)}\Bigg\} \ + \ \mathcal{O}^3(\alpha,\beta,\gamma,\epsilon_{S,L},Y_{\{ {}^{+-}_{00}}) \ .
\end{aligned} \tag{B.5}$$

$$\begin{aligned}
\mathcal{P}(3\pi,\tau_1;3\pi,\tau_2) &= \frac{|X_{3\pi}|^4}{2} \\
&\times\Bigg\{ R_S\big[e^{-\Gamma_S\tau_1-\Gamma_L\tau_2} + e^{-\Gamma_L\tau_1-\Gamma_S\tau_2}\big] \\
&\quad - 2|\bar{\eta}_{3\pi}|^2\cos(\Delta m(\tau_1-\tau_2))e^{-(\bar{\Gamma}+\alpha-\gamma)(\tau_1+\tau_2)} \\
&\quad + 4\frac{\beta}{|d|}|\bar{\eta}_{3\pi}|\sin(\Delta m\tau_1 - \phi_{3\pi} + \phi_{SW})e^{-(\bar{\Gamma}+\alpha-\gamma)\tau_1}e^{-\Gamma_L\tau_2} + (1\leftrightarrow 2) \\
&\quad + 2\left(\frac{\gamma}{\Delta\Gamma} + 4\frac{\beta}{\Delta\Gamma}\mathrm{Im}\big[\bar{\eta}_{3\pi} - Y_{3\pi}\big]\right)e^{-\Gamma_L(\tau_1+\tau_2)}\Bigg\} \ + \ \mathcal{O}^3(\alpha,\beta,\gamma,\epsilon_{S,L},Y_{3\pi}).
\end{aligned} \tag{B.6}$$

where $|\bar{\eta}_{3\pi}|e^{i\phi_{3\pi}} = \epsilon_S^+ + Y_{3\pi}$ and

$$R_S = -\gamma/\Delta\Gamma + |\bar{\eta}_{3\pi}|^2 - 4(\beta/\Delta\Gamma)\mathrm{Im}\big[\bar{\eta}_{3\pi}d/d^* - Y_{3\pi}\big]\,. \tag{B.7}$$

$$
\begin{aligned}
\mathcal{P}(\pi^+\pi^-,\tau_1;\ell^\pm,\tau_2) = &\frac{|X_{+-}|^2}{2}\frac{|a|^2}{2} \\
&\times\Big\{R_L e^{-\Gamma_L\tau_1-\Gamma_S\tau_2} + \big[1\pm\delta_L+\frac{\gamma}{\Delta\Gamma}\big]e^{-\Gamma_S\tau_1-\Gamma_L\tau_2} \\
&\quad\mp 2|\bar\eta_{+-}|\cos(\Delta m(\tau_1-\tau_2)+\phi_{+-})e^{-(\bar\Gamma+\alpha-\gamma)(\tau_1+\tau_2)} \\
&\quad\pm 4\frac{\beta}{|d|}\sin(\Delta m\tau_2-\phi_{SW})e^{-(\bar\Gamma+\alpha-\gamma)\tau_2}e^{-\Gamma_S\tau_1} \\
&\quad-2\Big(\frac{\gamma}{\Delta\Gamma}+4\frac{\beta}{\Delta\Gamma}\mathrm{Im}\,\bar\eta_{+-}\Big)e^{-\Gamma_S(\tau_1+\tau_2)}\Big\}.
\end{aligned}
\tag{B.8}
$$

## APPENDIX C: Formulae for the measurement of $\epsilon'/\epsilon$

In this Appendix, we present formulae for extracting $\epsilon'/\epsilon$ from the integrated distributions at fixed time interval $\Delta\tau = \tau_1 - \tau_2$ of the asymmetric decay into charged and neutral pion final states.

$$
\mathcal{A}(\pi^+\pi^-;\pi^0\pi^0;\Delta\tau) = \frac{\bar{\mathcal{P}}(\pi^+\pi^-;\pi^0\pi^0;\Delta\tau)-\bar{\mathcal{P}}(\pi^0\pi^0;\pi^+\pi^-;\Delta\tau)}{\bar{\mathcal{P}}(\pi^+\pi^-;\pi^0\pi^0;\Delta\tau)+\bar{\mathcal{P}}(\pi^0\pi^0;\pi^+\pi^-;\Delta\tau)}\,. \tag{C.1}
$$

$$
\mathcal{A}(\pi^+\pi^-;\pi^0\pi^0;\Delta\tau) = 3\mathrm{Re}\,\epsilon'/\epsilon\times\frac{\mathcal{N}_R}{\mathcal{D}} - 3\mathrm{Im}\,\epsilon'/\epsilon\times\frac{\mathcal{N}_I}{\mathcal{D}}\,, \tag{C.2}
$$

with

$$
\begin{aligned}
\mathcal{N}_R = &\, e^{-\Gamma_L\Delta\tau}\left[1+2\frac{\beta}{|d||\bar\eta_{+-}|}\sin(\phi_{SW}-\phi_{+-})\right] \\
&-e^{-\Gamma_S\Delta\tau}\left[1+2\frac{\beta}{|d||\bar\eta_{+-}|}\Big(\sin(\phi_{SW}-\phi_{+-})-|z|\sin(\phi_{SW}+\phi_z-\phi_{+-})\Big)\right]\,, \\
&+e^{-\bar\Gamma\Delta\tau}\left[2\frac{\beta}{|d||\bar\eta_{+-}|}|z|\sin(\Delta m\Delta\tau+\phi_{+-}-\phi_{SW}-\phi_z)\right]
\end{aligned}
\tag{C.3}
$$

$$
\begin{aligned}
\mathcal{N}_I = e^{-\Gamma_L \Delta\tau} & \left[ 2\frac{\beta}{|d||\bar{\eta}_{+-}|} \cos(\phi_{SW} - \phi_{+-}) \right] \\
-e^{-\Gamma_S \Delta\tau} & \left[ 2\frac{\beta}{|d||\bar{\eta}_{+-}|} \Big( \cos(\phi_{SW} - \phi_{+-}) - |z| \cos(\phi_{SW} + \phi_z - \phi_{+-}) \Big) \right] \quad , \\
+e^{-\bar{\Gamma}\Delta\tau} & \left[ 2 \sin \Delta m \Delta\tau - 2\frac{\beta}{|d||\bar{\eta}_{+-}|} |z| \cos(\Delta m \Delta\tau + \phi_{+-} - \phi_{SW} - \phi_z) \right]
\end{aligned}
\tag{C.4}
$$

$$
\begin{aligned}
\mathcal{D} = e^{-\Gamma_L \Delta\tau} & \left[ 1 + \frac{\gamma}{\Delta\Gamma |\bar{\eta}_{+-}|^2} + 2\frac{\beta}{|d||\bar{\eta}_{+-}|} \frac{\sin(2\phi_{SW} - \phi_{+-})}{\cos\phi_{SW}} \right] \\
+e^{-\Gamma_S \Delta\tau} & \left[ 1 - \frac{\gamma}{\Delta\Gamma |\bar{\eta}_{+-}|^2} \frac{\Gamma_L}{\Gamma_S} + 2\frac{\beta}{|d||\bar{\eta}_{+-}|} \Big( \frac{\sin(2\phi_{SW} - \phi_{+-})}{\cos\phi_{SW}} - 2|z| \sin(\phi_{SW} + \phi_z - \phi_{+-}) \Big) \right] \\
-e^{-\bar{\Gamma}\Delta\tau} & \left[ 2 \cos \Delta m \Delta\tau - 4\frac{\beta}{|d||\bar{\eta}_{+-}|} |z| \sin(\Delta m \Delta\tau + \phi_{+-} - \phi_{SW} - \phi_z) \right]
\end{aligned}
\tag{C.5}
$$

We have defined

$$
|z| e^{i\phi_z} = \frac{2\bar{\Gamma}}{\Gamma_S + \bar{\Gamma} + i\Delta m} \,. \tag{C.6}
$$


Acknowledgements:

We are grateful to Vera Lüth and to Paolo and Juliet Franzini for correspondence on experimental issues in the neutral kaon system, and to Philippe Eberhard for valuable correspondence and conversations. We also thank our colleagues in the SLAC theory group for many helpful discussions.

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# FIGURE CAPTIONS

1) Constraints on the EHNS parameters $\beta$ and $\gamma$ from the comparison of determinations of the $\epsilon$ parameter from different observables of the $K_L$–$K_S$ system: (a) the systematics of expected discrepancies; (b) the current experimental situation.