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Wavefunction-Independent Relations between the Nucleon Axial-Coupling g_A and the Nucleon Magnetic Moments

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Abstract

We calculate the proton's magnetic moment μ_p and its axial-vector coupling g_A as a function of its Dirac radius R_1 using a relativisitic three-quark model formulated on the light-cone. The relationship between μ_p and g_A is found to be independent of the assumed form of the light-cone wavefunction. At the physical radius $R_1 = 0.76$ fm, one obtains the experimental values for both μ_p and g_A, and the helicity carried by the valence u and d quarks are each reduced by a factor $\simeq 0.75$ relative to their non-relativistic values. At large proton radius, μ_p and g_A are given by the usual non-relativistic formulae. At small radius, μ_p becomes equal to the Dirac moment, as demanded by the Drell-Hearn-Gerasimov sum rule. In addition, as $R_1 \to 0$, the constituent quark helicities become completely disoriented and $g_A \to 0$.

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In recent years the light-cone quantization of quantum-field theory has emerged as a promising method for solving relativistic bound-state problems in the strong coupling regime [1]. Light-cone quantization has a number of unique features that make it appealing, most notably, the ground state of the free theory is also a ground state of the full theory, and the Fock expansion constructed on this vacuum state provides a complete relativistic many-particle basis for diagonalizing the full theory. The method seems therefore to be well-suited to solving quantum chromodynamics. For practical calculations one approximates the field theory by truncating the Fock space [2]. The assumption is that a few excitations describe the essential physics, and that adding more excitations only refines this initial approximation. This is quite different from the instant formulation of QCD where an infinite number of gluons is essential for formulating even the vacuum. In this paper we restrict ourselves to an effective three-quark Fock description of the nucleon. In this effective theory, all additional degrees of freedom (including zero modes) are parameterized in an effective potential [3]. In such a theory the constituent quarks will also acquire effective masses and form factors.

After truncation, one could in principle obtain the mass M and light-cone wavefunction $|\Psi\rangle$ of the three-quark bound-states by solving the Hamiltonian eigenvalue problem

$$H_{\rm LC}^{\rm effective}|\Psi\rangle = M^2|\Psi\rangle. \tag{1}$$

Given the eigensolutions $|\Psi\rangle$ one could then compute the form factors and other properties of the baryons. Even without explicit solutions, one knows that the helicity and flavor structure of the baryon eigenfunctions must reflect the assumed global SU(6) symmetry and Lorentz invariance of the theory. However, since we do not have an explicit representation for the effective potential in the light-cone Hamiltonian $H_{\rm LC}^{\rm effective}$ for three-quarks, we shall have to proceed by making an ansatz for the momentum space structure of the wavefunction Ψ. This may seem quite arbitrary, but as we will show below, for a given size of the proton, the predictions and interrelations between observables at $Q^2 = 0$, such as the proton magnetic moment μ_p and its axial coupling g_A, turn out to be essentially independent of the shape of the wavefunction.

The light-cone model given in Ref. [4] provides a framework for representing the general structure of the effective three-quark wavefunctions for baryons. The wavefunction Ψ is constructed as the product of a momentum wavefunction, which is spherically symmetric and invariant under permutations, and a spin-isospin wave function, which is uniquely determined by SU(6)-symmetry requirements. A Wigner [5] (Melosh [6]) rotation is applied to the spinors, so that the wavefunction of the proton is an eigenfunction of J and J_z in its rest frame [7,8]. To represent the range of uncertainty in the possible form of the momentum wavefunction we choose two simple functions of the invariant mass $\mathcal{M}$ of the quarks:

$$\psi_{\rm H.O.}(\mathcal{M}^2) = N_{\rm H.O.} \exp(-\mathcal{M}^2/2\beta^2), \tag{2}$$

$$\psi_{\rm Power}(\mathcal{M}^2) = N_{\rm Power}(1+\mathcal{M}^2/\beta^2)^{-p} \tag{3}$$

where β sets the scale of the nucleon size. Perturbative QCD predicts a nominal power-law fall off at large $k_\perp$ corresponding to $p = 3.5$ [3]. The invariant mass $\mathcal{M}$ can be written as

$$\mathcal{M}^2 = \sum_{i=1}^{3} \frac{\vec{k}^2_{\perp i} + m^2}{x_i} \tag{4}$$

where we used the longitudinal light-cone momentum fractions $x_i = p_i^+/P^+$ (P and p_i are the nucleon and quark momenta, respectively, with $P^+ = P_0 + P_z$). The internal momentum variables $\vec{k}_{\perp i}$ are given by $\vec{k}_{\perp i} = \vec{p}_{\perp i} - x_i \vec{P}_\perp$ with the constraints $\sum \vec{k}_{\perp i} = 0$ and $\sum x_i = 1$. The Melosh rotation has the matrix representation [6]

$$R_M(x_i, k_{\perp i}, m) = \frac{m + x_i\mathcal{M} - i\vec{\sigma}\cdot(\vec{n}\times\vec{k}_i)}{\sqrt{(m+x_i\mathcal{M})^2 + \vec{k}^2_{\perp i}}}, \tag{5}$$

with $\vec{n} = (0,0,1)$, and it becomes the unit matrix if the quarks are collinear

$$R_M(x_i, 0, m) = 1. \tag{6}$$

Thus the internal transverse momentum dependence of the light-cone wavefunctions also affects its helicity structure.

The Dirac and Pauli form factors $F_1(Q^2)$ and $F_2(Q^2)$ of the nucleons are given by the spin-conserving and the spin-flip vector current J_V^+ matrix elements ($Q^2 = -q^2$) [9]

$$F_1(Q^2) = \langle p+q, \uparrow | J_V^+ | p, \uparrow \rangle, \tag{7}$$

$$(Q_1 - iQ_2) F_2(Q^2) = -2M \langle p+q, \uparrow | J_V^+ | p, \downarrow \rangle. \tag{8}$$

We then can calculate the anomalous magnetic moment $a = \lim_{Q^2 \to 0} F_2(Q^2)$. [The total proton magnetic moment is $\mu_p = \frac{e}{2M}(1+a_p)$.] The same parameters as in Ref. [4] are chosen; namely $m = 0.263$ GeV (0.26 GeV) for the up- and down-quark masses, and $\beta = 0.607$ GeV (0.55 GeV) for $\psi_{\rm Power}$ ($\psi_{\rm H.O.}$) and $p = 3.5$. The quark currents are taken as elementary currents with Dirac moments $\frac{e_q}{2m_q}$. All of the baryon moments are well-fit if one takes the strange quark mass as 0.38 GeV. With the above values, the proton magnetic moment is 2.81 nuclear magnetons, the neutron magnetic moment is -1.66 nuclear magnetons[1] and the radius of the proton is 0.76 fm; i.e., $M_p R_1 = 3.63$ [4].

In Figure 1 we show the functional relationship between the anomalous moment a_p and its Dirac radius predicted by the three-quark light-cone model. The value of $R_1^2 = -6 dF_1(Q^2)/dQ^2|_{Q^2=0}$ is varied by changing β in the light-cone wavefunction while keeping the quark mass m fixed. The prediction for the power-law wavefunction $\psi_{\rm Power}$ is given by the broken line; the continuous line represents $\psi_{\rm H.O.}$. Figure 1 shows that when one plots the dimensionless observable a_p against the dimensionless observable MR_1 the prediction is essentially independent of the assumed power-law or Gaussian form of the three-quark light-cone wavefunction. Different values of $p > 2$ do also not affect the functional dependence of $a_p(M_p R_1)$ shown in Fig. 1. In this sense the predictions of the three-quark light-cone model relating the $Q^2 \to 0$ observables are essentially model-independent. The only parameter controlling the relation between the dimensionless observables in the light-cone three-quark model is m/M_p which is set to 0.28. For the physical proton radius $M_p R_1 = 3.63$ one obtains the empirical value for $a_p = 1.79$ (indicated by the dotted lines in Figure 1).

The prediction for the anomalous moment a can be written analytically as $a = \langle \gamma_V \rangle a^{\rm NR}$, where $a^{\rm NR} = 2M_p/3m$ is the non-relativistic ($R \to \infty$) value and γ_V is given as [10]

[1]The neutron value can be improved by relaxing the assumption of isospin symmetry.

$$\gamma_V(x_i, k_{\perp i}, m) = \frac{3m}{\mathcal{M}} \left[\frac{(1-x_3)\mathcal{M}(m+x_3\mathcal{M}) - \vec{k}_{\perp 3}^2/2}{(m+x_3\mathcal{M})^2 + \vec{k}_{\perp 3}^2} \right]. \tag{9}$$

The expectation value $\langle \gamma_V \rangle$ is evaluated as[2]

$$\langle \gamma_V \rangle = \frac{\int [d^3k] \gamma_V |\psi|^2}{\int [d^3k] |\psi|^2}. \tag{10}$$

We now take a closer look at the two limits $R \to \infty$ and $R \to 0$. In the non-relativistic limit we let $\beta \to 0$ and keep the quark mass m and the proton mass M_p fixed. In this limit the proton radius $R_1 \to \infty$ and $a_p \to 2M_p/3m = 2.38$ since $\langle \gamma_V \rangle \to 1$[3]. Thus the physical value of the anomalous magnetic moment at the empirical proton radius $M_p R_1 = 3.63$ is reduced by 25% from its non-relativistic value due to relativistic recoil and nonzero $k_\perp$[4].

To obtain the ultra-relativistic limit we let $\beta \to \infty$ while keeping m fixed. In this limit the proton becomes pointlike $M_p R_1 \to 0$ and the internal transverse momenta $k_\perp \to \infty$. The anomalous magnetic momentum of the proton goes linearly to zero as $a = 0.43 M_p R_1$ since $\langle \gamma_V \rangle \to 0$. Indeed, the Drell-Hearn-Gerasimov (DHG) sum rule [11] demands that the proton magnetic moment becomes equal to the Dirac moment at small radius. For a spin-$\frac{1}{2}$ system

$$a^2 = \frac{M^2}{2\pi^2 \alpha} \int_{s_{th}}^{\infty} \frac{ds}{s} \left[\sigma_P(s) - \sigma_A(s) \right], \tag{11}$$

where $\sigma_{P(A)}$ is the total photoabsorption cross section with parallel (antiparallel) photon and target spins. If we take the point-like limit, such that the threshold for inelastic excitation becomes infinite while the mass of the system is kept finite, the integral over the

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[2] $[d^3k] = d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3)$. The third component of $\vec{k}$ is defined as $k_{3i} = \frac{1}{2}(x_i \mathcal{M} - \frac{m^2 + \vec{k}_{\perp i}^2}{x_i \mathcal{M}})$. This measure differs from the usual one used in Ref. [3] by the Jacobian $\prod \frac{dk_{3i}}{dx_i}$ which can be absorbed into the wavefunction.

[3] This differs slightly from the usual non-relativistic formula $1 + a = \sum_q \frac{e_q}{e} \frac{M_p}{m_q}$ due to the non-vanishing binding energy which results in $M_p \neq 3m_q$.

[4] The non-relativistic value of the neutron magnetic moment is reduced by 31%.

photoabsorption cross section vanishes and $a = 0$ [9]. In contrast, the anomalous magnetic moment of the proton does not vanish in the non-relativistic quark model as $R \to 0$. The non-relativistic quark model does not reflect the fact that the magnetic moment of a baryon is derived from lepton scattering at non-zero momentum transfer [12]; i.e., the calculation of a magnetic moment requires knowledge of the boosted wavefunction. The Melosh transformation is also essential for deriving the DHG sum rule and low energy theorems (LET) of composite systems [12].

A similar analysis can be performed for the axial-vector coupling measured in neutron decay. The coupling g_A is given by the spin-conserving axial current J_A^+ matrix element

$$g_A(0) = \langle p, \uparrow | J_A^+ | p, \uparrow \rangle. \tag{12}$$

The value for g_A can be written as $g_A = \langle \gamma_A \rangle g_A^{\rm NR}$ with $g_A^{\rm NR}$ being the non-relativistic value of g_A and with γ_A as [10,13]

$$\gamma_A(x_i, k_{\perp i}, m) = \frac{(m + x_3 \mathcal{M})^2 - \vec{k}_{\perp 3}^2}{(m + x_3 \mathcal{M})^2 + \vec{k}_{\perp 3}^2}. \tag{13}$$

In Fig. 2 the axial-vector coupling is plotted against the proton radius $M_p R_1$. The same parameters and the same line representation as in Fig. 1 are used. The functional dependence of $g_A(M_p R_1)$ is also found to be independent of the assumed wavefunction. At the physical proton radius $M_p R_1 = 3.63$ one predicts the value $g_A = 1.25$ (indicated by the dotted lines in Figure 2) since $\langle \gamma_A \rangle = 0.75$. The measured value is $g_A = 1.2573 \pm 0.0028$ [14]. This is a 25% reduction compared to the non-relativistic SU(6) value $g_A = 5/3$, which is only valid for a proton with large radius $R_1 >> 1/M_p$. As shown by Ma and Zhang [13] the Melosh rotation generated by the internal transverse momentum spoils the usual identification of the $\gamma^+ \gamma_5$ quark current matrix element with the total rest-frame spin projection s_z, thus resulting in a reduction of g_A.

Thus given the empirical values for the proton's anomalous moment a_p and radius $M_p R_1$, its axial-vector coupling is automatically fixed at the value $g_A = 1.25$. This prediction is an essentially model-independent prediction of the three-quark structure of the proton in QCD.

The Melosh rotation of the light-cone wavefunction is crucial for reducing the value of the axial coupling from its non–relativistic value 5/3 to its empirical value. In Figure 3 we plot $g_A/g_A(R_1 \to \infty)$ versus $a_p/a_p(R_1 \to \infty)$ by varying the proton radius R_1. The near equality of these ratios reflects the relativistic spinor structure of the nucleon bound state, which is essentially independent of the detailed shape of the momentum-space dependence of the light-cone wavefunction.

We emphasize that at small proton radius the light-cone model predicts not only a vanishing anomalous moment but also

$$\lim_{R_1 \to 0} g_A(M_p R_1) = 0. \tag{14}$$

One can understand this physically: in the zero radius limit the internal transverse momenta become infinite and the quark helicities become completely disoriented. This is in contradiction with chiral models which suggest that for a zero radius composite baryon one should obtain the chiral symmetry result $g_A = 1$.

The helicity measures Δu and Δd of the nucleon each experience the same reduction as g_A due to the Melosh effect. Indeed, the quantity Δq is defined by the axial current matrix element

$$\Delta q = \langle p, \uparrow | \bar{q} \gamma^+ \gamma_5 q | p, \uparrow \rangle, \tag{15}$$

and the value for Δq can be written analytically as $\Delta q = \langle \gamma_A \rangle \Delta q^{\rm NR}$ with $\Delta q^{\rm NR}$ being the non-relativistic or naive value of Δq and with γ_A given in Eq. (13).

Figure 4 shows the prediction of the light-cone model for the quark helicity sum $\Delta\Sigma = \Delta u + \Delta d$ as a function of the proton radius R_1. The same parameters and the same line representation as in Fig. 1 are used. This figure shows that the helicity sum $\Delta\Sigma$ defined from the light-cone wavefunction depends on the proton size, and thus it cannot be identified as the vector sum of the rest-frame constituent spins. As emphasized by Ma [13], the rest-frame spin sum is not a Lorentz invariant for a composite system. Empirically, one can measures Δq from the first moment of the leading twist polarized structure function $g_1(x, Q)$. In the light-cone and parton model descriptions, $\Delta q = \int_0^1 dx[q^{\uparrow}(x) - q^{\downarrow}(x)]$, where $q^{\uparrow}(x)$ and $q^{\downarrow}(x)$ can be interpreted as the probability for finding a quark or antiquark with longitudinal momentum fraction x and polarization parallel or antiparallel to the proton helicity in the proton's infinite momentum frame [3]. [In the infinite momentum there is no distinction between the quark helicity and its spin-projection s_z.] Thus Δq refers to the difference of helicities at fixed light-cone time or at infinite momentum; it cannot be identified with $q(s_z = +\frac{1}{2}) - q(s_z = -\frac{1}{2})$, the spin carried by each quark flavor in the proton rest frame in the equal time formalism.

One sees from figure 4 that the usual SU(6) values $\Delta u^{\rm NR} = 4/3$ and $\Delta d^{\rm NR} = -1/3$ are only valid predictions for the proton at large MR_1. At the physical radius the quark helicities are reduced by the same ratio 0.75 as $g_A/g_A^{\rm NR}$ due to the Melosh rotation. Qualitative arguments for such a reduction have been given in Refs. [15,16]. Thus for $M_p R_1 = 3.63$, the three-quark model predicts $\Delta u = 1$, $\Delta d = -1/4$, and $\Delta\Sigma = \Delta u + \Delta d = 0.75$. Although the gluon contribution $\Delta G = 0$ in our model, the general sum rule [17]

$$\frac{1}{2}\Delta\Sigma + \Delta G + L_z = \frac{1}{2} \tag{16}$$

is still satisfied, since the Melosh transformation effectively contributes to L_z.

Suppose one adds polarized gluons to the three-quark light-cone model. Then the flavor-singlet quark-loop radiative corrections to the gluon propagator will give an anomalous contribution $\delta(\Delta q) = -\frac{\alpha_s}{2\pi}\Delta G$ to each light quark helicity [18]. The predicted value of $g_A = \Delta u - \Delta d$ is of course unchanged. For illustration we shall choose $\frac{\alpha_s}{2\pi}\Delta G = 0.20$. The gluon-enhanced quark model then gives the values in Table 1, which agree well with the present experimental values. Note that the gluon anomaly contribution to Δs has probably been overestimated here due to the large strange quark mass. One could also envision other sources for this shift of Δq such as intrinsic flavor [16].

In summary, we have shown that relativistic effects are important for understanding the spin structure of the nucleons. By plotting dimensionless observables against dimensionless observables we obtain model-independent relations independent of the momentum-space form of the three-quark light-cone wavefunctions. For example, the value of $g_A \simeq 1.25$ is correctly predicted from the empirical value of the proton's anomalous moment. For the physical proton radius $M_p R_1 = 3.63$ the inclusion of the Wigner (Melosh) rotation due to the finite relative transverse momenta of the three quarks results in a $\simeq 25\%$ reduction of the non-relativistic predictions for the anomalous magnetic moment, the axial vector coupling, and the quark helicity content of the proton.

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TABLES

TABLE I. Comparison of the quark content of the proton in the non-relativistic quark model (NR), in our three-quark model (3q), in a gluon-enhanced three-quark model (3q+g), and with experiment [19].

Quantity	NR	3q	3q+g	Expt.
Δu	$\frac{4}{3}$	1	0.80	0.80 ± 0.04
Δd	$-\frac{1}{3}$	$-\frac{1}{4}$	–0.45	-0.46 ± 0.04
Δs	0	0	–0.20	-0.13 ± 0.04
$\Delta\Sigma$	1	$\frac{3}{4}$	0.15	0.22 ± 0.10

FIGURES

FIG. 1. The anomalous magnetic moment $a = F_2(0)$ of the proton as a function of $M_p R_1$: broken line, pole type wavefunction; continuous line, gaussian wavefunction. The experimental value is given by the dotted lines. Our model is independent of the wavefunction for $Q^2 = 0$.

FIG. 2. The axial vector coupling g_A of the neutron to proton decay as a function of $M_p R_1$: line code as in Fig. 1. The experimental value is given by the dotted lines.

FIG. 3. $g_A/g_A(R_1 \to \infty)$ versus $a_p/a_p(R_1 \to \infty)$ by varying the proton radius R_1.: line code as in Fig. 1.

FIG. 4. The quantity $\Delta\Sigma = \Delta u + \Delta d$ of the proton as a function of $M_p R_1$: line code as in Fig. 1.

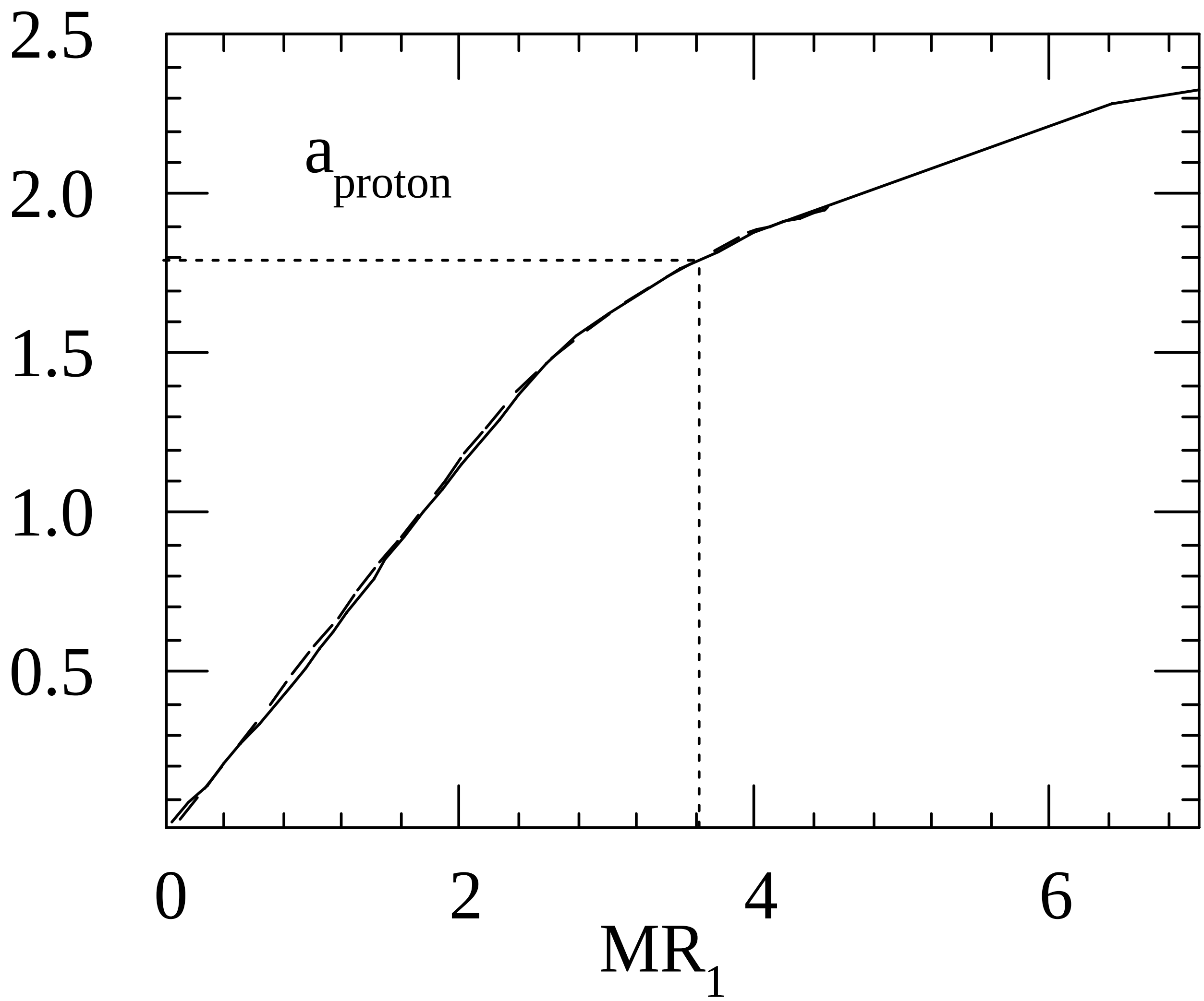

Fig. 1

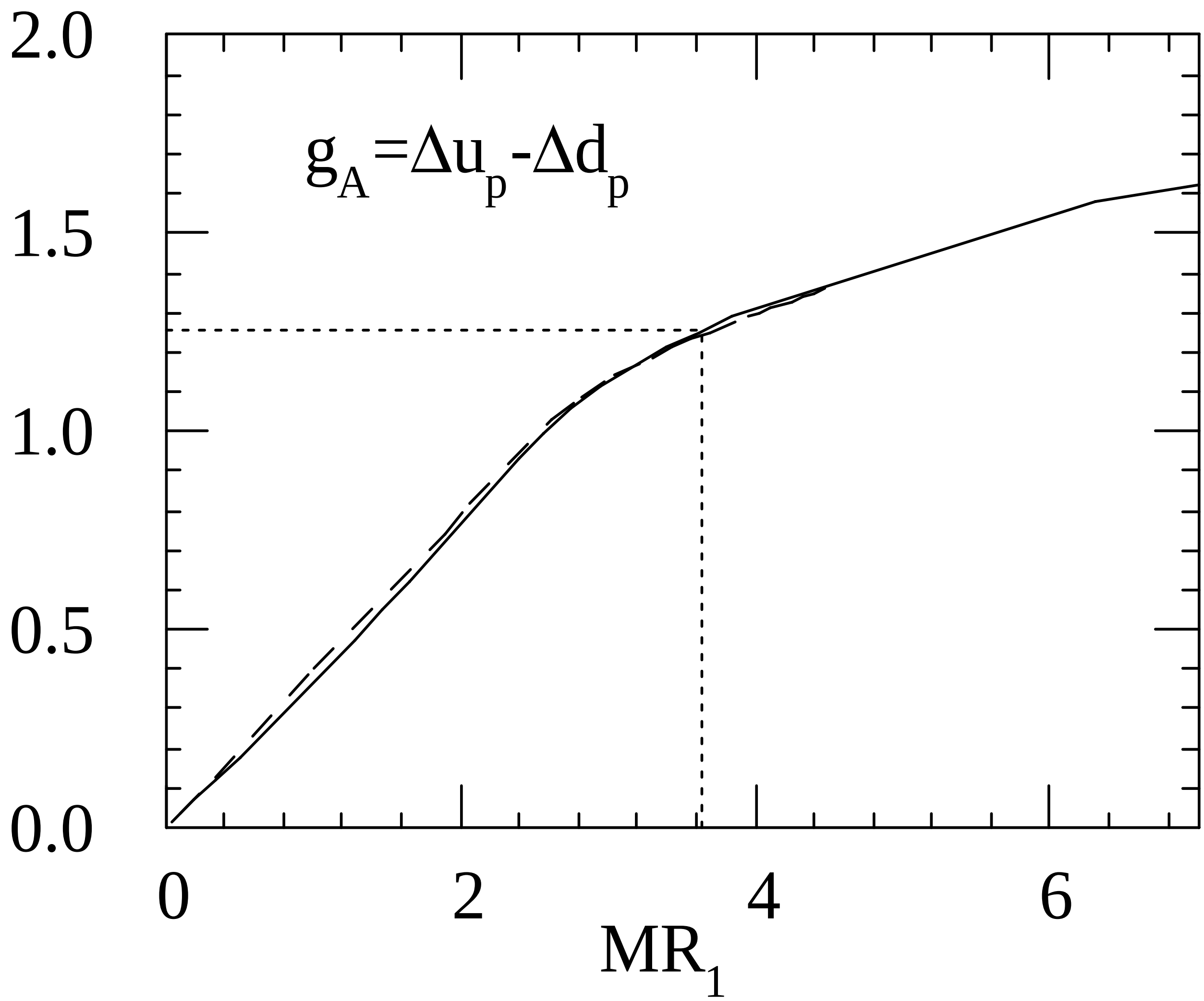

Fig. 2

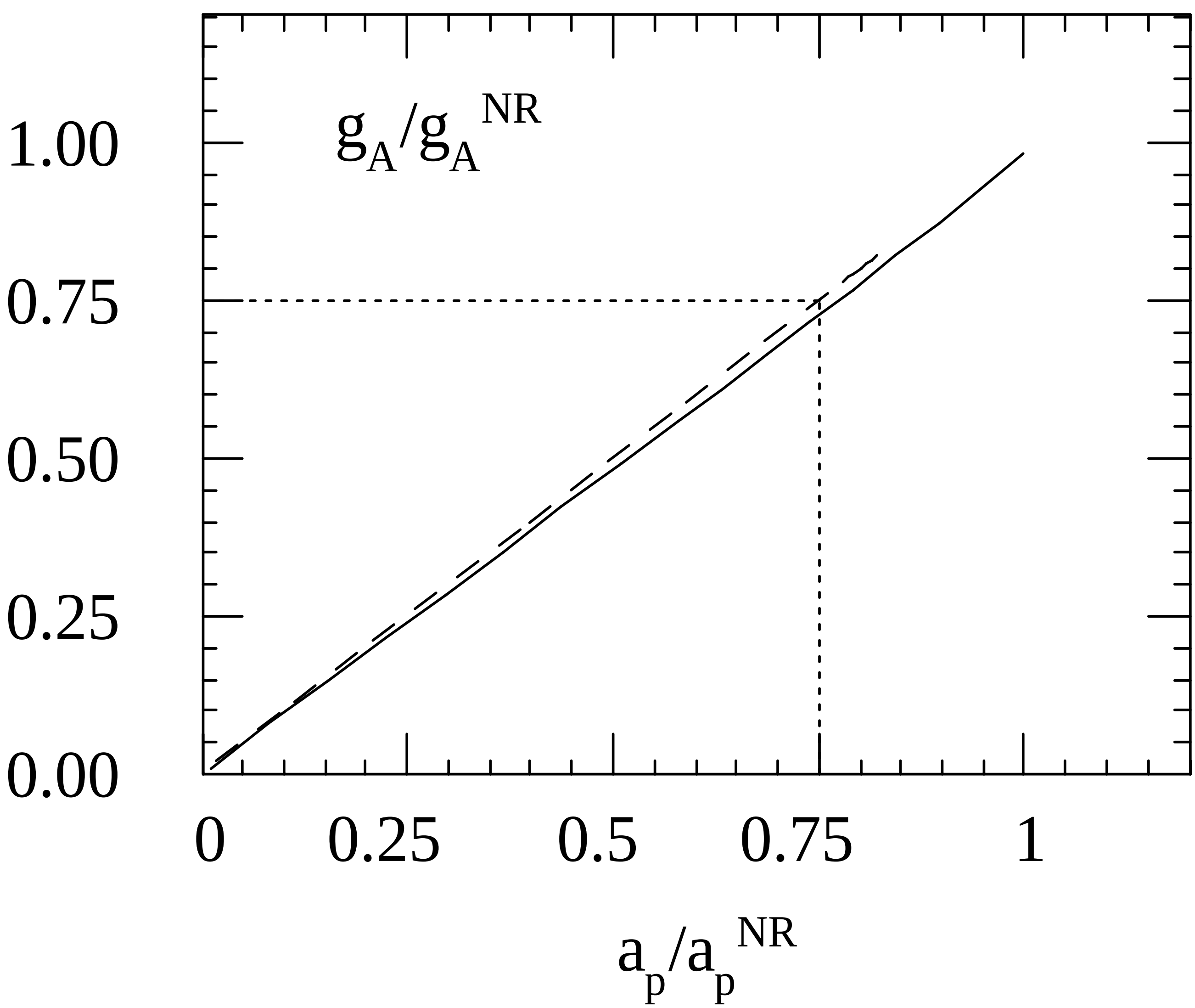

Fig. 3

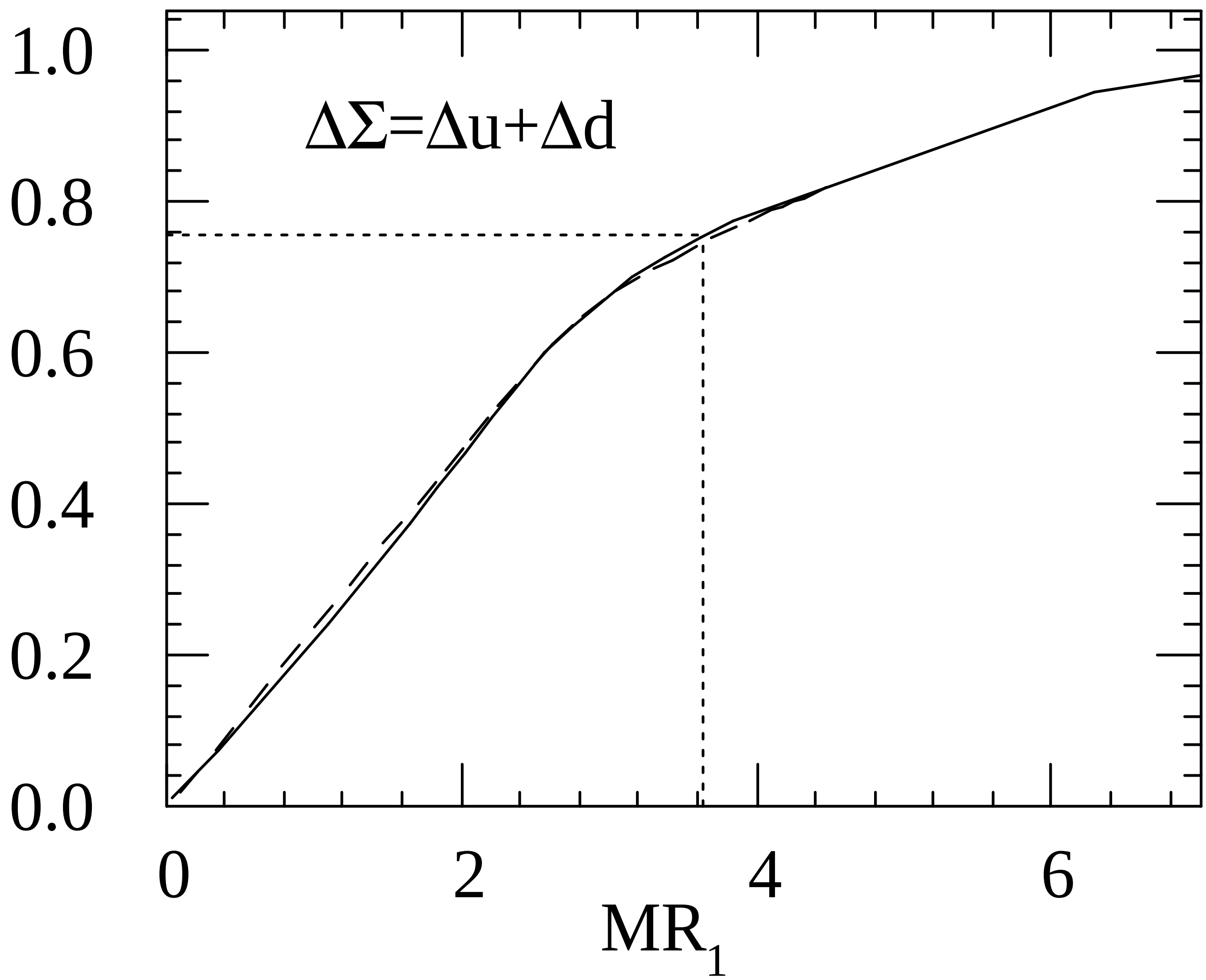

Fig. 4