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Rotational Invariance of Light Cone Fragmentation Amplitudes[⋆]

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ABSTRACT

Hadron wavefunctions are most naturally defined in the framework of light-cone quantization, a Hamiltonian formulation of QCD quantized at equal light-cone 'time' $\tau \equiv t + z$. Following earlier work of Hyer, we explore the constraints imposed on these wavefunctions by the required rotational symmetry of the full theory. We obtain nontrivial, powerful and general constraints on both wavefunctions and the corresponding fragmentation amplitudes.

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I. INTRODUCTION AND REVIEW

In the computation of amplitudes for processes involving scattering into bound states, which is an indispensable part of any quantitative approach to Quantum Chromodynamics (QCD), it is necessary to have a formalism in which the bound-state wavefunctions can be written in some compact and universal form. The most tractable such formalism is that of Light–Cone Quantization (LCQ); it thus provides the most attractive foundation for the description of hadrons in terms of their partonic constituents.

LCQ is a Hamiltonian theory, quantized at equal light-cone 'time' $\tau \equiv t + z$. The distribution of partons within hadrons is described by process-independent light-cone wavefunctions [1], which are invariant under boosts and under rotations about the preferred axis $\hat{z}$. However, as the rotational invariance of QCD is not manifest in the light-cone formulation, rotations about the other axes are dynamical in nature. It is thus a challenging task to extract the properties of wavefunctions under such rotations.

Such an analysis is further complicated by the presence of instantaneous interactions [2], which arise from the fact that the quantization surface is not strictly spacelike, so that interactions between points at the same 'time' coordinate τ are allowed. Hyer [3] has shown that the instantaneous interactions of hadrons can be represented by effective wavefunctions analogous to those of [1], and has related these effective wavefunctions at leading twist to the familiar noninstantaneous wavefunctions.

Reference [3] considers diagrams like that of Fig. 1. The central observation is that, while no perturbative relations between the effective wavefunction suggested in Fig. 1(a) and the higher Fock states of Fig. 1(b) can be extracted, the Dirac

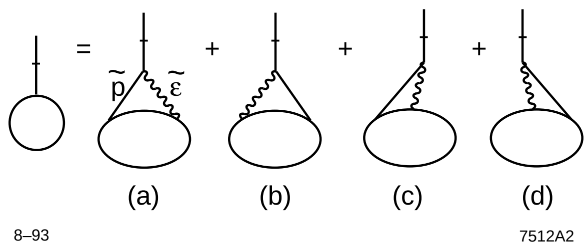

Fig. 1. Underlying processes which contribute to interactions like that shown in Figs. 2(b)–4. We must account for the possibility of the 'invisible' internal quark and gluon being either forward- or backward-moving.

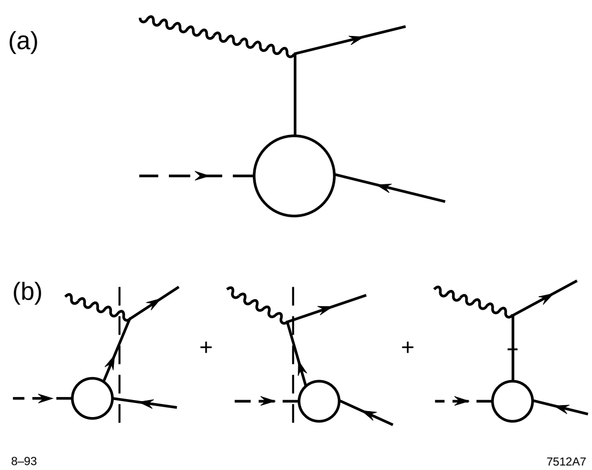

Fig. 2. Part of the K^* photodissociation amplitude: (a) shows the Feynman diagram; (b) shows the associated LCPTh diagrams.

structure of the former is independent of the dynamics of the latter. Thus the nonperturbative physics which enters into instantaneous interactions of hadrons can be absorbed into process-independent effective wavefunctions.

The amplitude shown in Fig. 2 is then computed in terms of the unknown wavefunction $\psi_{h\to q\bar{Q}}$, fragmentation amplitude $\psi_{q\to hQ}$, and instantaneous effective wavefunction $\tilde{\psi}_{h(Q)q}$. When the photon probe is prepared in the rotationally invariant Coulomb gauge, this amplitude is itself rotationally invariant [4]. It thus provides relations among the three nonperturbative quantities above, which prove sufficient to write the instantaneous effective wavefunction $\psi_{h(Q)q}$ entirely in terms of the wavefunction and fragmentation amplitude defined in Ref. [1].

In addition, Ref. [3] derives at leading twist the sum rule

$$\frac{x}{\bar{x}}\ \psi_{h\to\bar{q}Q}(x, \quad \overline{x}k_\perp) - \frac{1}{\bar{x}}\ \psi^*_{\bar{q}\to h\bar{Q}}(x, \quad \overline{x}k_\perp) \ = \ M_1(k_\perp^2)\ , \tag{1}$$

where we have introduced the notation $\bar{x} \equiv 1 - x$. Here the unknown constant

$$M_1 \equiv \frac{s}{-t}\ \mathcal{F}_{\gamma h\to\bar{q}Q}(s, t, u)$$

depends only on $k_\perp^2 = -t$ because of the Regge behavior of the amplitude $\mathcal{M}$.

II. THE FRAGMENTATION AMPLITUDE

In this paper, we extend the calculations of Ref. [3] to the s-channel graph shown in Fig. 3, which contributes to the amplitude $\mathcal{F}_{\phi Q\to hq}$ for a scalar probe ϕ. For this purpose, a new instantaneous wavefunction $\psi_{(Q)hq}$ is required, as well as the fragmentation amplitude $\psi_{Q\to hq}$.

It is clear that the Dirac structure associated with the instantaneous Q line is the same factor $\zeta_\pm$ derived in Ref. [3]:

$$\zeta_+ \ = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}\ ; \qquad \zeta_- \ = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}\ .$$

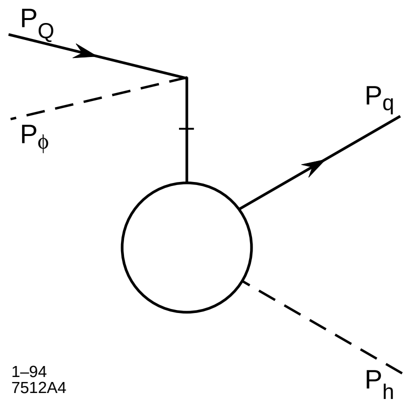

Fig. 3. A process involving the instantaneous wavefunction $\tilde{\psi}_{(Q)hq}$; the arrows indicate the direction of fermion flow.

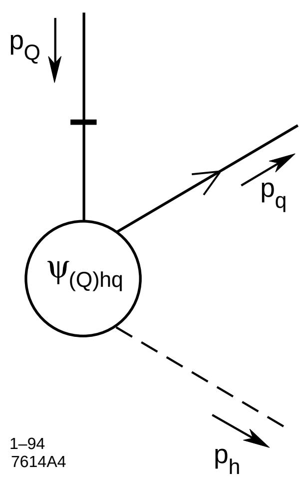

Fig. 4. Instantaneous insertions of the 's-channel' type, corresponding to the effective wavefunction $\tilde{\psi}_{(Q)hq}$.

We may thus represent an insertion like that of Fig. 4 by the factor

$$\mathrm{d}x\ \frac{\mathrm{d}^2 k_\perp}{16\pi^3}\ \tilde{\psi}_{(Q)hq}(x, k_\perp)\ \left[\frac{k_1 \mp i k_2}{p_h^+}\bar{\zeta}_\pm\right]\ \frac{u(p_q)}{\overline{\overline{p_q^+}}}\quad ,$$

where the hadron momentum $p_h = xp_Q + k_\perp$. Additional kinematic terms have been extracted from $\tilde{\psi}$ and included in the square brackets; their purpose is to give the quantity in brackets the same properties as $\bar{u}_\pm(p_Q)/\sqrt{p_Q^+}$ under boosts and under rotations about $\hat{z}$, so that the instantaneous effective wavefunction will in turn have the same transformation properties as the analogous fragmentation amplitude.

As in Ref. [3], we set up the most general kinematics in the center-of-mass frame. In terms of the light-cone momentum fractions x and y, the three-momenta are

$$\vec{p}_Q \;=\; \left(k_\perp, x - \frac{1}{2}\right) \;, \qquad \vec{p}_\phi \;=\; \left(-k_\perp, \frac{1}{2} - x\right) \;,$$

$$\vec{p}_h \;=\; \left(l_\perp, y - \frac{1}{2}\right) \;, \qquad \text{and} \qquad \vec{p}_q \;=\; \left(-l_\perp, \frac{1}{2} - y\right) \;.$$

For brevity, we introduce the notations

$$\hat{\epsilon}_{R} \;=\; \frac{1}{\sqrt{2}}\,(1, i) \;=\; \hat{\epsilon}_{L}{}^{*} \;, \qquad l_{R(L)} \equiv l_\perp \cdot \hat{\epsilon}_{R(L)} \;, \qquad \text{and} \qquad \vec{v} \;=\; (v_\perp, v_z) \;.$$

In these terms, the leading-twist contribution to the amplitude of the graph of Fig. 3 for negative external quark helicity is (in units $s = 1$)

$$\begin{aligned} \mathcal{F}(s,t,u) \;&=\; -g\,\sqrt{x\bar{y}}\,\Bigg\{ \bar{u}_+(P)\,\frac{u_-(p_Q)}{\sqrt{x}}\,\psi_{Q\to hq}(y, l_\perp) \\ &\qquad + \left(\frac{\sqrt{2}\,l_{L}}{y}\right)\bar{\zeta}_+\,\frac{u_-(p_Q)}{\sqrt{x}}\,\tilde{\psi}_{(Q)hq}(y, l_\perp) \Bigg\} \\ &=\; -\sqrt{2}\,g\,\Bigg\{ \frac{\sqrt{\bar{y}}}{x}\,k_{L}\,\psi(y, l_\perp) + \frac{\sqrt{x\bar{y}}}{y}\,l_{L}\,\tilde{\psi}(y, l_\perp) \Bigg\} \;. \end{aligned}$$

Of course, x and y are constrained by the values of the Mandelstam invariants s, t, u. We specialize to two particularly informative cases:

$$\text{CASE 1:} \qquad t = 0,\ u = -s \qquad \Rightarrow \quad x = y,\ k_\perp = l_\perp \ ,$$

$$\text{and} \qquad \mathcal{F} = M_2(s) \ = \ \bar{y}\ \psi(y, l_\perp) + \bar{y}\ \tilde{\psi}(y, l_\perp) \ .$$

$$\text{CASE 2:} \qquad u = 0,\ t = -s \qquad \Rightarrow \quad x = \bar{y},\ k_\perp = -l_\perp \ ,$$

$$\text{and} \qquad \mathcal{F} = M_3(s) \ = \ - \sqrt{y\bar{y}}\ \psi(y, l_\perp) + \bar{y} \sqrt{\frac{\bar{y}}{y}}\ \tilde{\psi}(y, l_\perp) \ .$$

From these two equations, it is a simple matter to extract $\psi_{Q\rightarrow hq}(y, k_\perp)$ in terms of the two unknown amplitudes $M_{2,3}$; we obtain the general form

$$\begin{aligned} \psi_{Q\rightarrow hq}(y, l_\perp) &= M_2\left(\frac{l_\perp^2}{y\bar{y}}\right) - \sqrt{\frac{y}{\bar{y}}}\ M_3\left(\frac{l_\perp^2}{y\bar{y}}\right) \ , \\ \tilde{\psi}_{(Q)hq}(y, l_\perp) &= \frac{y}{\bar{y}}\ M_2\left(\frac{l_\perp^2}{y\bar{y}}\right) + \sqrt{\frac{y}{\bar{y}}}\ M_3\left(\frac{l_\perp^2}{y\bar{y}}\right) \ . \end{aligned} \qquad (2)$$

Substituting these forms back into our expression for the amplitude, we obtain (modulo a phase arising from our spinor conventions, which vanishes when the $\hat{z}$ axis lies in the scattering plane) $\mathcal{F} = M_2 \sqrt{-u/s} + M_3 \sqrt{-t/s} = M_2 \cos(\theta_{\rm cm}/2) + M_3 \sin(\theta_{\rm cm}/2)$, which is manifestly independent of x and y.

Thus in the region of large momentum transfer, where our neglect of higher-twist contributions to the amplitude is an accurate approximation, the fragmentation amplitude and its instantaneous counterpart have very simple few-parameter representations. Indeed, we will shortly demonstrate that $M_2 = 0$ for pointlike 'hadrons'.

We may also replace the scalar probe with a photon probe quantized in the Coulomb gauge, as in [3]. The resulting constraints on ψ and $\tilde{\psi}$ are identical.

It must be emphasized here that the hadron h is considered to be in an eigenstate of helicity in the center-of-mass frame in question; the mixing of helicity states to form boost-invariant states can invalidate the above relations for vector mesons polarized along the z-axis. Thus the conclusions of Eq. (2) hold only for fragmentation into scalars, where no such subtleties arise. We will return to the vector case in Sec. IV.

We have also ignored the possibility of higher-order corrections to the amplitude. Since the wavefunction mixes terms of all orders in the coupling constant, there is no good reason to suppose that diagrams involving wavefunction terms should be order-by-order invariant; the constraints we derive are subject to corrections of order $\alpha_s(k_\perp^2)$ [5]. This is also true of the results of Ref. [3].

III. THE WAVEFUNCTION

We now turn to the interesting question of the hadronic wavefunction. Equation (1) can be written in the form

$$\frac{y}{\bar{y}}\ \psi_{h\to Q\bar{q}}(y,\quad \overline{y}\ k_\perp) = \frac{1}{\bar{y}}\ \psi_{Q\to hq}(y,\quad \overline{y}\ k_\perp) + M_1(k_\perp^2)\ .$$

We make the further assumption that, in the region of large momentum transfer, the wavefunction scales as $|k_\perp|^{-2n}$. We thus obtain the constraint

$$y\bar{y}^{n-1}\ \psi_{h\to Q\bar{q}}(y,l_\perp)\ =\ \bar{y}^{n-1}\ \psi_{Q\to hq}(y,l_\perp) + M_1\left(\frac{l_\perp^2}{y\bar{y}}\right)\ . \qquad (3)$$

Introducing the notation $\delta\mathcal{M}^2 = l_\perp^2/y\bar{y}$ to denote the light-cone virtuality, we substitute Eq. (2) into (3) to obtain

$$\psi_{h\to Q\bar{q}}(y,l_\perp)\ =\ \frac{\bar{y}^{1-n}\ M_1(\delta\mathcal{M}^2) + M_2(\delta\mathcal{M}^2)}{y} + \frac{M_3(\delta\mathcal{M}^2)}{\overline{y\bar{y}}}\ .$$

For pointlike vertices, $n = 1/2$; thus ψ can only remain finite as $y \to 0$ if we also have $M_2 = -M_1$. It follows that rotationally invariant pointlike wavefunctions must have the form

$$\psi_{h\to Q\bar{q}}(y, l_\perp) \;=\; \frac{C_3 + C_1(\bar{y} - \sqrt{\bar{y}})/\sqrt{\bar{y}}}{|k_\perp|} \,, \tag{4}$$

where $C_i \equiv (\delta\mathcal{M}^2)^n\, M_i(\delta\mathcal{M}^2)$. However, we can repeat the above derivation with $Q \leftrightarrow \bar{q}$, so C_1 (whose corresponding term is not symmetric under $y \leftrightarrow \bar{y}$) must vanish as well. Thus in Eq. (2), $M_2 = 0$ for pointlike scalars.

However, we are most interested in bound states, for which $n = 1$ [6]. For these states, we again expect $M_2 = -M_1$, so that the wavefunction $\psi(y, k_\perp)$ will vanish as $y \to 0, 1$ for fixed $k_\perp$. We thus derive the general form

$$\psi_{h\to Q\bar{q}}(y, k_\perp) \;=\; C_3\, \frac{\sqrt{y\bar{y}}}{k_\perp^2} \tag{5}$$

for a scalar bound state.

This result is somewhat surprising, especially to an intuition shaped by the nonrelativistic expectation that ψ depends only on $\delta\mathcal{M}^2 = k_\perp^2/y\bar{y}$. The difference from the relativistic case lies in the spinor normalizations $u(p)/\sqrt{p^+}$ which are used in Ref. [1]. In the nonrelativistic case, p^+ is essentially determined by the mass, and can be treated as a constant. Here, however, we work in the opposite limit, where the masses are considered negligible.

IV. THE VECTOR CASE

The results of Eqs. (4) and (5) cannot be used to derive boost-invariant wavefunctions for vector mesons, since the amplitudes we consider have been prepared with meson polarizations that are not themselves aligned with the $\hat{z}$-axis.

However, we note that in every amplitude we have computed (as shown in Figs. 2 and 3), the angle between the hadron and the boost axis is determined by either y or $\bar{y}$ [7]. Since the helicity eigenstates are formed from a superposition of boost-invariant eigenstates with coefficients y, $\bar{y}$, and $\overline{y\bar{y}}$, we have the general form for pointlike vector particles:

$$\psi_{h\to Q\bar{q}}(y, k_\perp) \; = \; \frac{C_+ y + C_0 \quad \overline{y\bar{y}} + C_- \bar{y}}{|k_\perp|} \; , \tag{6}$$

and for vector mesons:

$$\psi_{h\to Q\bar{q}}(y, k_\perp) \; = \; (C_+ y + C_0 \quad \overline{y\bar{y}} + C_- \bar{y}) \; \frac{\overline{y\bar{y}}}{k_\perp^2} \; . \tag{7}$$

The coefficient C_0, which represents mixing with the helicity-zero state, vanishes for massless particles.

For transversely polarized mesons, one of the coefficients $C_\pm$ is expected to vanish when the Q and $\bar{q}$ helicities are opposite, since a quark which inherits nearly the entire momentum of a hadron should also share its helicity. Similarly, for longitudinally polarized mesons, symmetry under reflections in the xy-plane implies $C_+ = C_-$. Thus, we have obtained a two-parameter form for the wavefunctions of vector mesons.

V. CONCLUSIONS

The lack of manifest rotational invariance in the light-cone formulation of physical theories is a potentially serious drawback. However, it can be circumvented in part by extracting the hidden consequences of rotational invariance, which is what we have attempted here. We find that the wavefunction in the region of large momentum transfer must have the general form

$$
\begin{aligned}
\psi_{h\to Q\bar q}(x,k_\perp) &= C\,\frac{\sqrt{x\bar x}}{k_\perp^2} \quad \text{for scalar mesons}\,,\\
\psi_{h_\uparrow\to Q_+\bar q_-}(x,k_\perp) &= (C_+\sqrt{x} + C_0\sqrt{\bar x})\,\frac{x\sqrt{\bar x}}{k_\perp^2} \quad \text{for some transverse vector mesons}\,,\\
\psi_{h\to Q\bar q}(x,k_\perp) &= (C_\pm + C_0\sqrt{x\bar x})\,\frac{\sqrt{x\bar x}}{k_\perp^2} \quad \text{for other vector mesons}\,.
\end{aligned}
\tag{8}
$$

The second form holds only for the 'asymmetric' helicity combination in which one quark shares the meson polarization, while the other does not; in all other cases, the symmetric form should be used.

The forms given in Eq. (8) should serve as a guide to the formulation of realistic model wavefunctions and as a check on wavefunctions extracted numerically through some discretization procedure. We must reiterate, however, that the relations we have derived are valid only at leading order and leading twist, so that the numerical precision with which they can be applied is limited.

Also, the wavefunctions are themselves gauge-dependent. The results above, as well as the factorization of instantaneous contributions derived in [3], depend on the use of light-cone gauge $A^+ = 0$.

The derivation we have given depends on the assumption that $\psi \propto k_\perp^{-2}$ in the large-$k_\perp$ region; given this assumption, the corresponding x-dependence is almost entirely determined.

The coefficients $C_\pm$ and C_0 of Eq. (7) may be further constrained in a rotationally invariant theory. The extraction of such a relation, however, will require more subtlety than has been necessary to obtain the above results.

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2. S. J. Brodsky, R. J. Roskies, and R. Suaya, *Phys. Rev.* **D8**, 4574 (1974).
3. T. Hyer, SLAC–PUB–6279 (to appear in *Phys. Rev.* **D**).
4. Note that QCD is still quantized in the light-cone gauge $A^+ = 0$; this gauge condition is necessary to obtain the instantaneous effective wavefunctions of Ref. [3]. In addition, the wavefunctions of Ref. [1] are themselves gauge-dependent; however, it is only in light-cone gauge that they have the desired intuitive correspondence with the parton model.
5. This fact was pointed out to the author by G. P. Lepage.
6. See Appendix A of Ref. [1]. The resulting logarithmic divergence of the integral $\mathrm{d}^2 k_\perp \psi(x, k_\perp)$, which enters into the distribution amplitude, is cancelled by the quark anomalous dimension, so that the distribution amplitude remains finite.
7. Due to the Regge behavior of the amplitude of Fig. 2, we can always obtain $x = 1$ by setting $s = -t/\bar{y}$; see [3]. Thus the momentum fraction z probed in the scattering is the same as the momentum fraction y which determines the angle between p_h and $\hat{z}$.