hep–th/9312167
SLAC–PUB–6411
December 1993
(T)

Cosmological String Solutions by Dimensional Reduction†

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Invited talk presented at the
27th Symposium on the Theory of Elementary Particles
Wendisch–Rietz, September 7–11, 1993

Abstract

We obtain cosmological four-dimensional solutions of the low-energy effective string theory by reducing a five-dimensional black hole, and black hole–de Sitter solution of the Einstein gravity down to four dimensions. The appearance of a cosmological constant in the five dimensional Einstein–Hilbert action produces a special dilaton potential in the four-dimensional effective string action. Cosmological scenarios implemented by our solutions are discussed.

†Work supported by the Department of Energy, contract DE–AC03–76SF00515.
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We describe a procedure to obtain new cosmological solutions by dimensional reduction of a known theory. We ignore all terms $\mathcal{O}(\alpha'^2)$ and consider only curvature and dilaton terms. Then we can rewrite the effective string action in four dimensions (4-D) as a five-dimensional (5-D) action without a dilaton. This 5-D theory is our starting point in the construction of our solutions. We take solutions of this 5-D theory, reduce this theory to 4-D and interpret the 4-D fields as possible string background fields. Firstly, we want to discuss only the 5-D Einstein-Hilbert (EH) action. In a second step we add a cosmological constant in the 5-D theory. Because we start with black hole solutions in 5-D, this procedure has the advantage that spatially non–flat geometries ($k \neq 0$, e.g., S_3) are automatically included. The standard approach, i.e., to solve the string equations of motion (vanishing of the β functions), is discussed, for instance, in [1, 2, 3]. But there the generalization to $k \neq 0$ is rather difficult and a complete analytical solution for arbitrary k was not yet found. In the present paper the 5-D theory is used only as a technical tool providing us with a new nontrivial cosmological solution of pure dilaton gravity. This approach differs from the Kaluza–Klein philosophy[4], where a moduli field appears as a remnant of a higher dimensional theory that has its own physical meaning.

In the first part of this talk we present the procedure and the solution [5]; in a second part we interpret and discuss this solution.

So, restricting ourselves on curvature and dilaton terms, the 4-D effective action in the lowest order in the α' expansion is given by

$$S = \frac{1}{2}\int d^4x \sqrt{|G|}\, e^{-2\phi}\left(R + 4(\partial\phi)^2\right) \ , \tag{1}$$

where we assume that the central charge term vanishes. Let us now transform this 4-D action into a 5-D Einstein-Hilbert (EH) action. First we define a 5-D metric (not depending on the fifth coordinate),

$$G_{\mu\nu} \to \tilde{G}_{ab} = \left(\begin{array}{c|c} e^{\alpha\phi} & \\ \hline & e^{\beta\phi}\, G_{\mu\nu} \end{array}\right) \quad , \quad \text{with:} \qquad \alpha = \pm\frac{4}{\sqrt{3}} \quad , \quad \beta = -2\left(1 \pm \sqrt{\frac{1}{3}}\right) , \tag{2}$$

and Latin (Greek) indices are running from one to five (four). Using the 5-D metric (2) and adding one dummy integration we get for the action (1) the 5-D EH action

$$S \to \tilde{S} = \int d^5x \sqrt{|\tilde{G}|}\, \tilde{R} \ . \tag{3}$$

We want to use this procedure for the construction of new cosmological solutions. Therefore the 4-D metric should be spatially isotropic and homogeneous (Friedmann-Robertson-Walker),

$$ds^2 = -d\tau^2 + a^2(\tau)\left[dr^2 + \left(\frac{\sin\sqrt{k}r}{\sqrt{k}}\right)^2\left(\sin^2\theta d\phi^2 + d\theta^2\right)\right] = -d\tau^2 + a^2(\tau)d\Omega^2_{3,k} \ . \tag{4}$$

The whole dynamics of this metric are contained in the world radius $a(\tau)$, which has to be determined by the field equations.

First we demonstrate how to get a cosmological solution for $k = 1$. In this case the spatial part of the space time is a S_3 manifold with $a(\tau)$ as the time-dependent radius. The most general 5-D metric respecting the S_3 symmetry is given by a Schwarzschild metric that can be written as

$$\tilde{ds}^2 = e^{\nu(t)}dx^2 - e^{\lambda(t)}dt^2 + t^2 d\Omega^2_{3,k=1} \tag{5}$$

where x is our fifth coordinate that the theory should not depend on, and t corresponds to the time in the 4-D theory. In comparison to the usual Schwarzschild metric our time corresponds to the radius and x to the time, however, with opposite signs in front of dx^2 and dt^2. Since we have no matter in the 5-D theory a nontrivial vacuum solution satisfying the desired S_3 symmetry is given by a 5-D black hole. The generalization of this solution for arbitrary k is given by

$$e^{-\lambda} = C\, e^{\nu} = -k + \frac{2m}{t^2}\ . \tag{6}$$

After performing the reduction to the 4-D theory (2) we obtain for the dilaton and the 4-D metric

$$\begin{aligned}
\phi &= \pm\tfrac{\sqrt{3}}{4}\nu \\
ds^2 &= e^{-\beta\phi}\left(-e^{\lambda}dt^2 + t^2 d\Omega^2_{3,k}\right) \\
&= -\left(e^{-\lambda}\right)^{\frac{-1\pm\sqrt{3}}{2}} dt^2 + \left(e^{-\lambda}\right)^{\frac{1\pm\sqrt{3}}{2}} t^2\, d\Omega^2_{3,k}\ .
\end{aligned} \tag{7}$$

Unfortunately, we can perform the integration $(e^{-\lambda})^{\frac{-1\pm\sqrt{3}}{2}}\, dt^2 = d\tau^2$ only numerically and cannot find an analytic expression for (4). At the end of this talk we will discuss some special cases and present some numerical results. In addition, in order to have a real metric in (7) we have the restriction that (6) has to remain positive, i.e., $\frac{2m}{t^2} > k$ defines the physical t region. Therefore, in general (e.g., for $m > 0$) the universe starts at $t = 0$ and ends at the zero of (6), i.e., at the horizon of the 5-D theory. If (6) has no zero (for $k = -1, 0$) we do not need to restrict the t region. With horizons we mean always the 5-D black hole horizons. They do not correspond to horizons in the 4-D theory, instead, they are singular points, namely the end or the beginning of the universe. The singularity at these points is caused by the Weyl transformation in (7).

Before we discuss the solution (7) in detail let us describe a possible generalization of the 5-D theory. The simplest extension is given by adding a cosmological constant, i.e.

$$\tilde{S} \to \tilde{S} = \int d^5x\sqrt{|\tilde{G}|}\left(\tilde{R} - \Lambda\right)\ . \tag{8}$$

Again we look for a static black hole solution and find for arbitrary k

$$e^{-\lambda} = C\, e^{\nu} = -k + \frac{2m}{t^2} + \frac{\Lambda}{12}t^2\ . \tag{9}$$

For $k = 1$ this solution corresponds to the known 5-D Schwarzschild–de Sitter metric (after interpreting x as time and t as radius). The constant C can be eliminated by a constant rescaling of x, or equivalently by a constant shift in the dilaton (cf. (7)), i.e., the constant part of the dilaton ($\phi \sim \phi_0 + \phi(t)$) is fixed by the x scale. Another useful parameterization is given by

$$e^{-\lambda} = \frac{\Lambda}{12} \frac{(t^2 - t_+)(t^2 - t_-)}{t^2} \quad , \qquad \text{with} \qquad t_\pm = \frac{6k}{\Lambda} \left(1 \pm \sqrt{1 - \frac{2}{3} \frac{m\Lambda}{k^2}} \right) \ . \tag{10}$$

where $t_\pm$ are the two horizons (black hole and de Sitter) of the Schwarzschild–de Sitter metric. Both horizons coincide at the critical limit $3k^2 = 2m\Lambda_{cr}$. In order to get a real 4-D metric we have again (as for $\Lambda = 0$) the restriction that $e^{-\lambda} > 0$. If we reduce the 5-D action (8) in terms of (2) to the 4-D theory we see that the cosmological constant produces a special dilaton potential in 4-D

$$\tilde{S} = \frac{1}{2} \int d^5x \sqrt{|\tilde{G}|} \left(\tilde{R} - \Lambda \right) \quad \to \quad S = \frac{1}{2} \int d^4x \sqrt{|G|}\, e^{-2\phi} \left(R + 4(\partial\phi)^2 - \Lambda e^{\beta\phi} \right) \ , \tag{11}$$

where β is defined in (2).

To get standard cosmology let us now consider the solution (9) in the Einstein frame. This frame is defined by the Weyl transformation: $G^{(E)}_{\mu\nu} = e^{-2\phi} G_{\mu\nu}$. For the effective action (11) we find

$$S = \int d^4x \sqrt{|G^{(E)}|} \left[R^{(E)} - 2(\partial\phi)^2 - \Lambda\, e^{\mp \frac{2}{\sqrt{3}}\phi} \right] \ . \tag{12}$$

The action (12) contains the well-known Einstein–Hilbert term describing the gravitational part of the theory. Therefore, the Einstein frame is more popular from the point of view of the Einstein gravity. But there are also some arguments in favor of the string frame [6], e.g., the free motion of a string follows geodesics in the string frame and not in the Einstein frame. In the following we consider both frames on an equal footing. The matter part of (12) is given by the dilaton terms (however, the kinetic term has the wrong sign) and the corresponding energy-momentum tensor is

$$T^{matter}_{\mu\nu} = 2 \left(\partial_\mu\phi \partial_\nu\phi - \frac{1}{2} (\partial\phi)^2 G^{(E)}_{\mu\nu} \right) - \frac{1}{2} \Lambda\, e^{\mp \frac{2}{\sqrt{3}}\phi}\, G^{(E)}_{\mu\nu} \ . \tag{13}$$

In this frame the 4-D metric is given by

$$ds^2_E = -e^{\lambda/2} dt^2 + t^2 e^{-\lambda/2} d\Omega^2_{3,k} \tag{14}$$

and we see that the $\pm$ ambiguity of the string metric (7) dropped out. The origin of this ambiguity is discussed below.

In order to obtain statements about the evolution of the 4-D universe we have to bring the solution (7) and (14) into the standard form (4), where the world radius $a\,(t(\tau))$ and the function $t(\tau)$ are defined by

$$\begin{array}{llll} a^2 = t^2 \left(e^{-\lambda}\right)^{\frac{1 \pm \sqrt{3}}{2}} & , & \dot{t}^2(\tau) = \left(e^{-\lambda}\right)^{\frac{1 \mp \sqrt{3}}{2}} & \text{(in the string frame)} \ , \\ a^2_{(E)} = t^2\, e^{-\lambda/2} & , & \dot{t}^2(\tau) = e^{-\lambda/2} & \text{(in the Einstein frame)} \ , \\ e^{-\lambda} = -k + \frac{2m}{t^2} + \frac{\Lambda}{12} t^2 & . & & \end{array} \tag{15}$$

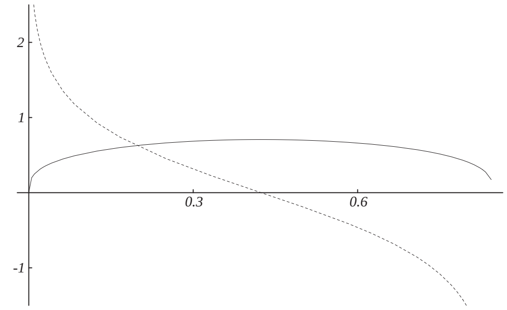

Fig. 1. $a(\tau)$ and $\phi(\tau)$ (dashed) in the Einstein frame for $k = 1$, $m = \frac{1}{2}$, $\Lambda = 0$.

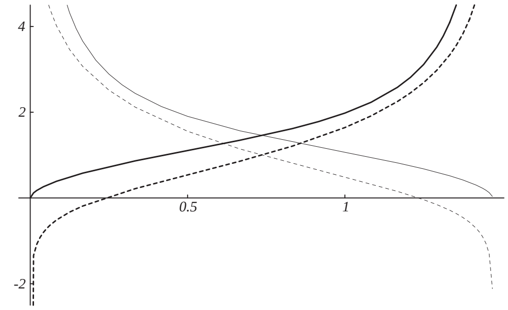

Fig. 2. $a(\tau)$ and $\phi(\tau)$ (dashed) in the string frame for $k = 1$, $m = \frac{1}{2}$, $\Lambda = 0$.

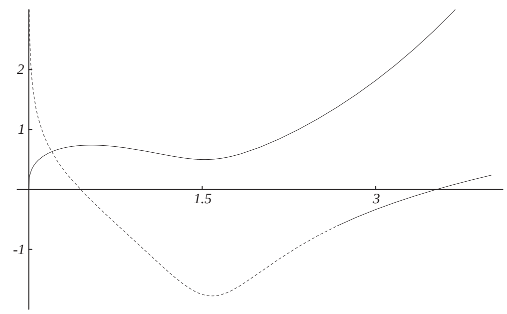

Fig. 3. $a(\tau)$ and $\phi(\tau)$ (dashed) in the Einstein frame for $k = 1$, $m = \frac{1}{2}$, $\Lambda = 3.1$.

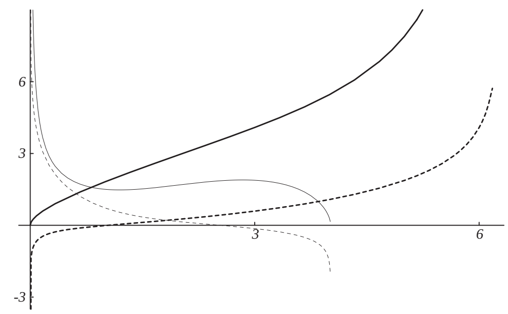

Fig. 4. $a(\tau)$ and $\phi(\tau)$ (dashed) in the string frame for $k = -1$, $m = \frac{1}{2}$, $\Lambda = -1$.

Unfortunately we were not able to solve these equations analytically. Instead, we plotted some numerical results. Figure 1 is an example for a closed universe ($k = 1$) with finite lifetime and vanishing dilaton potential. The expansion starts at $\tau = 0$, reaches the maximum at $t^2(\tau) = m$, and shrinks to zero at the horizon $t^2(\tau) = 2m$. The corresponding lifetime of the universe is according to (15), given by

$$\tau_0 = \int_0^{\sqrt{2m}} \left(\frac{t^2}{2m - t^2}\right)^{\frac{q}{2}} dt = \sqrt{\frac{2m}{\pi}}\, \Gamma(\frac{q+1}{2})\, \Gamma(1 - \frac{q}{2}) \tag{16}$$

where: $q = \frac{1}{2}(1 \mp \sqrt{3})$ in the string frame and $q = \frac{1}{2}$ in the Einstein frame. The left-right symmetry of this solution is simple a consequence of the symmetry: $t^2 \leftrightarrow 2m - t^2$ for (14). Figure 2 shows the same universe in the string frame. In this frame we have to take into account the $\pm$ ambiguity. The thinner line corresponds to the upper sign in (7), whereas the bold line is the solution for the lower sign. In these cases we have either a hyper inflation or a hyper deflation. This means the universe expands in a finite proper time τ from zero to infinity or vice verse (see also ([3])). In Figure 3 we can see the influence of a non-vanishing dilaton potential. At the beginning it looks similar to Figure 1, but in the contracting phase the potential becomes relevant and an accelerated expansion starts. In the last figure we have plotted an example for an open universe in the string frame. For the lower sign the world radius goes from zero to infinity and for the other sign it is vice verse. Again, the lifetime is finite.

Now we want to investigate some special questions in detail.

1. Asymptotic behavior. From (15) we find that it is always possible to fix the integration constant by $t(0) = 0$. Then, for $\tau \to 0$ (or $t \to 0$) we get

$$\begin{array}{lcl} a(\tau) \sim \tau^{\mp\frac{1}{\sqrt{3}}} & , & a_E(\tau) \sim \tau^{\frac{1}{3}} \\ e^{2\phi} \sim \tau^{-(1\pm\sqrt{3})} & , & e^{2\phi_E} \sim \tau^{\mp\frac{2}{\sqrt{3}}} \ . \end{array} \tag{17}$$

Although the dilaton is not transformed if we go to the Einstein frame, we have to perform different time redefinitions in both frames. For $k = \Lambda = 0$ (17) is an exact expression for all τ and coincides with the solution of Mueller [7]. The asymptotic behavior at the end (or late times) of the universe depends on whether the lifetime is finite or infinite. In the second case we find ($\tau \propto \infty$)

$$\begin{array}{lcl} a(\tau) \sim \begin{cases} \tau^{\sqrt{3}} & \Lambda > 0 \\ \tau & \Lambda = 0 \end{cases} & , & a_E \sim \begin{cases} \tau^{3} & \Lambda > 0 \\ \tau & \Lambda = 0 \end{cases} \\ e^{2\phi} \sim \begin{cases} \tau^{3-\sqrt{3}} & \Lambda > 0 \\ 1 & \Lambda = 0 \end{cases} & , & e^{2\phi_E} \sim \begin{cases} \tau^{2\sqrt{3}} & \Lambda > 0 \\ 1 & \Lambda = 0 \end{cases} \ . \end{array} \tag{18}$$

For vanishing potential ($\Lambda = 0$) this limit corresponds to $k = -1$ ($e^{-\lambda} > 0$), and we have the remarkable consequence that in the asymptotic limit our solution in both frames is a flat space time ($K = \tau$) with constant dilaton. This is a result of the asymptotic flatness of the 5-D black hole. In addition, we have no $\pm$ ambiguity in the string frame, because for the lower sign the lifetime of the universe is always finite for $\Lambda \neq 0$, and for $\Lambda = 0$ one gets for both signs the flat limit. Finally, at the end of a finite lifetime, i.e., near a horizon or for the lower sign in the string frame for $t \to \infty$, the asymptotic behavior is given by

$$\begin{array}{lcl} a(\tau) \sim (\tau_0 - \tau)^{\pm\frac{1}{\sqrt{3}}} & , & a_E(\tau) \sim (\tau_0 - \tau)^{\frac{1}{3}} \\ e^{2\phi} \sim (\tau_0 - \tau)^{-(1\mp\sqrt{3})} & , & e^{2\phi_E} \sim (\tau_0 - \tau)^{\pm\frac{2}{\sqrt{3}}} \ , \end{array} \tag{19}$$

where $\tau \to \tau_0$ (τ_0 is the lifetime of the universe). If we compare the behavior at the beginning and the end of the universe we get the statement for finite lifetime, that the world radius $a(\tau)$ in the string frame always has a singularity, either at $\tau = 0$ or $\tau = \tau_0$, i.e., there is a hyper(de)inflation. On the other hand a_E remains finite and vanishes at $\tau = 0$ and $\tau = \tau_0$ (see also [2, 3]). One example for this is plotted in Fig. 2.

2. Dilaton behavior. Apart from the flat limit ($\Lambda = 0$, $\tau \propto \infty$) the dilaton is always infinite at the beginning and the end of the universe. The reason is that these points are given by the zeroes of e^{λ} or $e^{-\lambda}$ and the dilaton is: $\phi = \mp\frac{\sqrt{3}}{4}\lambda$ (see (7)). Thus, at the beginning or the end we have either a strong- or weak-string coupling limit ($g_s \sim e^{2\phi}$), and the $\pm$ ambiguity switches between both. Because ds_E (14) does not depend on this ambiguity the Einstein metric is invariant under a change of the strong to the weak coupling limit and vice versa. Let us now investigate whether the dilaton has maxima or minima. At these points the first derivative vanishes, the energy momentum tensor (13) is given by a cosmological constant, and we can expect a phase of exponential expansion. These points are defined by: $\dot{\phi}(t(\tau)) = \phi'(t)\,\dot{t}(\tau) = 0$. From (15) we know that $\dot{t}$ is a non-vanishing function (because: $e^{-\lambda} > 0$) and $\phi' = 0$ yields

$$t^2_{extr} = \sqrt{\frac{24m}{\Lambda}} \tag{20}$$

or

$$e^{-\lambda} = -k + \sqrt{\frac{2\Lambda m}{3}} \quad , \quad \phi(t_{extr}) = \pm\frac{\sqrt{3}}{4}\log\left(-k + \sqrt{\frac{2\Lambda m}{3}}\right) \ . \tag{21}$$

Since $e^{-\lambda}$ has to be positive and finite in the physical region, we have the result that the dilaton has maxima or minima only if: $\Lambda > \Lambda_{cr} = \frac{3}{2m}$ for $k = 1$ or $\Lambda > 0$ for $k = -1, 0$. All these cases are fulfilled only if the 5-D theory contains no horizons, and therefore the t region is infinite. Furthermore, since the second derivative at this point does not vanish, this extremum is the global maximum or minimum of the dilaton. Thus we have the result that the dilaton has at most one extremum, and in this case the dilaton is $+\infty$ (strong coupling limit) at the beginning *and* at the end of the universe, or the dilaton is $-\infty$ (weak coupling limit) at both ends of the evolution of the universe. Figure 3 shows an example.

3. $\pm$ ambiguity. Mathematically this ambiguity appeared at the reduction of the 5-D Einstein–Hilbert theory to the 4-D string effective action. But physically there is another possible interpretation. If we set the cosmological constant to zero the 5-D theory can also be considered as a string theory with vanishing dilaton. To get a vanishing dilaton β function it is necessary that $\Lambda = 0$. In contrast to our 4-D theory, the 5-D theory posseses an abelian isometry corresponding to the independence of the fifth coordinate. Consequently we can construct a new 5-D string solution by a duality transformation [8]. Our 5-D Einstein–Hilbert action then becomes

$$\int d^5x\sqrt{\tilde{G}}\tilde{R} \rightarrow \int d^5x\sqrt{\bar{G}}e^{-2\psi}\left(\bar{R} + 4(\partial\psi)^2\right) \ , \tag{22}$$

and the new 5-D solution is connected with the old one by

$$\bar{G}_{55} = \frac{1}{\tilde{G}_{55}} \quad , \quad \psi = -\frac{1}{2}\log\tilde{G}_{55} \ , \tag{23}$$

where ψ is now a 5-D dilaton field. If we now reduce this dualized 5-D string theory to a 4-D string theory we again get the solution (7) but with switched $\pm$ signs. Thus, for $\Lambda = 0$ both solutions (7) are connected by dualizing the 5-D theory and both solutions can be generalized to $\Lambda \neq 0$. In addition, for $k = 0$ it is possible to relate both solutions by dualizing the spatial isometries [9]. Physically this duality transformation changes the sign of the dilaton and thus switches between the strong and weak coupling limit at the beginning or the end of the universe. In the string frame both solutions are physically different (see Fig. 2), whereas the 4-D Einstein metric is invariant under this transformation (e.g., Fig. 1).

To summarize, in the present paper we have obtained various cosmological solutions (7) of the low-energy effective string action. Although the method presented is very simple (reduction of a 5-D black hole solution) our solution, as far as we know, has not been obtained before. The reason is that we did not use the standard parameterization of the Robertson-Walker metric (4) and it seems to be impossible to solve the string equations of motion (vanishing of the β function) for this coordinate system analytically. In our procedure this impossibility becomes manifest in the failure to find an analytic solution of (15). However, in most cases it is possible to get some impression about the features of our solutions. We have investigated the asymptotic behavior of the world radius near zero, near the horizons of the corresponding 5-D black hole, and in the infinite future. For vanishing dilaton potential and $k = -1$ we obtained a flat universe in the large time limit. In the case where we were able to get an analytic expression ($\Lambda = k = 0$) in the Robertson-Walker parameterization (4) our result coincides with Mueller's solution [7]. Furthermore, the dilaton is divergent at the beginning and at the end of the universe and a duality transformation in the 5-D theory switches between the strong ($\phi \to +\infty$) and the weak ($\phi \to -\infty$) string coupling limit. While the 4-D string frame metric is not invariant under 5-D duality the 4-D Einstein frame metric is invariant.

The procedure developed in this paper should be applicable to more general 5-D theories. In this paper we have discussed the easiest case of pure dilaton gravity. The inclusion of further background fields like antisymmetric tensor and gauge field in the 5-D theory is in progress.

Acknowledgments

We would like to thank H. Dorn, D. Lüst and G. Weigt for helpful discussions.

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