SLAC–PUB–6409
UCLA/TEP/93/51
hep-ph/9312333
December, 1993
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One-Loop N Gluon Amplitudes with Maximal Helicity Violation via Collinear Limits *

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Abstract

We present a conjecture for the leading color part of the n-gluon one-loop amplitudes with maximal helicity violation. The conjecture emerges from the powerful requirement that the amplitudes have the correct behavior in the collinear limits of external momenta. One implication is that the corresponding amplitudes where three or more gluon legs are replaced by photons vanish for $n > 4$.

Submitted to Physical Review Letters

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*Research supported in part by the Department of Energy under grant DE-AC03-76SF00515.

[♭]Research supported in part by the US Department of Energy under grant DE-FG03-91ER40662 and in part by the Alfred P. Sloan Foundation under grant BR-3222.

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Multi-jet processes at colliders require knowledge of matrix elements with multiple final state partons. At tree-level concise formulae for maximally helicity violating amplitudes with an arbitrary number of external legs were first conjectured by Parke and Taylor [1], and later proven by Berends and Giele using recursion relations [2,3]. These have proven useful both as exact results and in approximation schemes [4].

In general amplitudes in gauge theories satisfy strong consistency conditions; they must be unitary, and must satisfy correct limits as the momenta of external legs become collinear [1,2,5]. In this letter we discuss the example of a one-loop amplitude which is sufficiently constrained that we can write down a form for an arbitrary number of external legs. The all-n conjecture which we present is for maximal helicity violation, that is with all (outgoing) legs of identical helicity, was originally displayed in ref. [6], and has just been confirmed by recursive techniques [7,8]. The construction is based upon extending the known one-loop four- and five-gluon [9] amplitudes which were first obtained using string-based methods [10].

The one-loop n-gluon partial amplitude $A_{n;1}(1^+, 2^+, \ldots, n^+)$ is associated with the color factor $N\,\mathrm{Tr}(T^{a_1}\cdots T^{a_n})$ and gives the leading contribution to the amplitude for a large N [11,5,12]. Its structure is particularly simple, making it an ideal candidate for finding an all n expression. (The corresponding amplitude with all negative helicities is given by spinor conjugation.) The all-plus helicity structure is cyclicly symmetric; and no logarithms or other functions containing branch cuts can appear. This can be seen by considering the cutting rules: the cut in a given channel is given by a phase space integral of the product of the two tree amplitudes obtained from cutting. One of these tree amplitudes will vanish for all assignments of helicities on the cut internal legs since $A_n^{\mathrm{tree}}(1^\pm, 2^+, 3^+, \ldots, n^+) = 0$, implying that all cuts vanish. Furthermore, the all plus helicity loop amplitude does not contain multi-particle poles; factorizing the amplitude on a multi-particle pole into lower point tree and loop amplitudes again yields a tree which vanishes for either helicity of the leg carrying the multi-particle pole. The only singularities are those where two (color-adjacent) momenta become collinear.

Another simplifying feature of the all-plus amplitude is the equality, up to a sign due to statistics, of the contributions of internal gluons, complex scalars and Weyl fermions. This is a consequence of the supersymmetry Ward identity [13] $A^{\mathrm{susy}}(1^\pm, 2^+, \ldots, n^+) = 0$. Since the $N = 1$ supersymmetry amplitude has one gluon and one gluino circulating in the loop, the gluino contribution must be equal and opposite to that of the gluon in order to yield zero for the total; similarly, the spectrum of an $N = 2$ supersymmetric theory contains two gluinos and one complex scalar in addition to the gluon, and the vanishing implies the equality of the contributions of complex scalars and gluons circulating in the loop. For Weyl fermions and complex scalars transforming under the fundamental rather than the adjoint representation (in a vector-like theory), the color factor is smaller by a factor of N, and *no* subleading color factors appear: in this case the amplitude presented here is in fact the complete answer rather than merely the leading-color piece.

At one loop the collinear limits of color-ordered one-loop QCD amplitudes are expected to have the form

$$A_{n;1}^{\rm loop} \xrightarrow{a\parallel b} \sum_{\lambda=\pm} \Bigg(\mathrm{Split}^{\rm tree}_{-\lambda}(a^{\lambda_a}, b^{\lambda_b})\, A^{\rm loop}_{n-1;1}(\ldots (a+b)^\lambda \ldots) \\ + \mathrm{Split}^{\rm loop}_{-\lambda}(a^{\lambda_a}, b^{\lambda_b})\, A^{\rm tree}_{n-1}(\ldots (a+b)^\lambda \ldots) \Bigg)\,, \tag{1}$$

in the limit where the momenta $k_a \to z k_P$ and $k_b \to (1-z) k_P$ with $k_P = k_a + k_b$. Here λ is the helicity of the intermediate state with momentum k_P. This is analogous to the form of tree-level collinear limits [1,2,5,14]. The explicit form of the one-loop splitting functions may be extracted from the known four- [15] and five-point [9] gluon amplitudes. All known one-loop amplitudes [9,16] satisfy eq. (1), though there is as yet no proof of its correctness for larger n. Because of the supersymmetry Ward identitity relating the gluon and fermion contribution to the scalar one, it suffices for our present purposes to prove it for the case of scalars in the loop, which turns out to be the easiest case. We shall give the outline of such a proof.

The one-loop all-plus helicity amplitudes have a simple collinear structure because the loop splitting function $\mathrm{Split}^{\rm loop}_{-\lambda}$ does not enter; it multiplies a tree amplitude which vanishes. The tree splitting functions that enter are [1,2,5]

$$\mathrm{Split}^{\rm tree}_{+}(a^+, b^+) \;=\; 0, \qquad\qquad \mathrm{Split}^{\rm tree}_{-}(a^+, b^+) = \frac{1}{\sqrt{z(1-z)}\,\langle a\, b\rangle}, \tag{2}$$

where we follow the notation of ref. [14] for the spinor inner products $\langle a\, b\rangle$ and $[a\, b]$ which are equal to $\sqrt{s_{ab}}$ up to a phase. In general, the non-vanishing splitting functions diverge as $1/\sqrt{s_{ab}}$ in the collinear limit $s_{ab} = (k_a + k_b)^2 \to 0$.

The outline of a proof of the universality of the scalar-loop contributions to the collinear splitting functions follows. We divide the diagrams into several sets, depending upon the topology of the two external collinear legs which, without loss of generality, we label 1 and 2. In a color-ordered diagram, only those legs which are nearest neighbors can have collinear singularities. It turns out that $\mathrm{Split}^{\rm tree}$ arises from the diagrams in fig. 1, $\mathrm{Split}^{\rm loop}$ from the diagrams in fig. 2 and diagrams without explicit poles in s_{12}, such as those of fig. 3, do not contribute to the splitting functions.

We begin with the diagrams in fig. 1. The only Feynman diagrams which can contribute to the tree splitting function are those containing explicit poles in s_{12}, as depicted in fig. 1; trees containing legs 1 and 2 but lacking this explicit pole will not contribute. The analysis is identical to the tree-level analysis and gives a similar result, yielding the first term in eq. (1) containing the tree splitting function.

The diagrams in fig. 2 also contain explicit collinear poles and give rise to the $\text{Split}^{\text{loop}}$ function. There are three groups of diagrams in this category depicted in figs. 2a–c. Evaluating and summing over the three types of diagrams in the collinear limit yields

$$\frac{1}{16\pi^2}\frac{1}{6}\,(k_1-k_2)^\mu \eta_{\mu\nu} A_{n-1}^{\text{tree}}(1+2,\ldots,n)^\nu \left(\frac{\sqrt{2}}{s_{12}}\right)\left[\varepsilon_1\cdot\varepsilon_2 - \frac{\varepsilon_1\cdot k_2 \varepsilon_2\cdot k_1}{k_1\cdot k_2}\right], \tag{3}$$

where ε_i are gluon polarization vectors. This will give the entire contribution to the loop splitting functions for internal scalars. Converting to a helicity basis [17] in a manner similar to that used at tree-level in ref. [2], one finds

$$\begin{aligned}
\text{Split}_{+}^{\text{loop }[0]}(a^+,b^+) &= -\frac{\sqrt{z(1-z)}}{48\pi^2}\frac{[a\,b]}{\langle a\,b\rangle^2}\,,\\
\text{Split}_{-}^{\text{loop }[0]}(a^+,b^+) &= \frac{\sqrt{z(1-z)}}{48\pi^2}\frac{1}{\langle a\,b\rangle}\,,
\end{aligned} \tag{4}$$

and other helicity combinations vanish.

The remaining diagrams do not have the required collinear pole arising from a tree propagator; it would have to emerge from the loop integral. One possibility is that one collinear leg is directly connected to the loop a via a three vertex while the other collinear leg is part of a tree or a four-vertex sewn onto the loop. These diagrams cannot have any collinear poles in s_{12} because the loop integral does not contain this kinematic invariant except as a sum with other kinematic invariants.

The next possibility, depicted in fig. 3a, is that both legs in the collinear pair are attached to a scalar loop by three-point vertices and are part of a loop with four or more legs. Since the splitting functions diverge as $s_{12}\to 0$, contributions come from regions where the three propagators $1/(l-k_2)^2$, $1/l^2$, and $1/(l+k_1)^2$ depicted in fig. 3a blow up. The leading singularities come from the region $l\approx \alpha k_1+\beta k_2$ where α and β are arbitrary constants. Near the special points $(\alpha,\beta)=(-1,0)$ and $(0,1)$ a fourth propagator blows up requiring a separate analysis, which will lead to the same conclusion as the generic case. In the generic case, in the region $l\approx\alpha k_1+\beta k_2$ the calculation reduces to a triangle integral. For a scalar propagating in the loop, each coupling to a gluon includes a power of the loop momentum, which allows the integral to be performed without dimensional regularization. An analysis of the integral [18] shows that there are no contributions to the splitting functions from fig. 3a; a similar analysis leads to the same conclusion for fig. 3b. For gluons or fermions circulating in the loop, loop-momentum-independent terms in the vertices of the diagrams in fig. 3 invalidate the above analysis, and these diagrams can contribute collinear poles. For the all-plus helicity, however, the supersymmetry identities extend the result to internal gluons and fermions as well, and the splitting functions appearing in eq. (1) are universal functions for arbitrary numbers of external legs.

The starting point in constructing our n-point expression is the known five-point one-loop helicity amplitude [9],

$$A_{5;1}(1^+,2^+,3^+,4^+,5^+) \;=\; \frac{iN_p}{192\pi^2}\,\frac{s_{12}s_{23}+s_{23}s_{34}+s_{34}s_{45}+s_{45}s_{51}+s_{51}s_{12}\;+\;\varepsilon(1,2,3,4)}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle}\;, \tag{5}$$

where $\varepsilon(i,j,m,n) = 4i\varepsilon_{\mu\nu\rho\sigma}k_i^\mu k_j^\nu k_m^\rho k_n^\sigma \;=\; [i\,j]\,\langle j\,m\rangle\,[m\,n]\,\langle n\,i\rangle - \langle i\,j\rangle\,[j\,m]\,\langle m\,n\rangle\,[n\,i]$, and N_p is the number of color-weighted bosonic states minus fermionic states circulating in the loop; for QCD with n_f quarks, $N_p = 2(1-n_f/N)$ with $N=3$.

Using eqs. (1) and (2) and $A_n^{\rm tree}(1^\pm,2^+,\cdots,n^+)=0$, we can construct higher point amplitudes by writing down general forms with only two particle-poles, and requiring that they have the correct collinear limits. Generalizing to all n we have

$$A_{n;1}(1^+,2^+,\ldots,n^+) \;=\; \frac{iN_p}{192\pi^2}\,\frac{E_n+O_n}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle n\,1\rangle}\;, \tag{6}$$

where

$$O_n \;=\; \sum_{1\le i_1<i_2<i_3<i_4\le n-1} \varepsilon(i_1,i_2,i_3,i_4) \;=\; -\sum_{1\le i_1<i_2<i_3<i_4\le n} {\rm tr}(\not k_{i_1}\not k_{i_2}\not k_{i_3}\not k_{i_4}\gamma^5)\;. \tag{7}$$

To describe E_n define $t_i^{[p]} = (k_i+k_{i+1}+\cdots+k_{i+p-1})^2$ (all indices mod n); note that $t_i^{[2]} = s_{i,i+1}$ and $t_i^{[1]}=0$. Then

$$\begin{aligned} E_{n=2m+1} \;&=\; \sum_{p=2}^{m}\sum_{i=1}^{n}\left(t_i^{[p]}t_{i+1}^{[p]} - t_{i+1}^{[p-1]}t_i^{[p+1]}\right) \\ E_{n=2m} \;&=\; \sum_{p=2}^{m-1}\sum_{i=1}^{n}\left(t_i^{[p]}t_{i+1}^{[p]} - t_{i+1}^{[p-1]}t_i^{[p+1]}\right) \;+\; \frac{1}{2}\sum_{i=1}^{n}\left(t_i^{[m]}t_{i+1}^{[m]} - t_{i+1}^{[m-1]}t_i^{[m+1]}\right)\;. \end{aligned} \tag{8}$$

or equivalently

$$E_n = -\sum_{1\le i_1<i_2<i_3<i_4\le n} {\rm tr}(\not k_{i_1}\not k_{i_2}\not k_{i_3}\not k_{i_4})\;. \tag{9}$$

The two terms O_n and E_n can be combined into a single trace, a form which agrees with ref. [7], but for the purposes of discussing symmetry properties, it is more convenient to keep them separate.

The O_n term (7) is not manifestly cyclicly symmetric; however, the difference between O_n and its cyclic permutation vanishes using momentum conservation. To verify that in the limit that two legs become collinear it reduces to the corresponding $(n-1)$-point term O_{n-1}, it suffices to check the limit $1 \parallel 2$. Terms of the form $\varepsilon(1, 2, j_3, j_4)$ clearly vanish. The remaining terms containing 1 and 2 may be paired as

$$\sum_{3 \le i_2 < i_3 < i_4 \le n-1} \Big(\varepsilon(1, i_2, i_3, i_4) + \varepsilon(2, i_2, i_3, i_4) \Big) \; = \sum_{3 \le i_2 < i_3 < i_4 \le n-1} \varepsilon(P, i_2, i_3, i_4), \qquad (10)$$

where $k_P = k_1 + k_2$. Adding these terms to the terms containing neither 1 nor 2, and relabeling $\{P, 3, 4, \ldots, n\} \to \{1, 2, 3, \ldots, n-1\}$, we see that $O_n \to O_{n-1}$ in the limit $1 \parallel 2$, as required. The cyclic symmetry of the E_n term (8) is manifest. The collinear limit of the equivalent form (9) follows the same argument as for the O_n terms.

One can argue that the expression (6) is uniquely determined by the collinear limits, using the fact that from the dimension of the amplitude, the n-point amplitude has $n-4$ more powers of momentum in the denominator than in the numerator. Because of this, a function which has vanishing collinear limits in all channels cannot be added to the amplitude. (The collinear limit of a five-point amplitude is special because one only has three independent momenta after taking the collinear limit, so $\varepsilon(1, 2, 3, 4)$ vanishes in all collinear limits.) Presumably this argument can be made more rigorous, and applied to more general helicity configurations as well. Indeed, it is not difficult to prove, assuming that the denominator is given by $\langle 1\,2 \rangle \cdots \langle n\,1 \rangle$, that the functions E_n and O_n are uniquely determined by the collinear limits for all $n > 5$.

In massless QED, through use of recursion relations [2,3], Mahlon has demonstrated that the one-loop n-photon helicity amplitudes $A_n(\gamma_1^\pm, \gamma_2^+, \cdots, \gamma_n^+)$ vanish for $n > 4$ [19]. It is easy to argue that the collinear limits are consistent with this result, and that many more "mixed" photon-gluon amplitudes should also vanish. Charge conjugation invariance implies that photon amplitudes with an odd number of legs vanish. This also implies that the amplitude with three photons and two gluons $A_{5;1}(\gamma_1, \gamma_2, \gamma_3, g_4, g_5) = 0$, since this amplitude is proportional to the corresponding photon amplitude: the two gluons have to be in a C even state. Using the collinear behavior (1) leads one to suspect that all six-point helicity amplitudes with three photons and three gluons may vanish, and continuing recursively in this way, that perhaps all amplitudes with three photons and additional gluons vanish.

One can confirm this suspicion in the all-plus case by explicit calculation, using the expression (6) and converting some of the gluons into photons. Amplitudes with r external photons and $(n-r)$ gluons have a color decomposition similar to that of the pure-gluon amplitudes, except that charge matrices are set to unity for the photon legs. The coefficients of these color factors, $A_{n;1}^{r\gamma}$, are given by appropriate cyclic sums over the pure-gluon partial amplitudes, retaining only the contributions from particles in the fundamental representation in the loop; e.g., for a single quark with electric charge Q, replace $N_p \to N_p^{\rm fund} = -2/N$, and the overall coupling factor $g^n \to g^{n-r}(eQ\sqrt{2})^r$.

Defining the short-hand

$$\mathcal{S}_n(i,j) \;=\; \frac{\langle i\, j\rangle}{\langle i\, n\rangle\,\langle n\, j\rangle}\,; \tag{11}$$

performing the cyclic sums, and making repeated use of the identity

$$\sum_{j=j_1+1}^{j_2} \mathcal{S}_n(j-1,j) \;=\; \mathcal{S}_n(j_1,j_2)\,, \tag{12}$$

we can write down simple forms for the all-plus partial amplitude with one or two external photons (legs $n \ldots n-r+1$), and any number of gluons,

$$A_{n;1}^{r\gamma} \;=\; \frac{iN_p^{\rm fund}}{192\pi^2}\,\frac{O_n^{r\gamma}+E_n^{r\gamma}}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle n-r-1,\,n-r\rangle\,\langle n-r,\,1\rangle}\,, \tag{13}$$

with

$$\begin{aligned}
O_n^{1\gamma} &= -2 \sum_{1\le i_1<i_2<i_3<i_4\le n-1} \varepsilon(i_1,i_2,i_3,i_4)\;\left[\mathcal{S}_n(i_1,i_2)+\mathcal{S}_n(i_3,i_4)\right]\,,\\
E_n^{1\gamma} &= 2 \sum_{1\le i_1<i_2<i_3\le n-1} \left[\mathcal{S}_n(i_1,i_2)\,s_{i_1i_2}s_{i_3n}+\mathcal{S}_n(i_2,i_3)\,s_{i_2i_3}s_{i_1n}+\mathcal{S}_n(i_3,i_1)\,s_{i_3i_1}s_{i_2n}\right]\,,\\
O_n^{2\gamma} &= 4 \sum_{1\le i_1<i_2<i_3\le n-2} \varepsilon(i_1,i_2,i_3,n-1)\;\left[\mathcal{S}_n(i_1,i_2)\mathcal{S}_{n-1}(i_2,i_3)-\mathcal{S}_{n-1}(i_1,i_2)\mathcal{S}_n(i_2,i_3)\right]\,,\\
E_n^{2\gamma} &= 4 \sum_{1\le i_1<i_2\le n-2} \left[\mathcal{S}_{n-1}(i_1,i_2)\mathcal{S}_n(i_1,i_2){\rm tr}(\not k_{i_1}\not k_n\not k_{i_2}\not k_{n-1})-s_{i_1i_2}\frac{[n-1\,n]}{\langle n-1\,n\rangle}\right]\,.
\end{aligned} \tag{14}$$

For three or more external photons, an even more striking result emerges: the amplitude vanishes,

$$A_{n>4}^{\rm loop}(\gamma_1^+,\gamma_2^+,\gamma_3^+,g_4^+,\ldots,g_n^+)=0. \tag{15}$$

Since amplitudes with even more photon legs are obtained by further sums over permutations of legs, this implies that for the all plus helicity configuration all amplitudes with three or more photon legs vanish (for $n>4$) in agreement with the expectation from the collinear limits.

In order to extend these methods to other helicity amplitudes one would first need a general proof of the collinear limits for particles circulating in the loop other than scalars [18] (which sufficed for the all-plus case because of the supersymmetry identities). The loop splitting functions appearing in equation (1) can already be extracted from five-parton amplitudes [9,16]. We expect that collinear limits will be a useful tool in constructing one-loop helicity amplitudes besides those presented here.

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Figure Captions.

Fig. 1: Diagrams that contribute to the tree splitting functions.

Fig. 2: Diagrams that contribute to the loop splitting functions.

Fig. 3: Two of the remaining diagram types which have no collinear poles for scalars in the loop.

Contact author for figures.