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Parity and Front-Form Quantization of Field Theories*

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ABSTRACT

Recently, we proposed a new front-form quantization which treated both the x^+ and the x^- coordinates as front-form 'times.' This quantization was found to preserve parity explicitly. In this paper we extend this construction to fermion fields in the context of the Yukawa theory. We quantize this theory using a method proposed originally by Faddeev and Jackiw . We find that P^- *and* P^+ become dynamical and that the theory is manifestly invariant under parity.

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1. Introduction

Front-form quantization is usually done by quantization along the front $x^+ = const$, though this causes some uneasiness[1] . Usually this is done by quantizing a system with constraints[2] [3] [4] [5] . In a previous paper[6] , we introduced a quantization which treated x^+ and x^- on equal footing. The main argument given was that this new approach was manifestly parity invariant. We also pointed out that this new approach had the same number of degrees of freedom as the equal-time approach. For the scalar case, the second order differential equation was specified by the two boundary conditions in both the equal time, as well as in the front-form case, where we took both x^+ and x^- as front-form 'times'[7] .

We want to present here a new argument, due to Robertson and McCartor[8] , which we feel is even more compelling. For this we will consider the Yukawa model with scalar mass μ and fermion mass m in $3+1$-dimensions:

$$\mathcal{L} = \frac{1}{2}(\partial_\nu\phi\partial^\nu\phi - \mu^2\phi^2) + \frac{i}{2}\partial^\nu\overline{\psi}\gamma_\nu\psi - \overline{\psi}\frac{i}{2}\partial^\nu\gamma_\nu\psi - m\overline{\psi}\psi - g\overline{\psi}\psi\phi \qquad 8(1.1)$$

The equations of motions for the fermion field are

$$i\partial^-\psi_+ = -i\gamma_0\gamma_i\partial^i\psi_- + m\gamma_0\psi_- + g\gamma_0\phi\psi_-$$

$$i\partial^-\psi_- = -i\gamma_0\gamma_i\partial^i\psi_+ + m\gamma_0\psi_+ + g\gamma_0\phi\psi_+ \qquad 8(1.2)$$

where $\psi_\pm = \Lambda_\pm\psi$ and $\Lambda_\pm = \frac{1}{2}\gamma^0\gamma^\pm$. In the usual front-form quantization, ψ_+ is taken to be the independent degree of freedom and ψ_- the *dependent* degree of freedom.

Let us look now at the scalar field. The equations of motions for it are

$$\partial^+\partial^-\phi - \partial_i^2\phi + \mu^2\phi = -g\overline{\psi}\psi \qquad 8(1.3)$$

and we want to point out that the fermionic current term is of the form

$$\overline{\psi}\psi = \psi_+^\dagger\gamma_0\psi_- + \psi_-^\dagger\gamma_0\psi_+ \qquad 8(1.4)$$

Robertson and McCartor invite us to look at the evolution of ϕ along x^+

$$\partial_-\phi_{x^+=\delta x^+} = \partial_-\phi_{x^+=0} + (\delta x^+)\frac{1}{4}(\partial_i^2\phi - \mu^2\phi - g\overline{\psi}\psi)_{x^+=0} \qquad 8(1.5)$$

As they point out , as we follow the evolution of ϕ, in order to determine ϕ everywhere we need to know ϕ along **both** $x^+ = 0$ and $x^- = 0$ surfaces[9] .

We'd like to point out that in some work involving initial value problems in gravity using front-form coordinates[10] [11] [12] [13] [14] , the initial data for these coordinates is also specified along **both** $x^+ = const$ and $x^- = const$ surfaces as well as at $x^+ = x^- = 0$. Furthermore, R. Penrose [12] points out that in this approach there are *no* constraints[15] .

2. Reduced Phase Space Quantization in Front-Form

We will apply the reduced phase space quantization of Faddeev and Jackiw[16] . to the Yukawa model in $3+1$ dimensions with boson mass μ and fermion mass m:

$$\mathcal{L} = \frac{1}{2}(\partial_\nu\phi\partial^\nu\phi - \mu^2\phi^2) + \frac{i}{2}\partial^\nu\overline{\psi}\gamma_\nu\psi - \overline{\psi}\frac{i}{2}\partial^\nu\gamma_\nu\psi - m\overline{\psi}\psi + \mathcal{L}_I \qquad 8(2.1)$$

where $\mathcal{L}_I$ is the interaction part of the Lagrangean. Let us write $\mathcal{L}$ out explicitly :

$$\mathcal{L}d^4x = \{\frac{1}{2}\partial^+\phi\partial^-\phi - \frac{1}{2}(\partial_i\phi)^2 - \frac{1}{2}\mu^2\phi^2 + \mathcal{L}_I + \psi_+^\dagger\frac{i\partial^-}{2}\psi_+ - \frac{i}{2}(\partial^-\psi_+^\dagger)\psi_+$$

$$+\psi_-^\dagger\frac{i\partial^+}{2}\psi_- - \frac{i}{2}(\partial^+\psi_-^\dagger)\psi_- + -\psi_+^\dagger\gamma_0\gamma_i\frac{i\partial_i}{2}\psi_- + \frac{i}{2}(\partial_i\psi_-^\dagger)\gamma_0\gamma_i\psi_+$$

$$-\psi_-^\dagger\gamma_0\gamma_i\frac{i\partial_i}{2}\psi_+ + \frac{i}{2}(\partial_i\psi_+^\dagger)\gamma_0\gamma_i\psi_- - m\psi_+^\dagger\frac{\gamma^-}{2}\psi_- - m\psi_-^\dagger\frac{\gamma^+}{2}\psi_+\}\frac{dx^-dx^+d^2x_\perp}{2} \quad 8(2.2)$$

where $\psi_\pm = \Lambda_\pm\psi$ as before. Note also that $\partial_\nu = \frac{\partial}{\partial x^\nu}$ so that $\partial^- = 2\partial_+ = 2\frac{\partial}{\partial x^+}$ and $\partial^+ = 2\partial_- = 2\frac{\partial}{\partial x^-}$. The corresponding conjugate momenta for x^+-derivatives are

$$\pi_\phi = \frac{\partial \mathcal{L}}{\partial(\partial^- \phi)} = \frac{1}{2}\partial^+ \phi \qquad 8(2.3)$$

$$\pi_\psi(x) = \frac{\partial \mathcal{L}}{\partial(\partial^- \psi)} = \frac{i}{2}\psi_+^\dagger \qquad 8(2.4)$$

$$\pi_{\psi^\dagger}(x) = \frac{\partial \mathcal{L}}{\partial(\partial^- \psi^\dagger)} = -\frac{i}{2}\psi_+ \qquad 8(2.5)$$

For the momenta corresponding to x^--derivatives we get similar forms:

$$\rho_\phi = \frac{\partial \mathcal{L}}{\partial(\partial^+ \phi)} = \frac{1}{2}\partial^- \phi \qquad 8(2.6)$$

$$\rho_\psi(x) = \frac{\partial \mathcal{L}}{\partial(\partial^+ \psi)} = \frac{i}{2}\psi_-^\dagger \qquad 8(2.7)$$

$$\rho_{\psi^\dagger}(x) = \frac{\partial \mathcal{L}}{\partial(\partial^+ \psi^\dagger)} = -\frac{i}{2}\psi_- \qquad 8(2.8)$$

We rewrite $\mathcal{L}d^4x$ in the following way

$$\mathcal{L}d^4x = \frac{1}{2}\big\{\pi_\phi\partial^-\phi + \pi_\phi\partial^-\phi + \pi_\psi\partial^-\psi_+ + \pi_\psi\partial^-\psi_+ + (\partial^-\psi_+^\dagger)\pi_{\psi^\dagger}$$

$$+(\partial^-\psi_+^\dagger)\pi_{\psi^\dagger} + \rho_\phi\partial^+\phi + \rho_\phi\partial^+\phi + \rho_\psi\partial^+\psi_- + \rho_\psi\partial^+\psi_- + (\partial^-\psi_-^\dagger)\rho_{\psi^\dagger} + (\partial^-\psi_-^\dagger)\rho_{\psi^\dagger}\big\}d^4x$$

$$-\mathcal{H}dx^+ - \mathcal{K}dx^- + \mathcal{M}d^4x \qquad 8(2.9)$$

The meaning of these terms is as follows : the first bracket represents the p-q-dot terms which go into the definitions of the canonical commutation relations; the second and third term are the Hamiltonians which define the evolution of the system along x^+ , given by $\mathcal{H}$, and along x^- given by $\mathcal{K}$[17] ; finally, the last term contains the remaining pieces which give the 'constraints', though are not 'true constraints' [16] as are consistent with the ϕ equations of motions. The Hamiltonians $\mathcal{H}$ and $\mathcal{K}$ are :

$$\mathcal{H} = \frac{dx^- dx_\perp^2}{2}\{\frac{1}{2}(\partial_i\phi)^2 + \frac{1}{2}\mu^2\phi^2 - \mathcal{L}_I + \psi_+^\dagger\gamma_0\gamma_i\frac{i\partial_i}{2}\psi_- - \frac{i}{2}(\partial_i\psi_-^\dagger)\gamma_0\gamma_i\psi_+$$

$$+m\psi_+^\dagger\frac{\gamma^-}{4}\psi_- + m\psi_-^\dagger\frac{\gamma^+}{4}\psi_+\} \qquad 8(2.10)$$

$$\mathcal{K} = \frac{dx^+ dx_\perp^2}{2}\{\frac{1}{2}(\partial_i\phi)^2 + \frac{1}{2}\mu^2\phi^2 - \mathcal{L}_I + \psi_-^\dagger\gamma_0\gamma_i\frac{i\partial_i}{2}\psi_+ - \frac{i}{2}(\partial_i\psi_+^\dagger)\gamma_0\gamma_i\psi_-$$

$$+m\psi_-^\dagger\frac{\gamma^+}{4}\psi_+ + m\psi_+^\dagger\frac{\gamma^-}{4}\psi_-\} \qquad 8(2.11)$$

and for the 'constraints' we get

$$\mathcal{M} = \{ -\frac{1}{2}\partial^+\phi\partial^-\phi + \frac{1}{2}(\partial_i\phi)^2 + \frac{1}{2}\mu^2\phi^2 - \mathcal{L}_I\} \qquad 8(2.12)$$

In what sense is the last term $\mathcal{M}$ not a constraint ? Well, we can rewrite is as

$$\mathcal{M}d^4x = \frac{1}{2}\mathcal{M}d^4x + \frac{1}{2}\mathcal{N}d^4x \qquad 8(2.13)$$

with $\mathcal{M} = \mathcal{N}$. Now we rewrite each of these terms thus

$$\mathcal{M} = \int \delta\mathcal{M}, \quad \mathcal{N} = \int \delta\mathcal{N} \qquad 8(2.14)$$

where the variations δ are all possible variations over the field ϕ . Using the usual Euler-Lagrange equations, this gives (up to total derivatives which we can discard):

$$\mathcal{M}d^4x = \frac{1}{2}\int \delta\phi\{\partial^\nu\frac{\partial\mathcal{M}}{\partial(\partial^\nu\phi)} - \frac{\partial\mathcal{M}}{\partial\phi}\}d^4x + \frac{1}{2}\int \delta\phi\{\partial^\nu\frac{\partial\mathcal{N}}{\partial(\partial^\nu\phi)} - \frac{\partial\mathcal{N}}{\partial\phi}\}d^4x \quad 8(2.15)$$

For the form $\mathcal{M}$, this gives

$$\mathcal{M}d^4x = \frac{1}{2}\pi_\phi C_\pi d^4x + \frac{1}{2}\rho_\phi C_\rho d^4x \qquad 8(2.16)$$

where we used the definitions

$$\delta\phi = \pi_\phi \delta x^+ = \rho_\phi \delta x^- \qquad 8(2.17)$$

and the 'constraints' C_π and C_ρ are

$$C_\pi = \int \delta x^+ \{ -\partial^+\partial^-\phi + \partial_i^2\phi - \mu^2\phi + \frac{\partial \mathcal{L}_I}{\partial \phi} \} \qquad 8(2.18)$$

$$C_\rho = \int \delta x^- \{ -\partial^+\partial^-\phi + \partial_i^2\phi - \mu^2\phi + \frac{\partial \mathcal{L}_I}{\partial \phi} \} \qquad 8(2.19)$$

We see that $C_\pi = C_\rho = 0$ identically by the equation of motion, so in that sense these are not new conditions, so are not 'true constraints' [16] .

Let us write the $\mathcal{L}dx^4$ with the explicit momenta dependence (up to total derivatives which we can discard [16],[18]), so as to make the resulting commutation relation clear :

$$\mathcal{L}d^4x = \frac{1}{2}2\{\pi_\phi d\phi - \phi d\pi - \phi + \pi_\psi d\psi_+ - d\pi_\psi \psi_+ + d\psi_+^\dagger \pi_{\psi^\dagger} - \psi_+^\dagger d\pi_{\psi^\dagger}\} \frac{dx^- dx_\perp}{2}$$

$$+\frac{1}{2}2\{\rho_\phi d\phi - \phi d\rho_\phi + \rho_\psi d\psi_- - d\rho_\psi \psi_- + d\psi_-^\dagger \rho_{\psi^\dagger} - \psi_-^\dagger d\rho_{\psi^\dagger}\} \frac{dx^+ dx_\perp}{2}$$

$$-\mathcal{H}dx^+ - \mathcal{K}dx^- + \frac{1}{2}\pi_\phi C_\pi d^4x + \frac{1}{2}\rho_\phi C_\rho d^4x \qquad 8(2.20)$$

We see now that we have two types of evolutions, one along x^+, for which the first term in equation 8(2.20) gives the commutation relations along surfaces $x^+ = y^+$ according to the form:

$$[\xi^a, \xi^b] = \Gamma_{ab}^{-1} \quad a, b = 1, ..6 \qquad 8(2.21)$$

with

$$\xi^1 = \pi_\phi, \xi^2 = \pi_\psi, \xi^3 = \pi_{\psi^\dagger}, \xi^4 = \phi, \xi^5 = \psi_+, \xi^6 = \psi_+^\dagger \qquad 8(2.22)$$

and

$$\Gamma_{14} = \Gamma_{25} = \Gamma_{36} = 2 = -\Gamma_{41} = -\Gamma_{52} = -\Gamma_{63} \qquad 8(2.23)$$

and all the other Γ's are 0 . The second term in equation 8(2.20) gives the commutation relations along surfaces $x^- = y^-$ according to the form :

$$[\eta^a, \eta^b] = \Delta^{-1}_{ab} \quad a, b = 1, ..6 \qquad 8(2.24)$$

with

$$\eta^1 = \rho_\phi, \eta^2 = \rho_\psi, \eta^3 = \rho_{\psi^\dagger}, \eta^4 = \phi, \eta^5 = \psi_-, \eta^6 = \psi^\dagger_- \qquad 8(2.25)$$

and

$$\Delta_{14} = \Delta_{25} = \Delta_{36} = 2 = -\Delta_{41} = -\Delta_{52} = -\Delta_{63} \qquad 8(2.26)$$

and all the other Δ's are 0 . Going now to the quantum commutators, we get the following relations for fields at equal $x^+ = y^+$, the usual front-form 'time' :

$$[\phi(x^+, x^-, x_\perp), \pi_\phi(y^+, y^-, y_\perp)]_{x^+=y^+} = \frac{i}{2}\delta(x^- - y^-)\delta^2(x_\perp - y_\perp) \qquad 8(2.27)$$

$$\left\{\psi_+(x^+, x^-, x_\perp), \pi_\psi(y^+, y^-, y_\perp)\right\}_{x^+=y^+} = +\frac{i}{2}\Lambda_+\delta(x^- - y^-)\delta^2(x_\perp - y_\perp) \quad 8(2.28)$$

$$\left\{\psi^\dagger_+(x^+, x^-, x_\perp), \pi_{\psi^\dagger}(y^+, y^-, y_\perp)\right\}_{x^+=y^+} = -\frac{i}{2}\Lambda_+\delta(x^- - y^-)\delta^2(x_\perp - y_\perp) \quad 8(2.29)$$

For fields at equal $x^- = y^-$, a new front-form 'time', we get:

$$[\phi(x^+, x^-, x_\perp), \rho_\phi(y^+, y^-, y_\perp)]_{x^-=y^-} = \frac{i}{2}\delta(x^- - y^-)\delta^2(x_\perp - y_\perp) \qquad 8(2.30)$$

$$\left\{\psi_-(x^+, x^-, x_\perp), \rho_\psi(y^+, y^-, y_\perp)\right\}_{x^-=y^-} = +\frac{i}{2}\Lambda_-\delta(x^+ - y^+)\delta^2(x_\perp - y_\perp) \quad 8(2.31)$$

$$\left\{\psi_-^\dagger(x^+,x^-,x_\perp),\rho_{\psi^\dagger}(y^+,y^-,y_\perp)\right\}_{x^-=y^-} = -\frac{i}{2}\Lambda_-\delta(x^+-y^+)\delta^2(x_\perp-y_\perp) \quad 8(2.32)$$

The equations of motions are now like in Faddeev and Jackiw [16]

$$\Gamma_{ab}\partial^-\xi^b = \frac{\partial\mathcal{H}}{\partial\xi^a} \qquad 8(2.33)$$

for the x^+ variation, and

$$\Delta_{ab}\partial^-\eta^b = \frac{\partial\mathcal{K}}{\partial\eta^a} \qquad 8(2.34)$$

for the x^- variation . For $a=4$ and $b=1$, equation 8(2.33) gives

$$\partial^+\partial^-\phi = -\partial_i^2\phi + \mu^2\phi - \frac{\partial\mathcal{L}_I}{\partial\phi} \qquad 8(2.35)$$

For $a=5$ and $b=2$ we recover the equation of motion for $\psi_+^\dagger$

$$i\partial^-\psi_+^\dagger = i\frac{\partial_i\psi_-^\dagger}{2}\gamma_0\gamma_i - m\psi_-^\dagger\frac{\gamma_0}{2} - \frac{\partial\mathcal{L}_I}{\partial\psi_+} \qquad 8(2.36)$$

We get similar results from 8(2.34). Note that for $a=1$ and $b=4$ we get a seeming contradiction :

$$2\partial^-\phi = 4\pi_\phi = 0, 2\partial^+\phi = 4\rho_\phi = 0 \qquad 8(2.37)$$

But this is just why we have the $\mathcal{M}$ term, which contains the 'constraints' of the theory [16], [18] : it is of the form 8(2.16) , where the π_ϕ and ρ_ϕ are the Lagrange multipliers and the C's are the 'constraints'.

3. Quantization of the Fields

Now that we have the commutation relations, we are ready to define the fields ϕ and ψ. According to [12], using two null hyperplanes, the initial data must be specified on each of the hyperplanes as well as on their intersection . In this case, we will have initialization on the two surfaces $x^+ = 0$ and $x^- = 0$. We will require, though, that on the intersection of these surfaces, at $x^+ = x^- = 0$ these fields satisfy certain consistency conditions. This works out as follows.

On $x^+ = 0$ we have then :

$$\phi(x^+ = 0, x^-, x_\perp) = \int \frac{d^2k_\perp}{(2\pi)^3}\frac{dk^+}{2k^+}\left\{a(k^+, k_\perp)e^{-ik.x} + a^\dagger(k^+, k_\perp)e^{+ik.x}\right\} \quad 8(3.1)$$

$$\psi_+(x^+ = 0, x^-, x_\perp) = \int \frac{d^2k_\perp}{(2\pi)^3}\frac{dk^+}{2k^+}\sum_\lambda \left\{b(k^+, k_\perp)u_+(k^+, k_\perp, \lambda)e^{-ik.x}\right.$$

$$\left. + d^\dagger(k^+, k_\perp)v_+(k^+, k_\perp, \lambda)e^{+ik.x}\right\} \quad 8(3.2)$$

In this case, $ik.x = ik^+x^- - ik_\perp.x_\perp$.

On the other hyperplane, $x^- = 0$ we get similar forms:

$$\phi(x^- = 0, x^+, x_\perp) = \int \frac{d^2k_\perp}{(2\pi)^3}\frac{dk^-}{2k^-}\left\{\hat{a}(k^-, k_\perp)e^{-i\hat{k}.x} + \hat{a}^\dagger(k^-, k_\perp)e^{+i\hat{k}.x}\right\} \quad 8(3.3)$$

$$\psi_-(x^- = 0, x^+, x_\perp) = \int \frac{d^2k_\perp}{(2\pi)^3}\frac{dk^-}{2k^-}\sum_\mu \left\{\hat{b}(k^-, k_\perp)u_-(k^-, k_\perp, \mu)e^{-i\hat{k}.x}\right.$$

$$\left. + \hat{d}^\dagger(k^-, k_\perp)v_-(k^-, k_\perp, \mu)e^{+i\hat{k}.x}\right\} \quad 8(3.4)$$

Here , $i\hat{k}.x = ik^-x^+ - ik_\perp.x_\perp$.

We require now that the fields be consistent at $x^+ = x^- = 0$. This means that we have

$$\phi(x^+ = 0, x^- = 0, x_\perp) = \phi(x^- = 0, x^+ = 0, x_\perp) \qquad 8(3.5)$$

a tautology, obviously true. This implies

$$\int \frac{d^2 k_\perp}{(2\pi)^3} \frac{dk^+}{2k^+} \{a(k^+, k_\perp) e^{+ik_\perp . x_\perp} + a^\dagger(k^+, k_\perp) e^{-ik_\perp . x_\perp}\}$$

$$= \int \frac{d^2 k_\perp}{(2\pi)^3} \frac{dk^-}{2k^-} \{\hat{a}(k^-, k_\perp) e^{+ik_\perp . x_\perp} + \hat{a}^\dagger(k^-, k_\perp) e^{-ik_\perp . x_\perp}\} \qquad 8(3.6)$$

As k^+ and k^- are just dummy variables here, we get that

$$a(k^+, k_\perp) = \hat{a}(k^+, k_\perp), \quad a^\dagger(k^+, k_\perp) = \hat{a}^\dagger(k^+, k_\perp) \qquad 8(3.7)$$

and we need to point out that the variables are the **same** for both creation operators. So this means that

$$a(k^+, k_\perp) \neq \hat{a}(k^-, k_\perp) \qquad 8(3.8)$$

hence the field ϕ has different effects on the two surfaces. On $x^+ = 0$, $\phi(x^+ = 0, x^-, x_\perp)$ creates or destroys scalar quanta with momentum $k = (k^+, k_\perp)$. On $x^- = 0$, $\phi(x^- = 0, x^+, x_\perp)$ creates or destroys quanta with momentum $\hat{k} = (k^-, k_\perp)$.

A similar though more involved analysis goes for the fermion fields. Equating $\psi_+(x^+ = 0, x^-, x_\perp)$ and $\psi_-(x^- = 0, x^+, x_\perp)$ at $x^+ = x^- = 0$ we get

$$\psi_+(x^+ = 0, x^- = 0, x_\perp) = \psi_-(x^- = 0, x^+ = 0, x_\perp) \qquad 8(3.9)$$

which gives

$$\int \frac{d^2 k_\perp}{(2\pi)^3} \frac{dk^+}{2k^+} \sum_\lambda \{b(k^+, k_\perp) u_+(k^+, k_\perp, \lambda) e^{+ik_\perp . x_\perp}$$

$$+d^\dagger(k^+,k_\perp)v_+(k^+,k_\perp,\lambda)e^{-ik_\perp.x_\perp}\}$$

$$= \int \frac{d^2k_\perp}{(2\pi)^3}\frac{dk^-}{2k^-}\sum_\mu \{b(k^-,k_\perp)u_-(k^-,k_\perp,\mu)e^{+ik_\perp.x_\perp}$$

$$+d^\dagger(k^-,k_\perp)v_-(k^-,k_\perp,\mu)e^{-ik_\perp.x_\perp}\} \quad 8(3.10)$$

After some substitutions, this gives the following relations for the fermion fields and the associate spinors

$$b(k^+,k_\perp) = \hat{b}(k^+,k_\perp), \quad d^\dagger(k^+,k_\perp) = \hat{d}^\dagger(k^+,k_\perp) \quad 8(3.11)$$

$$u_+(k^+,k_\perp) = u_-(k^+,k_\perp) = u(k^+,k_\perp) \quad 8(3.12)$$

$$v_+(k^+,k_\perp) = v_-(k^+,k_\perp) = v(k^+,k_\perp) \quad 8(3.13)$$

Note again that the variables for the creation, destruction operators as well as for the spinors are all the **same** . On the other hand, $\psi_+(x^+=0,x^-,x_\perp)$ and $\psi_-(x^-=0,x^+,x_\perp)$ act differently on the two surfaces: on $x^+=0$, $\psi_+(x^+=0,x^-,x_\perp)$ creates or destroys fermion quanta of momentum $k=(k^+,k_\perp)$. On $x^-=0$, $\psi_-(x^-=0,x^+,x_\perp)$ creates or destroys quanta with momentum $\hat{k}=(k^-,k_\perp)$.

Let us also point out that these relationships between a and $\hat{a}$, b and $\hat{b}$, d and $\hat{d}$, u_+ and u_-, v_+ and v_- guarantee that we have no doubling of the independent degrees of freedom , a possibility due to the presence of **two** initializing surfaces: we have the same number of creation, destruction operators and of spinors as in the equal-time quantization case.

4. Parity in Front-Form Quantization

We are ready now to study how the fields ϕ , ψ_+ and ψ_- transform under parity. For this we use the usual definition (Bjorken and Drell for instance[19]) :

$$\mathcal{P}\phi(x^+,x^-,x_\perp)\mathcal{P}^{-1} = \pm\phi(x^-,x^+,-x_\perp) \qquad 8(4.1)$$

since under parity $(x^+,x^-,x_\perp) \to (x^-,x^+,-x_\perp)$. The $\pm$ in front of the scalar field represent the intrinsic parity of the field. For the scalar field we get

$$\mathcal{P}\phi(x^+=0,x^-,x_\perp)\mathcal{P}^{-1} = \mathcal{P}\int \frac{d^2k_\perp}{(2\pi)^3}\frac{dk^+}{2k^+}\{a(k^+,k_\perp)e^{-ik.x}+a^\dagger(k^+,k_\perp)e^{+ik.x}\}\mathcal{P}^{-1} \qquad 8(4.2)$$

This becomes

$$\mathcal{P}\phi(x^+=0,x^-,x_\perp)\mathcal{P}^{-1} = \int \frac{d^2(-k_\perp)}{(2\pi)^3}\frac{dk^-}{2k^-}\{a(k^-,-k_\perp)e^{-ik'.x'}+a^\dagger(k^-,-k_\perp)e^{+ik'.x'}\} \qquad 8(4.3)$$

if

$$\mathcal{P}a(k^+,k_\perp)\mathcal{P}^{-1} = a(k^-,-k_\perp), \quad \mathcal{P}a^\dagger(k^+,k_\perp)\mathcal{P}^{-1} = a^\dagger(k^-,-k_\perp) \qquad 8(4.4)$$

and $ik'.x' = ik^-x^+ - ik_\perp x_\perp$. Redefining variables $(k^-,-k_\perp) \to (l^-,l_\perp)$, we get the result

$$\mathcal{P}\phi(x^+=0,x^-,x_\perp)\mathcal{P}^{-1} = \phi(x^-=0,x^+,-x_\perp) \qquad 8(4.5)$$

Let us consider the fermion fields now. In this case we have [19]

$$\mathcal{P}\psi(x^+,x^-,x_\perp)\mathcal{P}^{-1} = \gamma_0\psi(x^-,x^+,-x_\perp) \qquad 8(4.6)$$

and we expect that fields defined on x^+ will be mapped into fields defined on x^-

by parity. Indeed, that is what we find for ψ_+ :

$$\mathcal{P}\psi_+(x^+=0,x^-,x_\perp)\mathcal{P}^{-1} = \mathcal{P}\int \frac{d^2k_\perp}{(2\pi)^3}\frac{dk^+}{2k^+}\sum_\lambda \{b(k^+,k_\perp)u(k^+,k_\perp,\lambda)e^{-ik.x}$$

$$+d^\dagger(k^+,k_\perp)v(k^+,k_\perp,\lambda)e^{+ik.x}\}\mathcal{P}^{-1} \qquad 8(4.7)$$

This becomes

$$\mathcal{P}\psi_+(x^+=0,x^-,x_\perp)\mathcal{P}^{-1} = \int \frac{d^2(-k_\perp)}{(2\pi)^3}\frac{dk^-}{2k^-}\sum_\lambda \{b(k^-,-k_\perp)u(k^-,-k_\perp,\lambda)e^{-ik'.x'}$$

$$+d^\dagger(k^-,-k_\perp)v(k^-,-k_\perp,\lambda)e^{+ik'.x'}\} \qquad 8(4.8)$$

if creation, destroying operators transform under parity thus :

$$\mathcal{P}b(k^+,k_\perp)\mathcal{P}^{-1} = b(k^-,-k_\perp), \quad \mathcal{P}d^\dagger(k^+,k_\perp)\mathcal{P}^{-1} = -d^\dagger(k^-,-k_\perp) \qquad 8(4.9)$$

and if the spinors transform thus :

$$\gamma_0 u(k^+,k_\perp) = +u(k^-,-k_\perp), \quad \gamma_0 v(k^+,k_\perp) = -v(k^-,-k_\perp) \qquad 8(4.10)$$

A rather long and subtle but essentially straightforward calculation show that this is indeed the case[20] . After manipulations similar to the scalar case and after using the fact that $ik'.x' = ik^-x^+ - ik_\perp x_\perp$ as well as redefining variables , we obtain the following result :

$$\mathcal{P}\psi_+(x^+=0,x^-,x_\perp)\mathcal{P}^{-1} = \gamma_0\psi_-(x^-=0,x^+,-x_\perp) \qquad 8(4.11)$$

We derive now these relations for arbitrary x^+ and x^-. Note that for the x^+

evolution we have

$$\phi(x^+, x^-, x_\perp) = e^{-iP^-x^+}\phi(x^+ = 0, x^-, -x_\perp) \qquad 8(4.12)$$

or

$$\psi_-(x^+, x^-, x_\perp) = e^{-iP^-x^+}\psi_-(x^+ = 0, x^-, -x_\perp) \qquad 8(4.13)$$

so that the parity-transformed field is

$$\mathcal{P}\phi(x^+, x^-, x_\perp)\mathcal{P}^{-1} = \mathcal{P}e^{-iP^-x^+}\mathcal{P}^{-1}\mathcal{P}\phi(x^+ = 0, x^-, -x_\perp)\mathcal{P}^{-1} \qquad 8(4.14)$$

which becomes

$$\mathcal{P}\phi(x^+, x^-, x_\perp)\mathcal{P}^{-1} = e^{-iP^+x^-}\phi(x^- = 0, x^+, -x_\perp) \qquad 8(4.15)$$

since

$$\mathcal{P}P^-\mathcal{P}^{-1} = \mathcal{P}\int \mathcal{H}\mathcal{P}^{-1} = \int \mathcal{K} = P^+ \qquad 8(4.16)$$

by use of the equations 8(2.10) and 8(2.11). A similar result holds for the fermion case. We also get the generator of x^- evolutions to transform properly as well since

$$\mathcal{P}P^+\mathcal{P}^{-1} = \mathcal{P}\int \mathcal{K}\mathcal{P}^{-1} = \int \mathcal{H} = P^- \qquad 8(4.17)$$

again, by use of equations 8(2.11) and 8(2.10).

Since now the generators of evolution along x^+ and x^- ($\mathcal{H}$ and $\mathcal{K}$ respectively), transform properly under parity , we can evolve the parity relations obtained at $x^+ = 0$ and $x^- = 0$ to relations for arbitrary x^+ and x^-. For the scaler case we get

$$\mathcal{P}\phi(x^+, x^-, x_\perp)\mathcal{P}^{-1} = \phi(x^-, x^+, -x_\perp) \qquad 8(4.18)$$

as expected from previous work [6] .

For the fermion case, we get

$$\mathcal{P}\psi_+(x^+, x^-, x_\perp)\mathcal{P}^{-1} = \gamma_0\psi_-(x^-, x^+, -x_\perp) \qquad 8(4.19)$$

which show very clearly that parity maps independent fields on $x^+ = 0$ $[\psi_+(x^+ = 0, x^-, x_\perp)]$, to independent fields on $x^- = 0$ $[\psi_-(x^- = 0, x^+, x_\perp)]$, demonstrating the it is crucial that we take **both** $x^+ = 0$ and $x^- = 0$ as quantizing surfaces if we desire to have fields with parity as an explicit symmetry.

Thus far we have looked at transformation properties of independent fields on $x^+ = 0$. It is quite straightforward to show that we get similar results for the fields which are initialized on $x^- = 0$:

$$\mathcal{P}\phi(x^-, x^+, x_\perp)\mathcal{P}^{-1} = \phi(x^+, x^-, -x_\perp) \qquad 8(4.20)$$

for the scalar field and

$$\mathcal{P}\psi_-(x^-, x^+, x_\perp)\mathcal{P}^{-1} = \gamma_0\psi_+(x^+, x^-, -x_\perp) \qquad 8(4.21)$$

for the fermion field. This completes our demonstration that fields defined on $x^+ = 0$ and $x^- = 0$ transform properly under parity, and are defined consistently.

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9. Another argument is as follows: In equal-time quantization, one specifies the initial data along a space-like surface $t = 0$. Evolution of that system corresponds to specifying the system at other surfaces $t = c_1$, $t = c_2$ and so on. Now for the type of equation we are considering (hyperbolic, second order), the characteristic curve lies along the surfaces $x^+ = 0$ and $x^- = 0$. For $t \neq 0$, our equal-time quantization surface **will** intersect both characteristic surfaces, hence the solution will be well defined. Now, when we consider

quantization along $x^+ = 0$, (one of the characteristic surfaces) evolving the solution along x^- for different constant x^+'s will **never** intersect the characteristic surface $x^+ = 0$. This seems to say that we will not have a consistent solution to our equations. Therefore, if we intend to quantize our theory along characteristic surfaces, we **must** take all these surfaces as our quantization surfaces. In our case, this means that we need to use both $x^+ = 0$ and $x^- = 0$ as quantization surfaces on the front-form.

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ordering amount to making it dynamic. The mystery is why is that so, since P^+ is supposed to be kinematic. An attractive feature of the present work is that P^+ becomes dynamical naturally.

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