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Commensurate Scale Relations: Relating Observables in QCD without Renormalization Scale or Scheme Ambiguity*

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ABSTRACT

We use the BLM method to show that all perturbatively-calculable observables in QCD, including the annihilation ratio $R_{e^+e^-}$, the heavy quark potential, and radiative corrections to structure function sum rules, are related to each other at fixed relative scales. The commensurate scale relations connecting the effective charges for observables A and B have the form

$$\alpha_A(Q_A) = \alpha_B(Q_B)\left(1 + r_{A/B}\frac{\alpha_B}{\pi} + \cdots\right),$$

where the coefficient $r_{A/B}$ is independent of the number of flavors n_F contributing to coupling constant renormalization. The ratio of scales $\lambda_{A/B} = Q_A/Q_B$ is unique at leading order and guarantees that the observables A and B pass through new quark thresholds at the same physical scale. We also show that the commensurate scales satisfy the transitivity rule

$$\lambda_{A/B} = \lambda_{A/C}\lambda_{C/B},$$

which is the renormalization group property which ensures that predictions in PQCD are independent of the choice of an intermediate renormalization scheme C. In particular, scale-fixed predictions can be made without reference to theoretically-constructed renormalization schemes such as $\overline{\text{MS}}$. QCD can thus be tested in a new and precise way by checking that the observables track both in their relative normalization and in their commensurate scale dependence.

One of the most serious difficulties preventing precise tests of QCD is the scale ambiguity of its perturbative predictions. Consider a measurable quantity such as $\rho = R_{e^+e^-}(s) - 3\Sigma e_q^2$. The PQCD prediction is of the form

$$\rho = r_0 \alpha_s(\mu) \left[1 + r_1(\mu) \frac{\alpha_s(\mu)}{\pi} + r_2(\mu) \frac{\alpha_s^2(\mu)}{\pi^2} + \cdots \right] .$$

Here $\alpha_s(\mu) = g_s^2/4\pi$ is the renormalized coupling defined in a specific renormalization scheme such as $\overline{\rm MS}$, and μ is a particular choice of renormalization scale. Since ρ is a physical quantity, its value must be independent of the choice of μ as well as the choice of renormalization scheme. Nevertheless, since we only have truncated PQCD predictions to a given order in α_s^N, the predictions do depend on μ. In the specific case of $R_{e^+e^-}$, where we have predictions [1,2] through order α_s^3, the sensitivity to μ has been shown to be less than 10% over a large range of $\ell n\, \mu$ [2]. However, in the case of the hadronic beauty production cross section $(d\sigma/d^2p_T)(\bar{p}p \to B + X)$, which has been computed to next-to-leading order in α_s, the prediction [3] for the normalization of the heavy quark p_T distribution at hadron colliders ranges over a factor of 4 if one chooses one "physical value" such as $\mu = \frac{1}{4}\sqrt{m_B^2 + p_T^2}$ rather than an equally well motivated choice $\mu = \sqrt{m_B^2 + p_T^2}$.

There is, in fact, no consensus on how to estimate the theoretical error due to the scale ambiguity, what constitutes a reasonable range of physical values, or indeed how to identify what the central value should be. Even worse, if we consider the renormalization scale μ as totally arbitrary, the next-to-leading coefficient $r_1(\mu)$ in the perturbative expansion can take on the value zero or any other value. Thus it is difficult to assess the convergence of the truncated series, and finite-order analyses cannot be meaningfully compared to experiment.

The μ dependence of the truncated prediction ρ_N is often used as a guide to assess the accuracy of the perturbative prediction, since this dependence reflects the presence of the uncalculated terms. However, the scale dependence of ρ_N only reflects one aspect of the total series. For example, consider the orthopositronium $J^{PC} = 1^{--}$ decay rate computed in quantum electrodynamics: $\Gamma_{(e^+e^-)} = \Gamma_0 \left[1 - 10.3\ (\alpha/\pi) + \cdots\right]$. The large next-to-leading coefficient, $r_1 = 10.3$ shows that there is important new physics beyond Born approximation. The magnitude of the higher order terms in the decay rate is not related to the renormalization scale since the QED coupling α does not run appreciably at the momentum transfers associated with positronium decay.

Thus we have a difficult dilemma: If we take μ as an unset parameter in PQCD predictions, then we have no reliable way to assess the accuracy of the truncated series or the parameters extracted from comparison with experiment. If we guess a value for μ and its range, we are left with a prediction without an objective guide to its theoretical precision. The problem of the scale ambiguity is compounded in multi-scale problems where several plausible physical scales enter.

In fact three quite distinct methods to set the renormalization scale in PQCD have been proposed in the literature:

1. *Fastest Apparent Convergence* (FAC) [4]. This method chooses the renormalization scale μ so that the next-to-leading order coefficient vanishes: $r_1(\mu) = 0$.
2. *The Principle of Minimum Sensitivity* (PMS) [5]. In this procedure, one argues that the best scale is the one that minimizes the scale dependence of the truncated prediction ρ_N, since that is a characteristic property of the entire series. Thus in this method one chooses μ at the stationary point

$d\rho_N/d\mu = 0$.

3. *Brodsky-Lepage-Mackenzie* (BLM) [6]. In the BLM scale-fixing method, the scale is chosen such that the coefficients r_i are independent of the number of quark flavors renormalizing the gluon propagators. In practice, one chooses the scale so that n_f does not appear in the next-to-leading order coefficient. That is, if $r_1(\mu) = r_{10}(\mu) + r_{11}(\mu)n_f$, where $r_{10}(\mu)$ and $r_{11}(\mu)$ are n_f independent, then one chooses the scale μ given by the condition $r_{11}(\mu) = 0$. This prescription ensures that, as in quantum electrodynamics, vacuum polarization contributions due to fermion pairs are all incorporated into the coupling constant $\alpha(\mu)$ rather than the coefficients.

These scale-setting methods can give strikingly different results in practical applications, For example, Kramer and Lampe have analyzed [7] the application of the FAC, PMS, and BLM methods for the prediction of jet production fractions in e^+e^- annihilation in PQCD. Jets are defined by clustering particles with invariant mass less than $\sqrt{ys}$, where y is the resolution parameter and $\sqrt{s}$ is the total center-of-mass energy. Physically, one expects the renormalization scale μ to reflect the invariant mass of the jets, that is, μ should be of order $\sqrt{ys}$. For example, in the analogous problem in QED, the maximum virtuality of the photon jet which sets the argument of the running coupling $\alpha(Q)$ cannot be larger than $\sqrt{ys}$. Thus one expects μ to decrease as the resolution parameter $y \to 0$. However, the scales chosen by the FAC and PMS methods do not reproduce this behavior (see Fig. 1): The predicted scales $\mu_{PMS}(y)$ and $\mu_{FAC}(y)$ rise without bound at small values for the jet fraction y. On the other hand, the BLM scale has the correct physical behavior as $y \to 0$. Since the argument of the running coupling becomes small using the BLM method, standard QCD perturbation theory in $\alpha_s[\mu_{BLM}(y)]$ will not be convergent in the low y domain [8]. In contrast, the scales chosen by PMS and FAC give no sign that the perturbative results break down in the soft region.

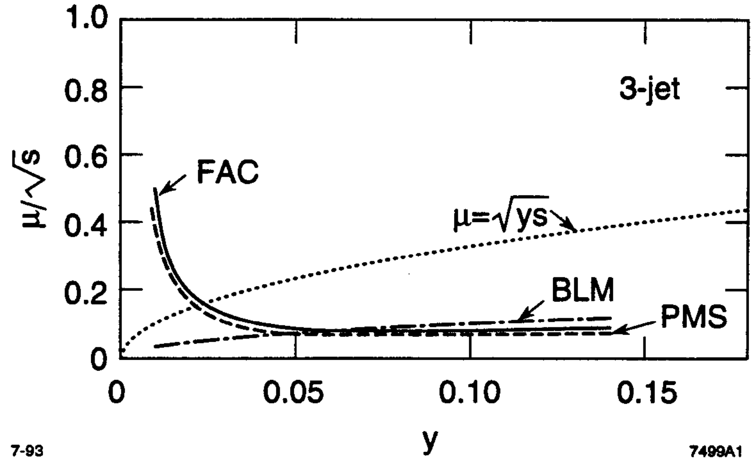

Figure 1. The scale $\mu/\sqrt{s}$ according to the BLM (dashed-dotted), PMS (dashed), FAC (full) and $\sqrt{y}$ (dotted) procedures for the three-jet rate in e^+e^- annihilation, as computed by Kramer and Lampe [7]. Notice the strikingly different behavior of the BLM scale from the PMS and FAC scales at low y. In particular, the latter two methods predict increasing values of μ as the jet invariant mass $\mathcal{M} < \sqrt{(ys)}$ decreases.

In this talk we shall use the BLM method to show that all perturbatively-calculable observables in QCD, including the annihilation ratio $R_{e^+e^-}(Q^2)$, the heavy quark potential, and the radiative corrections to the Bjorken sum rule can be related to each other at fixed relative scales. The "commensurate scale relation" for observables A and B in terms of their effective charges has the form

$$\alpha_A(Q_A) = \alpha_B(Q_B) \left(1 + r_{A/B}\frac{\alpha_B}{\pi} + \cdots\right) .$$

The ratio of the scales $\lambda_{A/B} = Q_A/Q_B$ is chosen so that the coefficient $r_{A/B}$ is independent of the number of flavors n_F contributing to coupling constant renormalization, which guarantees that the observables A and B pass through new quark thresholds at the same physical scale. We shall show that the value of $\lambda_{A/B}$ is unique at leading order, and that the relative scales satisfy the transitivity rule [9]

$$\lambda_{A/B} = \lambda_{A/C}\ \lambda_{C/B}\ .$$

This is equivalent to the group property defined by Peterman and Stückelberg [10] which ensures that predictions in PQCD are independent of the choice of an intermediate renormalization scheme C [11]. In particular, scale-fixed predictions can be made without reference to theoretically-constructed renormalization schemes such as $\overline{\rm MS}$; QCD can thus be tested by checking that the observables track both in their relative normalization and commensurate scale dependence.

It is interesting that the task of setting the renormalization scale has never been considered a problem or ambiguity in perturbative QED. For example, the leading-order parallel-helicity amplitude electron-electron scattering has the form

$$\mathcal{M}_{ee\to ee}(++;++) = \frac{8\pi s}{t}\ \alpha(t) + \frac{8\pi s}{u}\ \alpha(u)\ .$$

Here $\alpha(Q) = \alpha(Q_0)/(1 - \Pi[Q^2, Q_0^2, \alpha(Q_0)])$ is the QED running coupling which sums all vacuum polarization insertions Π into the renormalized photon propagator. The value $\alpha(0)$ is conventionally normalized by Coulomb scattering at $t = -Q^2 = 0$. Notice that both physical scales t and u appear in the argument of the running coupling constant in the cross-section; if one chooses any other scale for the running coupling, in either the direct or crossed graph amplitude, then one generates a spurious geometric series in $n_f\ (\alpha/\pi)\,\ell n(-t/\mu^2)$ or $n_f\ (\alpha/\pi)\,\ell n(-u/\mu^2)$, where n_f represents the number of fermions contributing to the vacuum polarization of the photon propagator.

In general, the "skeleton" expansion of Feynman amplitudes in QED guarantees that all dependence of an observable on the variable n_f is summed into the running coupling constant; the coefficients in QED perturbation series are thus always n_f-independent once the proper scale in α has been set. Note that the variable n_f is defined to count only vacuum polarization insertions, not light-by-light loops, since such contributions do not contribute to the coupling constant renormalization in QED.

The use of the running coupling constant $\alpha(Q)$ in QED allows one to sum in closed form all proper and improper vacuum polarization insertions to all orders, thus going well beyond ordinary perturbation theory. For example, consider the perturbative series for the lepton magnetic anomalous moment:

$$a_\ell = \frac{\alpha(Q_1^2)}{2\pi} + r_2 \frac{\alpha^2(Q_2^2)}{\pi^2} + r_3 \frac{\alpha^3(Q_3^2)}{\pi^3} + \cdots$$

the values $Q_1 = e^{-5/4} m_\ell$, etc., can be determined either by the explicit insertion of the running coupling into the integrand of the Feynman amplitude and the mean value theorem, or equivalently, by simply requiring that the coefficients r_n be independent of n_f. (Light-by-light scattering contributions are not related to coupling constant renormalization and thus enter explicitly in the order α^3 coefficient.) Thus the formula for the anomalous moment using the running coupling is form invariant, identical for each lepton $\ell = e, \mu, \tau$, since the dependence on lepton vacuum polarization insertions is implicitly contained in the dependence of the running coupling constant. These examples are illustrations of the general principle that observables such as the anomalous moments can be related to other observables such as the heavy lepton potential $V(Q^2) = -4\pi\alpha(Q^2)/Q^2$ which can be taken as the empirical definition of the on-shell scheme usually used to define $\alpha(Q^2)$.

The same procedure can easily be adapted [6] to non-Abelian theories such as QCD. One of the most useful observables in QCD is the heavy quark potential since it can be computed in lattice gauge theory from a Wilson loop, and it can be extracted phenomenologically from the heavy quarkonium spectrum. If the interacting quarks have infinite mass, then all radiative correction are associated with the exchange diagrams, rather than the vertex corrections. It is convenient to write the heavy quark potential as $V(Q^2) = -4\pi C_F \alpha_V(Q)/Q^2$. This defines the "effective charge" $\alpha_V(Q^2)$ where by definition the "self-scale" $Q^2 = -t$ is the momentum transfer squared. The subscript V indicates that the coupling is defined through the potential.

In fact, any perturbatively-calculable physical quantity can be used to define an effective charge [4] by incorporating the entire radiative correction into its definition; for example

$$R_{e^+e^-}(Q^2) \equiv R^0_{e^+e^-}(Q^2)\left[1 + \frac{\alpha_R(Q)}{\pi}\right] ,$$

where R^0 is the Born result and $Q^2 = s = E^2_{cm}$ is the annihilation energy squared. An important result is that all effective charges $\alpha_A(Q)$ satisfy the Gell-Mann-Low renormalization group equation with the same β_0 and β_1; different schemes or effective charges only differ through the third and higher coefficients of the β function. Thus, any effective charge can be used as a reference running coupling constant in QCD to define the renormalization procedure. More generally, each effective charge or renormalization scheme, including $\overline{\text{MS}}$, is a special case of the universal coupling function $\alpha(Q, \beta_n)$ [12]. Peterman and Stückelberg have shown [10] that all effective charges are related to each other through a set of evolution equations in the scheme parameters β_n. Physical results relating observables must of course be independent of the choice of any intermediate renormalization scheme.

Let us now consider expanding any observable or effective charge $\alpha_A(Q_A)$ in terms of α_V :

$$\alpha_A(Q_A) = \alpha_V(\mu)\left[1 + (A_{VP}\, n_F + B)\frac{\alpha_V}{\pi} + \cdots\right] .$$

Since α_V sums all vacuum polarization contributions by definition, no coefficient in the series expansion in α_V can depend on n_F; *i.e.* all vacuum polarization contributions are already incorporated into the definition of α_V. Thus we must shift the scale μ in the argument of α_V to the scale [6] $Q_V = e^{3A_{VP}(\mu)}\mu$:

$$\alpha_A(Q_A) = \alpha_V(Q_V)\left[1 + r_1^{A/V}\,\frac{\alpha_V}{\pi} + \cdots\right] ,$$

where $r_1^{A/V} = B + (33/2)\, A_{VP}$ is the next-to-leading coefficient in the expansion of the observable A in scheme V. Thus the relative scale between the two observables A and V, $\lambda_{A/V} = Q_A/Q_V$, is fixed by the requirement that the coefficients in the expansion in α_V scheme are independent of vacuum polarization corrections. Alternatively, one can derive the same result by explicitly integrating the one loop integrals in the calculation of the observable A using $\alpha_V(\ell^2)$ in the integrand, where ℓ^2 is the four-momentum transferred squared carried by the gluon. (In practice one only needs to compute the mean-value of $\ell n\ \ell^2 = \ell n\, Q_V^2$ [13].) One can eliminate the n_F vacuum polarization dependence that appears in the higher order coefficients by allowing a new scale to appear in each order of perturbation theory. In practice, only the leading order commensurate scale is required in order to test PQCD to good precision.

We can compute other observables B and even effective charges such as $\alpha_{\overline{\rm MS}}$ as an expansion in α_V scheme:

$$\alpha_B(Q_B) = \alpha_V(Q_V)\left[1 + r_1^{B/V}\,\frac{\alpha_V}{\pi} + \cdots\right] ,$$

where $Q_V = Q_B/\lambda_{B/V}$ and again $r_1^{B/V}$ must be independent of vacuum polarization contributions. We can now substitute and eliminate $\alpha_V(Q_V)$:

$$\alpha_B(Q_B) = \alpha_A(Q_A)\left[1 + r_1^{B/A}\,\frac{\alpha_A}{\pi} + \cdots\right] ,$$

where $Q_B/Q_A = \lambda_{B/A} = \lambda_{B/V}/\lambda_{A/V}$, and $r_1^{B/A} = r_1^{B/V} - r_1^{A/V}$. Note also the symmetry property $\lambda_{B/A}\lambda_{A/B} = 1$. Alternatively, we can compute the commensurate scale $Q_A = Q_B/\lambda_{B/A}$ directly by requiring $r_1^{B/A}$ to be n_F-independent. The result is in agreement with the transitivity rule: the BLM procedure for fixing the commensurate scale ratio between two observables is independent of the intermediate renormalization scheme. The scale-fixed relation between the heavy quark potential and $\alpha_{\overline{\rm MS}}$ is [6, 14] $\alpha_V(Q) = \alpha_{\overline{\rm MS}}(e^{-5/6}Q)[1 - 2(\alpha_{\overline{\rm MS}}/\pi) + \cdots]$.

The transitivity and symmetry properties of the commensurate scales are the scale transformations of the renormalization "group" as originally defined by Peterman and Stückelberg [10]. The predicted relation between observables must be independent of the order one makes substitutions; *i.e.* the algebraic path one takes to relate the observables. It is important to note that the PMS method, which fixes the renormalization scale by finding the point of minimal sensitivity to μ, does not satisfy these group properties [9]. The results are chaotic in the sense that the final scale depends on the path of applying the PMS procedure. Furthermore, any method which fixes the scale in QCD must also be applicable to Abelian theories such as QED, since in the limit $N_C \to 0$ the perturbative coefficients in QCD coincide with the perturbative coefficients of an Abelian analog of QCD [15].

The commensurate scale relations provide a new way to test QCD: One can compare two observables by checking that their effective charges agree both in normalization and in their scale dependence. The ratio of commensurate scales $\lambda_{A/B}$ is fixed uniquely: it ensures that both observables A and B pass through heavy quark thresholds at precisely the same physical point. Theoretical calculations are often performed most advantageously in $\overline{\mathrm{MS}}$ scheme, but all reference to such constructed schemes may be eliminated when comparisons are made between observables. This also avoids the problem that one need not expand observables in terms of couplings which have singular or ill-defined functional dependence.

The physical value of the commensurate scale in α_V scheme reflects the mean virtuality of the exchanged gluon. However, in other schemes, including $\overline{\mathrm{MS}}$, the argument of the effective charge is displaced from its physical value. The relative scale for a number of observables is indicated in Table I. For example, the physical scale for the branching ratio $\Upsilon \to \gamma X$ when expanded in terms of α_V is $(1/2.77)M_\Upsilon \sim (1/3)M_\Upsilon$, which reflects the fact that the final state phase space is divided among three vector systems. (When one expands in $\overline{\mathrm{MS}}$ scheme, the corresponding scale is $0.157 M_\Upsilon$.) Similarly, the physical scale appropriate to the hadronic decays of the η_b is $(1/1.67)M_{\eta_b} \sim (1/2)M_{\eta_b}$.

Table I

Leading Order Commensurate Scale Relations

		$\alpha_{\overline{\rm MS}}(0.435Q)$		
	$\alpha_{\eta_b}(1.67Q)$		$\alpha_{\Upsilon}(2.77Q)$	
$\alpha_{\tau}(1.36Q)$		$\alpha_V(Q)$		$\alpha_R(0.614Q)$
	$\alpha_{GLS}(1.18Q)$		$\alpha_{g_1}(1.18Q)$	
		$\alpha_{M_2}(0.904Q)$		

After scale-fixing, the ratio of hadronic to leptonic decay rates for the Υ has the form [6]

$$\frac{\Gamma(\Upsilon \to \text{hadrons})}{\Gamma(\Upsilon \to \mu^+\mu^-)} = \frac{10(\pi^2-9)}{81\pi e_b^2}\,\frac{\alpha^3_{\overline{\rm MS}}(0.157M_\Upsilon)}{\alpha^2_{\rm QED}}\left[1 - 14.0(5)\,\frac{\alpha_{\overline{\rm MS}}}{\pi} + \cdots\right] .$$

Thus as is the case of positronium decay, the next to leading coefficient is very large, and perturbation theory is not likely to be reliable for this observable. On the other hand, the commensurate scales for the second moment of the non-singlet structure function M_2 and the effective charges in the Bjorken Sum Rule (and the Gross-Llewellyn-Smith Sum Rule) are not far from the physical value Q when expressed in α_V scheme. At large n the commensurate scale for M_n is proportional to $1/\sqrt{n}$ at large n, reflecting the fact that the available phase-space for parton emission decreases as n increases. In multiple-scale problems, the commensurate scale can depend on all of the physical invariants. For example, the scale controlling the evolution equation for the non-singlet structure function depends on x_{Bj} as well as Q [16]. In the case of inclusive reactions which factorize at leading twist, each structure function, fragmentation function, and subprocess cross section can have its own commensurate scale.

A number of examples of commensurate scale relations between various single-scale observables based on published three-loop $\overline{\mathrm{MS}}$ calculations are given in Table II. For simplicity we have used the leading-order scale determined by eliminating the n_f dependence from the next-to-leading coefficient. The next-to-leading coefficient becomes a rational polynomial in N_c after scale fixing. We take $n_f = 3$ to fix the higher order term. In principle, one can improve these relations by requiring that all coefficients must be n_f-independent in α_V scheme. As in the example of the muon anomalous moment, the commensurate scale appearing in argument of the higher order contributions may differ from the scale of the next-to-leading order term. The extension of the BLM procedure to higher orders has also been discussed recently by Grunberg and Kataev [17] and by Samuel and Surguladze [2].

An interesting illustration of commensurate scale relations is the connection between the effective charge α_{g_1} for the Bjorken sum rule for the first moment of the isospin non-singlet helicity-dependent structure functions: $\Gamma^{p-n} \equiv (g_A/6)\ [1 - (\alpha_{g_1}(Q)/\pi)]$ and the effective charge for the annihilation cross section:

$$\alpha_{g_1}(Q) = \alpha_R(0.52Q)\ \left[1 - \frac{\alpha_R}{\pi} + \cdots\right]\ .$$

Mattingly and Stevenson [18] have recently obtained an empirical form for $\alpha_R(Q)$ by smearing the annihilation cross section data and fitting to the three loop form using the PMS scale. Since the PMS and BLM scale are nearly coincident in this case, we can use their determination for $\alpha_R(Q)$ to predict the Bjorken sum rule radiative corrections [19]. For example, at the scale appropriate to the E142 spin-dependent structure function measurements at SLAC, $Q^2 = 2\ \mathrm{GeV}^2$, one finds $\alpha_R(0.52Q)/\pi \simeq 0.16$ and hence $\alpha_{g_1}(1.4\,\mathrm{GeV})/\pi \simeq 0.14$ which corresponds to $\Gamma^{p-n} = 0.180$. The predictions for the Bjorken sum rule at EMC and SMC

Table II

Commensurate Scale Relations For Effective Charges to Order α_s^3

$$\alpha_R(Q) = \alpha_{\overline{MS}}(0.70759Q)\left[1 + (1/12)(\alpha_{\overline{MS}}/\pi) - 15.7331\,(\alpha^2_{\overline{MS}}/\pi^2) + \cdots\right]$$

$$\alpha_{g_1}(Q) = \alpha_{\overline{MS}}(0.36788Q)\left[1 - (11/12)(\alpha_{\overline{MS}}/\pi) + 0.21527\,(\alpha^2_{\overline{MS}}/\pi^2) + \cdots\right]$$

$$\alpha_R(Q) = \alpha_{g_1}(1.92344Q)\left[1 + (\alpha_{g_1}/\pi) - 14.115\,(\alpha^2_{g_1}/\pi^2) + \cdots\right]$$

$$\alpha_{g_1}(Q) = \alpha_R(0.519903Q)\left[1 - (\alpha_R/\pi) + 16.115\,(\alpha^2_R/\pi^2) + \cdots\right]$$

$$\alpha_R(Q) = \alpha_\tau(2.20707Q)\left[1 + 0(\alpha_\tau/\pi) - 5.94141\,(\alpha^2_\tau/\pi^2) + \cdots\right]$$

$$\alpha_\tau(Q) = \alpha_R(0.45309Q)\left[1 + 0(\alpha_R/\pi) + 5.94141\,(\alpha^2_R/\pi^2) + \cdots\right]$$

$$\alpha_{g_1}(Q) = \alpha_\tau(1.14746Q)\left[1 - (\alpha_\tau/\pi) + 10.1736\,(\alpha^2_\tau/\pi^2) + \cdots\right]$$

$$\alpha_\tau(Q) = \alpha_{g_1}(0.87149Q)\left[1 + (\alpha_{g_1}/\pi) - 8.17363\,(\alpha^2_{g_1}/\pi^2) + \cdots\right]$$

momentum transfers $Q^2 = 10.7$ GeV^2 and $Q^2 = 4.6$ GeV^2 are $\alpha_{g_1}(3.27\ \text{GeV})/\pi \simeq 0.09$ and $\alpha_{g_1}(2.14\ \text{GeV})/\pi \simeq 0.11$, corresponding to $\Gamma^{p-n} = 0.190$ and $\Gamma^{p-n} = 0.186$, respectively. Alternatively, for the E142 data, we can use the commensurate scale relation

$$\alpha_{g_1}(Q) = \alpha_\tau(1.145Q)[1 - \frac{\alpha_\tau}{\pi} + \cdots]\ ,$$

and the empirical determination [2] $\alpha_\tau(m_\tau)/\pi \simeq 0.19$ to find a consistent determination $\alpha_{g_1}(1.55\ \text{GeV})/\pi \simeq 0.15$. The uncertainty in the PQCD radiative corrections is thus considerable smaller than usually assumed [19].

The commensurate scale relations between observables can be tested at quite low momentum transfers, even where PQCD relationships would be expected to break down. It is likely that some of the higher twist contributions common to the two observables are also correctly represented by the commensurate scale relations. In contrast, expansions of any observable in $\alpha_{\overline{MS}}(Q)$ must break down at low momentum transfer since $\alpha_{\overline{MS}}(Q)$ becomes singular at $Q = \Lambda_{\overline{MS}}$. (For example, in the t'Hooft scheme where the higher order $\beta_n = 0$ for $n = 2, 3, \ldots$, $\alpha_{\overline{MS}}(Q)$ has a simple pole at $Q = \Lambda_{\overline{MS}}$.) The commensurate scale relations allow tests of QCD without explicit reference to schemes such as $\overline{\mathrm{MS}}$. It is thus reasonable to expect that the series expansions are more convergent when one relates finite observables to each other.

The BLM scale has also recently been used by Lepage and Mackenzie [13] and their co-workers to improve lattice perturbation theory. By using the BLM method one can eliminate $\alpha_{\mathrm{Lattice}}$ in favor of α_V thus avoiding an expansion with artificially large coefficients. The lattice determination, together with the empirical constraints from the heavy quarkonium spectra, promises to provide a well-determined effective charge $\alpha_V(Q)$ which could be adopted as the QCD standard coupling.

After one fixes the renormalization scale μ to the BLM value, it is still useful to compute the logarithmic derivative of the truncated perturbative prediction $d\ell n\, \rho_N / d\ell n \mu$ at the BLM-determined scale. If this derivative is large, or equivalently, if the BLM and PMS scales strongly differ, then one knows that the truncated perturbative expansion cannot be numerically reliable, since the entire series is independent of μ. Note that this is a necessary condition for a reliable series, not a sufficient one, as evidenced by the large coefficients in the positronium and quarkonium decay widths which appear when the scales are set correctly. In the case of the two and three jet decay fractions in e^+e^- annihilation, the BLM and PMS scales strongly differ at low values of the jet discriminant y. Thus, by using this criterion, we establish that the perturbation theory must fail in the small y regime, requiring careful resummation of the $\alpha_s \ell n\, y$ series. [8]

However, if we restrict the analysis to jets with invariant mass $\mathcal{M} < \sqrt{ys}$, with $0.14 > y > 0.05$, then we have an ideal situation, since both the PMS and FAC scales nearly coincide with the BLM scale when one computes jet ratios in the $\overline{\rm MS}$ scheme (See Fig. 1.) *i.e.*, the renormalization scale dependence in this case is minimal at the BLM scale, and the computed NLO coefficient is nearly zero. In fact, Kramer and Lampe [7] find that the BLM scale and the NLO PQCD predictions give a consistent description of the LEP 2-jet and 3-jet data for $0.14 > y > 0.05$ at the Z. Neglecting possible uncertainties due to hadronization effects, this allows a determination of α_s with remarkably small error: [7] $\alpha_{\overline{\rm MS}}(M_z) = 0.107 \pm 0.003$, which corresponds to $\Lambda^{(5)}_{\overline{\rm MS}} = 100 \pm 20$ MeV.

The BLM method and the commensurate scale relations presented in this talk can be applied to the whole range of QCD and standard model processes, making the tests of theory much more sensitive. The method should also improve precision tests of electroweak, supersymmetry and other non-Abelian theories.

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