

NEW QCD RESULTS FROM STRING THEORY*

ZVI BERN

Department of Physics, UCLA, Los Angeles, CA 90024

LANCE DIXON

Stanford Linear Accelerator Center, Stanford University, Stanford, CA 94309

and

DAVID A. KOSOWER

*Service de Physique Théorique de Saclay, Centre d'Etudes de Saclay
F-91191 Gif-sur-Yvette cedex, France*

ABSTRACT

We discuss new results in QCD obtained with string-based methods. These methods were originally derived from superstring theory and are significantly more efficient than conventional Feynman rules. This technology was a key ingredient in the first calculation of the one-loop five-gluon amplitude. We also present a conjecture for a particular one-loop helicity amplitude with an arbitrary number of external gluons.

1. Introduction.

Calculations beyond the leading order in quantum chromodynamics are important to refining our understanding of known physics in present-day and future collider experiments. The discovery of new physics relies to a large extent on the subtraction of known physics from the data. In particular, QCD loop corrections are important but are in general quite formidable to calculate. Intermediate expressions can be many thousands of times larger in size than final expressions. This explosion of terms has been a major obstacle in performing computations required by experiment. Here we discuss new techniques based on string theory which bypass much of the algebra associated with one-loop Feynman diagram computations in gauge theories [1–4]. With the new string-based techniques the one-loop five-gluon amplitudes have been computed yielding a compact form [3]. These amplitudes have not been obtained with traditional techniques.

Recent years have seen substantial progress in improving the situation in tree-level calculations [5–8]. Tree-level matrix elements have been essential for checks of QCD processes and for estimates of QCD backgrounds to new physics searches.

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Here we discuss the one-loop corrections to the amplitudes. The next-to-leading order corrections are important because leading tree-level calculations miss essential physics. Two problems are that the tree-level results exhibit a strong renormalization scale dependence (which does not make physical sense) and that the cone angle dependence is incorrect. The next-to-leading order corrections to a large extent remedy this situation.

The one-loop method discussed here was originally derived from string theory. Although based on string theory it has been summarized in terms of simple rules which require no knowledge of string theory [1]; the structure of these rules can also be understood from conventional field theory through particular gauge choices and organizations of the amplitude [9].

Using methods developed through string theory we have obtained results which have previously not been obtained through conventional means. Besides the five-gluon amplitudes [3], a first calculation of a four-point one-loop gravity amplitude has been performed [10]. We have used the explicit five-gluon results to jump-start a conjecture for a particular one-loop helicity amplitude but with arbitrary numbers of external legs [38].

Using direct string methods we have also calculated the one-loop four-point helicity amplitudes with two external quarks [11], which agree with the calculation of ref. [12].

Is string theory ‘required’ for field theory calculations? To develop and extend the methods string theory has been crucial. To actually evaluate amplitudes there is no need to turn to string theory. The main role of string theory is to provide a principle for discovering compact representations for field theory amplitudes. In particular, given the string-based rules for the one-loop n -gluon amplitudes and the understanding of these rules in field theory [9,13], there does not appear to be a clear way to extend the rules to multi-loops, or to gravity, without referring back to string theory to at least some extent. It is, however, possible to formulate a conventional field theory framework for obtaining much of the efficiency of the string-based method by working backwards from the string-based rules. These field theory ideas can then be applied more generally to gauge theory amplitude calculations which involve non-abelian vertices [9]. For example, some of the ideas obtained from string theory were used to aid the calculation of one-loop five-point amplitudes with external fermions and gluons [14]. Another example is the application of these ideas [15] to weak interaction processes such as $Z \rightarrow 3\gamma$ [16].

2. Difficulty of Loop Computations

An underlying cause of the complexity of QCD calculations is that the non-abelian vertices are relatively complicated. Since the vertices each contain six terms, one encounters a rapidly growing number of terms as one sews together vertices with propagators to form Feynman diagrams. Furthermore, the integrals associated with larger numbers of legs become increasingly complicated.

As a simple example consider the pentagon diagram one would encounter in

a brute force five-point computation. A naive count of the number of terms gives about 6^5 terms. (This count is reduced by the use of on-shell conditions but increased since each internal momentum turns into a sum of external momenta.) Each term is associated with an integral which evaluates to an expression on the order of a page in length. This means that one is faced with about 10^4 pages of algebra for this single diagram. As bad as this situation might seem, it is actually much worse because of the structure of the results. After evaluating the integrals and summing over diagrams one obtains expressions of the form

$$\frac{N_1}{D_1} + \frac{N_2}{D_2} + \dots \quad (1)$$

where the N_i and D_i are the numerators and denominators one encounters when performing the integrals. In general the denominators contain spurious singularities which cancel only after putting large numbers of terms on a common denominator; this unfortunately causes an explosion of terms in the numerators.

The basic observation for being able to improve on conventional computations is that Feynman diagram computations always involve large cancellations amongst the various terms. Anyone who has done a Feynman diagram computation has undoubtedly asked themselves why vast amounts of algebra are required when answers tend to be quite small. A nice example of this is the one-loop four-gluon helicity amplitude

$$A_{4;1}(1^-, 2^+, 3^+, 4^+) = -\left(1 + \frac{n_s}{N} - \frac{n_f}{N}\right) \frac{i}{48\pi^2} \frac{[24]^2 u}{[12] \langle 23 \rangle \langle 34 \rangle [41]} \quad (2)$$

where n_s and n_f are the number of massless complex scalars and Weyl fermions in the fundamental representation of an $SU(N)$ theory. The plus and minus signs associated with each leg denote the helicity and $u = 2k_1 \cdot k_3$ is a Mandelstam variable. The various brackets are spinor inner products defined by [7]

$$\langle j l \rangle = \langle k_j^- | k_l^+ \rangle, \quad [j l] = \langle k_j^+ | k_l^- \rangle$$

where the $|k_l^\pm\rangle$ are Weyl spinors. The spinor inner products are a convenient way to represent helicity amplitudes. (This amplitude has been decomposed using the one-loop Chan-Paton color decomposition [17].) Although this expression fits on a line, a brute force computation performed in the conventional way would start with expressions containing about 10^4 terms. Clearly there is considerable room for improving Feynman diagram computations at one loop. The claim is that string theory coupled with other ideas such as spinor helicity methods provides a way for doing this.

3. Why String Theory is Helpful.

It does not take much string theory for one to realize that string theory can be helpful in gauge theory computations. String theory satisfies a number of properties which indicates its usefulness in gauge theory computations.

- 1) String amplitudes are compact. At each order of perturbation theory there is only a single diagram; this provides for a compact organization, which might lead one to suspect (at least at a hand-waving level) that string theory might be useful for gauge theory calculations.
- 2) In string theory one can switch between fermions and bosons in the loop by changing the world-sheet boundary conditions. This means that whether bosons or fermions circulate in the loop, the same basic string equations describe the two situations. This leads one to suspect that it should be possible to use information from a calculation with fermions in the loop to aid calculations with bosons in the loop. This may be contrasted to the usual Feynman rules one finds in textbooks, where the fermion and gluon Feynman rules do not resemble each other.
- 3) The $N = 4$ superstring amplitude integrands are simple [18]. In particular for the four-point amplitude the kinematic expression factors out of the string integrand. In the field theory limit this implies that the various contributions to a gluon amplitude A satisfy

$$A^{\text{gluon}} + 4A^{\text{fermion}} + 3A^{\text{scalar}} = \text{simple} \quad (3)$$

where the particle labels refer to the particles circulating in the loop. (Note that the massless spectrum of $N = 4$ super-Yang-Mills is one gluon, four Weyl fermions and three complex scalars.) This relation then implies that not only are the integrands of the various contributions related via the previous point, but that they in fact satisfy a simple linear relation. These relations can be used to prevent duplication of effort when calculating the various contributions to a process.

- 4) In string theory (closed string) \sim (open string)². Since closed strings contain gravity and open strings contain Yang-Mills, one might expect that (gravity) \sim (Yang-Mills)². This can be made precise and be used to turn string theory into an extremely efficient computational tool for gravity [10].
- 5) In string theory the loop momentum is implicitly integrated out. This is useful because helicity techniques [7] are most efficient after loop momentum is integrated out.

The strategy is to build a string theory with a spectrum relevant for the field theory of interest, and then to take the infinite string tension limit in order to extract the organizational efficiency of the string in the field theory limit. Once the essential reorganization has been summarized in terms of a set of field theory rules or ideas there is no need to turn to string theory for every new calculation.

4. String Theory and Extraction of Field Theory Limit

The infinite tension limit of a string theory is a field theory [19,18]. In order to use string theory as a computational tool, control of the massless matter content of the string theory is required, because colored massless matter particles can run around the loops. It is possible to build consistent heterotic string theories [20] whose infinite-tension limit is a non-abelian gauge theory where one of the factors is an $SU(N)$ with no matter fields [21]. The technology needed for such a construction is the one used to construct four-dimensional string models [22]; the formulation of Kawai, Lewellen and Tye is particularly simple, although any of the other formulations can be used depending on one's taste. In the original derivation of the string-based rules [1], it was essential to use a consistent string in order to prevent extraneous problems from entering. Without full consistency there would be no guarantee that the final results obtained would be correct. A heterotic string was used in the original derivation of the string-based rules because bosonic strings always contain unwanted massless scalars and tachyons, while four-dimensional type II [23,24] and type I [25] superstrings do not have a rich enough variety of fully consistent models.

Bosonic string constructions are generally much simpler than super or heterotic string constructions so for pedagogical purposes that is what will be discussed here. Heterotic string constructions are needed to provide a consistent derivation of the rules [1], but the procedure for extracting the field theory limit is similar. The open bosonic string discussed here is identical to the one used by Metsaev and Tseytlin [26] to obtain the Yang-Mills β -function from string theory. This string is given by a naive truncation of an oriented open bosonic string to four-dimensions. In this way all massless colored scalars arising from the dimensional compactification are simply thrown away. This string is inconsistent as a fundamental string theory because of the naive truncation of the spectrum. Another technicality is that the string does contain a tachyon, which might be worrisome; however, one can handle this with the prescription that exponentially large terms due to the tachyon should be dropped in the same way that exponentially small terms from the higher mass states are dropped. These potential inconsistencies of the bosonic string are irrelevant in the field-theory limit, as can be verified explicitly using an independent calculation with the fully-consistent heterotic-string formalism. What is important here is the basic structure that emerges from string theory without facing the full technicalities of heterotic string constructions.

In general, an amplitude in string theory is evaluated by performing the Polyakov surface integral [27]

$$A_n \sim \int DX \exp \left[\frac{1}{\alpha'} \int d^2\nu \partial_\alpha X^\mu \partial_\alpha X_\mu \right] V_1 V_2 \cdots V_n \quad (4)$$

where the $V_i \sim \varepsilon_i \cdot \partial X e^{ik \cdot X}$ are the vertex operators for external gluons with polarizations ε . At one-loop this path integral is performed on a world-sheet annulus.

Since the world-sheet bosons are free, Wick's theorem can be used to evaluate the string n -gluon amplitude in terms of the two-point correlation on the annulus

$$\begin{aligned} \langle X_\mu(\nu_1) X^\nu(\nu_2) \rangle &= \delta_\mu^\nu G_B(\nu_{12}) = -\delta_\mu^\nu \left[\log |2 \sinh(\nu_{12})| - \frac{(\nu_{12})^2}{\tau} - 4q \sinh^2(\nu_{12}) \right] \\ &\quad + \mathcal{O}(q^2) \end{aligned} \tag{5}$$

where $\tau = -\log(q)/2$ is the real modular parameter of the annulus, ν_i represents the location of the vertex operator on the annulus and $\nu_{ij} = \nu_i - \nu_j$. (These parameters are π/i times the conventional one in refs. [23,28].) As discussed in ref. [1], in the field theory limit these parameters are proportional to sums of Schwinger proper time parameters. A repeated application of Wick's theorem to evaluate the Polyakov integral yields the string partial amplitude associated with the color trace $\text{Tr}(T^{a_1} T^{a_2} \dots T^{a_n})$

$$\begin{aligned} A_{n;1} &= i \frac{(4\pi)^{\epsilon/2}}{16\pi^2} (\sqrt{2})^n (\alpha')^{n/2-2} \int_0^\infty d\tau \int \prod_{i=1}^{n-1} d\nu_i \theta(\nu_i - \nu_{i+1}) \tau^{-2+\epsilon/2} Z \\ &\quad \times \prod_{i<j}^n \exp \left\{ \alpha' k_i \cdot k_j G_B(\nu_{ij}) + \sqrt{\alpha'} (k_i \cdot \varepsilon_j - k_j \cdot \varepsilon_i) \dot{G}_B(\nu_{ij}) \right. \\ &\quad \left. - \varepsilon_i \cdot \varepsilon_j \ddot{G}_B(\nu_{ij}) \right\} \Big|_{\text{multi-linear}} \end{aligned} \tag{6}$$

where

$$\dot{G}_B(\nu) = \frac{1}{2} \frac{\partial}{\partial \nu} G_B(\nu), \quad \ddot{G}_B(\nu) = \frac{1}{4} \frac{\partial^2}{\partial \nu^2} G_B(\nu) \tag{7}$$

and ν_n is fixed at τ . The 'multi-linear' signifies that after expanding the exponential only terms which are linear in all n polarizations vectors are to be kept. The string oscillator contributions to the partition function are

$$Z = q^{-1} \prod_{n=1}^{\infty} (1 - q^n)^{-2(1-\delta_R \epsilon/2)}. \tag{8}$$

Full consistency of the string demands that the dimension $D = 26$ [29], but for the purposes of obtaining field theory amplitudes $D = 4 - \epsilon$ where ϵ is the dimensional regularization parameter necessary to handle infrared divergences; the regularization parameter δ_R , included in the string partition function, determines the precise form of the regularization [1]. In order to obtain a sensible field theory limit, the leading q^{-1} has been maintained by hand independent of the number of dimensions. (A fully consistent heterotic string such as the one used in ref. [1] does not require any adjustments, such as this one.) The field theory limit of the amplitude (6) yields the pure Yang-Mills contributions to the amplitude including Faddeev-Popov

ghosts. The conventions have been adjusted so that in the field theory limit the number of π 's and 2's which need to be shuffled around are minimized and so that these normalizations agree with the ones used in the heterotic string analysis of refs. [1].

Partial amplitudes associated with two color traces are a bit different since the string vertex operators are located on both boundaries of the annulus; examples can be found in chapter 8 of ref. [28].

In order to take the infinite string tension limit $\alpha' \rightarrow 0$ of the string amplitude (6), it is convenient to first integrate by parts on the string world-sheet in order to remove all \ddot{G}_B from the kinematic factor [1]. (The analysis of the field theory limit can also be performed without the integration-by-parts step so it should not be taken as an essential ingredient to the string-based method.) In open string theory there are potential boundary terms, but these can be removed by an appropriate analytic continuation in external momenta since all the boundary terms contain a factor of $|\nu_i - \nu_j|^{-n - \alpha' k_i \cdot k_j} |_{\nu_i \rightarrow \nu_j} = 0$. (One technicality is that the periodicity on the annulus under $\nu \rightarrow \nu + \tau$ must be used to remove some of the surface terms.) In appendix B of ref. [30] it was proven that all \ddot{G}_B 's can always be eliminated from the kinematic function, by appropriate integration by parts.

In the field theory limit, the contributions to an integrated-by-parts one-loop amplitude can be classified in terms of tree and loop parts. The tree parts are obtained by first extracting the massless poles in the S -matrix before taking the field theory limit of the loop. Examples of these kinematic poles are found in the regions where $\nu_i \rightarrow \nu_j$ and are of the form

$$\int d\nu_i \frac{1}{\nu_{ij}^{1 + \alpha' k_i \cdot k_j}} \longrightarrow -\frac{1}{\alpha' k_i \cdot k_j} \quad (\alpha' \rightarrow 0). \quad (9)$$

In general, the kinematic poles extracted in this way correspond to the poles of a scalar ϕ^3 diagram.

After kinematic poles have been extracted, the field theory limit of the loop is needed. This is obtained by taking $\tau, |\nu| \rightarrow \infty$ which corresponds to squeezing the annulus down to a field theory loop. The values of the Green functions in this limit are

$$\begin{aligned} \exp(G_B(\nu)) &\rightarrow \exp\left(\frac{\nu^2}{\tau} - |\nu|\right) \times \text{constant} \\ \dot{G}_B(\nu) &\rightarrow \frac{\nu}{\tau} - \text{sign}(\nu)\left(\frac{1}{2} + e^{-2|\nu|} - qe^{2|\nu|}\right). \end{aligned} \quad (10)$$

The exponentiated bosonic Green function was not expanded beyond $\mathcal{O}(q^0)$; after carrying out the integration by parts procedure the higher order terms do not contribute since they carry too many explicit powers of α' . For \dot{G}_B , terms through $\mathcal{O}(q)$ should be kept due to the presence of the overall q^{-1} in the string amplitude (6).

In the field theory limit two types of loop contributions are obtained depending on whether a power of q is extracted from the string partition function or from the Green functions. For the former contribution one simply keeps the leading order contributions from the bosonic Green functions. This type of contribution is described by the bosonic zero-mode [28] or loop momentum integral of the string [9]. A product of \dot{G}_B 's contains exponentially growing and decaying terms as well as terms which are constant. In general, when terms proportional to $q = e^{-2\tau}$ are extracted from a product of \dot{G}_B in order to cancel the overall q^{-1} , a factor of the form

$$\exp\left[\left(|x_k - x_l| - \sum |x_i - x_j|\right)\tau\right] \quad (11)$$

is obtained where $x_i \equiv \nu_i/\tau$. In order to avoid exponential suppression or growth as $\alpha' \rightarrow 0$ the sum must add up to cancel within the exponential exactly. This will happen only if each x_i which appears with a positive sign also appears with a negative sign after expressing the absolute values in terms of the x_i s directly. The correct prescription for dealing with exponentially growing terms due to the tachyon is to simply drop them in the same way that exponentially decaying terms are dropped. (The exponential growth is an artifact of the Schwinger proper time representation of tachyonic propagators.)

The result of collecting those terms where the exponential terms completely cancel is that only those which form a *cycle* of \dot{G}_B 's, defined to be a product of \dot{G}_B 's with indices arranged in the form

$$\dot{G}_B(\nu_{i_1 i_2}) \dot{G}_B(\nu_{i_2 i_3}) \cdots \dot{G}_B(\nu_{i_m i_1}), \quad (12)$$

will not vanish. Furthermore, the cyclic ordering of the indices must follow the same ordering of the corresponding legs in the partial amplitude.

The analysis of the field theory limit of a superstring is quite similar. A superstring is essential in order to be able to include space-time fermions into the string formalism and provides a more consistent framework than bosonic strings. A detailed discussion of the field theory limit of a heterotic string has been given in ref. [1].

By organizing the contributions obtained in the field theory limit a set of string-based rules for calculating gluon amplitudes can be obtained [1,2]. The string-based rules work by specifying a set of substitution rules on the string kinematic expression

$$\mathcal{K} = \int \prod_{i=1}^n dx_i \prod_{i<j}^n \exp\left(k_i \cdot k_j G_B^{i,j} + (k_i \cdot \varepsilon_j - k_j \cdot \varepsilon_i) \dot{G}_B^{i,j} - \varepsilon_i \cdot \varepsilon_j \ddot{G}_B^{i,j}\right) \Big|_{\text{multi-linear}} \quad (13)$$

to obtain the contributions to various diagrams which represent the various corners of moduli space. This 'master formula' contains all information for all diagrams and particle types which can circulate in the loop. Although this is the kinematic formula for a bosonic string, one can use world-sheet supersymmetry in a superstring

to relate the contributions of the world-sheet fermions to those of the world-sheet bosons [2]; in this way the kinematic expression associated with the bosons can be used to summarize all contributions. Alternatively one can specify rules which act directly on the superstring kinematic expression, which was the original form of the rules presented in ref. [1]. An example of a substitution rule is

$$\dot{G}_B^{i,j} \rightarrow \frac{1}{2} + \sum_{k=i+1}^j a_k \quad (14)$$

where the a_k are ordinary Feynman parameters. There are a variety of other substitution rules which depend on the particular corner of string moduli space under consideration and are described more fully in refs. [1,4].

Because the contribution of any type of particle is contained in the master formula, relationships between fermion and boson contributions become apparent within the integrands of each diagram. This can be used to obtain even further simplifications; once the fermion loop contribution to the n -gluon amplitude has been computed, calculating the gluon loop contribution is relatively simple [3].

In this way one obtains a set of Feynman parameter polynomials which are far more compact than one would obtain by traditional Feynman diagram methods. The Feynman parameters must then be integrated in order to obtain the amplitudes.

5. Field Theory Interpretation

Since a conventional field theory interpretation of string-based rules has been developed [9], one can use string-motivated ideas directly in a conventional field theory setting. The following strategy incorporates the ideas that were extracted from the mapping between field theory and string theory and greatly improves the calculational efficiency over traditional Feynman diagram computations. First background field Feynman gauge [31] should be used in calculations where a non-abelian vertex appears in the loop. This gauge is useful for constructing the one-particle-irreducible diagrams, since the vertices are particularly well suited for loop calculations. These one-particle irreducible diagrams describe a gauge-invariant effective action. For sewing trees onto the one-particle irreducible loop diagrams the Gervais-Neveu gauge [32] is a particularly efficient gauge because of the simplicity of the three- and four-point vertices. (Although it might seem strange that two different gauge choices are used for the loop and tree parts of the Feynman diagrams, in the background field method this has been justified by Abbott, Grisaru and Schaeffer [31].) In general, one should use color ordered [5,30] vertices in order to minimize the number of diagrams which must be explicitly computed. For internal fermions it is best to use the second order formalism described in ref. [9] because then the gauge boson and fermion contributions are quite similar. In particular for $N = 4$ supersymmetry there are large cancellations between the various contributions to the loops [18]. In this way a good fraction of the work does not have to be duplicated

for each type of particle circulating in the loop in agreement with the string theory expectations.

Spinor helicity methods [7] are also important to help minimize the amount of required algebra. Since spinor helicity methods do not handle off-shell loop momentum efficiently it should be integrated out early in the calculation to obtain a representation in terms of Feynman parameters. In order to minimize the number of terms which appear, spinor helicity should be applied on a term-by-term basis in the numerator as one integrates out the loop momentum. An alternative approach which implicitly and systematically integrates out the loop momentum is the electric circuit analogy discussed in refs. [33].

In this way one can attain many of the simplifications of a more direct string approach. Gravity does, however, provide a concrete example where a direct string-based computation is significantly more efficient than a computation based on the above field theory ideas [10].

6. Explicit Calculations

Using the string-based methods discussed above we have performed a computation of the one-loop five gluon amplitudes [3]. Additional ingredients which enter into this calculation are a simple integration method for the pentagon parameter integrals [34] and improvements in the spinor helicity method.

The finite helicity one-loop amplitudes associated with the color trace $\text{Tr}(T^{a_1} \dots T^{a_5})$ are

$$\begin{aligned}
A_{5;1}(1^+, 2^+, 3^+, 4^+, 5^+) &= \left(1 + \frac{n_s}{N} - \frac{n_f}{N}\right) \\
&\times \frac{i}{48\pi^2} \frac{\langle 12 \rangle [12] \langle 23 \rangle [23] + \langle 45 \rangle [45] \langle 51 \rangle [51] + \langle 23 \rangle \langle 45 \rangle [25] [34]}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle}
\end{aligned} \tag{15}$$

$$\begin{aligned}
A_{5;1}(1^-, 2^+, 3^+, 4^+, 5^+) &= \left(1 + \frac{n_s}{N} - \frac{n_f}{N}\right) \\
&\times \frac{i}{48\pi^2} \frac{1}{\langle 34 \rangle^2} \left[-\frac{[25]^3}{[12][51]} + \frac{\langle 14 \rangle^3 [45] \langle 35 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 45 \rangle^2} - \frac{\langle 13 \rangle^3 [32] \langle 42 \rangle}{\langle 15 \rangle \langle 54 \rangle \langle 32 \rangle^2} \right].
\end{aligned} \tag{16}$$

The infrared-divergent amplitudes (which are the ones which interfere with the tree diagrams to produce the next-to-leading order corrections to the cross-section) are given in ref. [3] and are more complicated. These amplitudes have not been obtained with traditional techniques used, for example, by Ellis and Sexton [35].

We have also calculated four-point matrix elements with two external quarks by a direct string approach [11]. Five-point matrix elements have also been obtained,

using field theory. An example of one of the amplitudes is

$$\begin{aligned}
A_{5;1}(1_{\bar{q}}^-, 2_q^+, 3^+, 4^+, 5^+) &= \frac{i}{16\pi^2} \left[-\frac{1}{2} \left(1 + \frac{1}{N^2} \right) \frac{\langle 12 \rangle (\langle 12 \rangle [23] \langle 31 \rangle + \langle 14 \rangle [45] \langle 51 \rangle)}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \right. \\
&\quad \left. + \frac{1}{3} \left(1 - \frac{n_f}{N} + \frac{n_s}{N} \right) \left(\frac{\langle 13 \rangle [34] \langle 41 \rangle^2}{\langle 12 \rangle \langle 34 \rangle^2 \langle 45 \rangle \langle 51 \rangle} + \frac{\langle 14 \rangle \langle 24 \rangle [45] \langle 51 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle^2} + \frac{[23] [25]}{[12] \langle 34 \rangle \langle 45 \rangle} \right) \right].
\end{aligned} \tag{17}$$

The remaining five-point amplitudes with external fermions are more complicated and will be presented elsewhere [14]. The calculational ideas motivated by the string organization were useful in obtaining these results.

7. String-Based Methods for Gravity

Another application of the string-based technique is to gravity [10]. Roughly speaking the structure of string theory implies that

$$(\text{Closed String}) \sim (\text{Open String})^2. \tag{18}$$

Since closed strings contain gravity and open strings contain gauge theory one might expect that

$$(\text{Gravity}) \sim (\text{Yang-Mills})^2. \tag{19}$$

This relationship can be made precise and turned into an extremely efficient computational tool for perturbative gravity amplitudes. At tree-level Berends, Giele and Kuijf [36] have made use of this relationship to obtain tree-level gravity amplitudes from known Yang-Mills amplitudes. At one-loop this relationship can also be made precise [10]; in particular, the calculation of the one-loop four-graviton amplitude with one minus and three plus helicities is rather easy by making use of string-based rules modified for the case of gravity coupled to massless matter. The result of such a calculation is given by

$$A(1^-, 2^+, 3^+, 4^+) = \frac{i\kappa^4}{(4\pi)^2} \frac{1}{5760} (N_b - N_f) \frac{s^2 t^2}{u^2} (u^2 - st) \left(\frac{[24]^2}{[12] \langle 23 \rangle \langle 34 \rangle [41]} \right)^2 \tag{20}$$

where κ is the gravitational coupling, N_b is the number of physical bosonic states and N_f is the number of fermionic states in the particular theory of gravity under consideration. (Here s, t, u are Mandelstam variables.) The fact that any massless state gives an identical contribution up to a sign is in agreement with the supersymmetry identities [37].

This type of calculation would be exceedingly difficult with conventional techniques, given the complexity of the gravity three- and four-point field theory vertices. This may be compared to the string-based technique where the calculation of the above helicity amplitude is reduced to an elementary exercise. It is amusing that

the string-based gravity calculation is only slightly more difficult than the gluon calculation. It is intriguing that in terms of conventional field theory the required reorganization to obtain this simplicity is fairly difficult to guess without some input from string theory.

8. A Conjecture for an Arbitrary Number of External Legs.

It is possible to construct a conjecture for the n -point all plus gluon helicity amplitude [38], using properties of the amplitude as two legs become collinear, and using the explicitly calculated five-point amplitude $A_{5;1}(1^+, 2^+, 3^+, 4^+, 5^+)$ to jump-start the conjecture. This conjecture is analogous to the one for maximal helicity violating tree-amplitudes formulated by Parke and Taylor [39] and proven by Berends and Giele [6].

The conjecture for the gluon contribution to the loop is given by

$$A_{n;1}(1^+, 2^+, \dots, n^+) = \frac{i}{96\pi^2} \frac{E_n + O_n}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}, \quad (21)$$

where the parity-odd terms are given by

$$O_n = 4i \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq n-1} \varepsilon_{\mu\nu\rho\sigma} k_{i_1}^\mu k_{i_2}^\nu k_{i_3}^\rho k_{i_4}^\sigma. \quad (22)$$

To describe the even terms define

$$\begin{aligned} t_i^{(p)} &= (k_i + k_{i+1} + \cdots + k_{i+p-1})^2, \\ t_i^{(2)} &= (k_i + k_{i+1})^2 = s_{i,i+1}, \\ t_i^{(1)} &= 0, \end{aligned} \quad (23)$$

and with all indices mod n :

$$\begin{aligned} X_{p,p}^{(n)} &= \sum_{i=1}^n t_i^{(p)} t_{i+1}^{(p)}, \\ X_{p-1,p+1}^{(n)} &= \sum_{i=1}^n t_{i+1}^{(p-1)} t_i^{(p+1)}. \end{aligned} \quad (24)$$

Then the ansätze for odd and even n are

$$\begin{aligned} E_{2m+1} &= \sum_{p=2}^m \left(X_{p,p}^{(2m+1)} - X_{p-1,p+1}^{(2m+1)} \right), \\ E_{2m} &= \sum_{p=2}^{m-1} \left(X_{p,p}^{(2m)} - X_{p-1,p+1}^{(2m)} \right) + \frac{1}{2} \left(X_{m,m}^{(2m)} - X_{m-1,m+1}^{(2m)} \right). \end{aligned} \quad (25)$$

Note that $X_{1,3}^{(n)} = 0$, since $t_i^{(1)} = 0$. The expression (21) is cyclicly symmetric, and in the limit that two legs become collinear, is proportional to the corresponding $(n - 1)$ -point amplitude. This conjecture will be discussed more fully elsewhere [38]. (A very recent paper by Mahlon gives the result for the corresponding all- n expression in QED [40].)

9. Summary and Conclusions

In this talk, we reviewed the string-based method for evaluating one-loop n -gluon amplitudes [1]. The method was originally derived by taking the field theory limit of an appropriately constructed [21] four-dimensional string theory [22]. Using the string-based method, together with a simple integration method [34] and improvements in the spinor helicity method, a first calculation of all five-gluon helicity amplitudes has been performed [3]. The five-point amplitude can be used to jump-start a conjecture for the all plus helicity amplitude with an arbitrary number of external legs [38]. Using a direct string approach we have also calculated four-point helicity amplitudes with two external quarks and gluons [11] which agree with ref. [12]. The first calculation of a one-loop four-point graviton amplitude has also been performed in a direct string theory approach [10].

The string based rules have been reinterpreted in terms of field theory in refs. [9,13]. The field theory ideas obtained in this way can then be applied more widely to a variety of problems to aid in computations. This was used as an aid in the calculation of five-point one-loop amplitudes with external gluons and quarks [14]. Another example is the application of these ideas to certain weak interaction processes to significantly improve their calculational efficiency [15]. Some progress has also been made on the extension of string-based methods to multiloops [41].

In summary, string theory has been useful for calculations of gauge theory amplitudes required by experiments.

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References

- [1] Z. Bern and D.A. Kosower, Phys. Rev. Lett. 66:1669 (1991); Nucl. Phys. B379:451 (1992);
Z. Bern and D.A. Kosower, in *Proceedings of the PASCOS-91 Symposium*, eds. P. Nath and S. Reucroft.
- [2] Z. Bern, Phys. Lett. 296B:85 (1992).
- [3] Z. Bern, L. Dixon and D.A. Kosower, Phys. Rev. Lett. 70:2677 (1993).
- [4] Z. Bern, UCLA/93/TEP/5, hep-ph/9304249, proceedings of TASI 1992.
- [5] F.A. Berends and W.T. Giele, Nucl. Phys. B294:700 (1987);

- M. Mangano and S.J. Parke, Nucl. Phys. B299:673 (1988);
M. Mangano, S. Parke and Z. Xu, Nucl. Phys. B298:653 (1988);
M. Mangano, Nucl. Phys. B309:461 (1988).
- [6] F.A. Berends and W.T. Giele, Nucl. Phys. B306:759 (1988);
D.A. Kosower, Nucl. Phys. B335:23 (1990).
- [7] F. A. Berends, R. Kleiss, P. De Causmaecker, R. Gastmans and T. T. Wu,
Phys. Lett. 103B:124 (1981);
P. De Causmaecker, R. Gastmans, W. Troost and T. T. Wu, Nucl. Phys. B206:53
(1982);
R. Kleiss and W. J. Stirling, Nucl. Phys. B262:235 (1985);
J. F. Gunion and Z. Kunszt, Phys. Lett. 161B:333 (1985);
R. Gastmans and T.T. Wu, *The Ubiquitous Photon: Helicity Method for QED
and QCD* (Clarendon Press) (1990) ;
Z. Xu, D.-H. Zhang and L. Chang, Nucl. Phys. B291:392 (1987).
- [8] M. Mangano and S.J. Parke, Phys. Rep. 200:301 (1991).
- [9] Z. Bern and D.C. Dunbar, Nucl. Phys. B379:562 (1992).
- [10] Z. Bern, D.C. Dunbar and T. Shimada, Phys. Lett. B312:277 (1993).
- [11] Z. Bern, L. Dixon and D. Kosower, unpublished.
- [12] Z. Kunszt, A. Signer and Z. Trocsanyi, preprint ETH-TH/93-11, hep-ph/9305239,
to appear in Nucl. Phys. B.
- [13] M.J. Strassler, Nucl. Phys. B385:145 (1992);
M.G. Schmidt and C. Schubert, preprint HD-THEP-93-24, hep-th/9309055.
- [14] Z. Bern, L. Dixon and D.A. Kosower, in preparation.
- [15] Z. Bern and A.G. Morgan, preprint UCLA/93/TEP/36, DTP/93/80.
- [16] M. Baillargeon and F. Boudjema, Phys. Lett. B272:158 (1991);
X.Y. Pham, Phys. Lett. B272:373 (1991);
F.-X. Dong, X.-D. Jiang and X.-J. Zhou, Beijing preprint BIHEP-TH-92-32;
E.W.N. Glover and A.G. Morgan, DPT/93/4.
- [17] J.E. Paton and H.M. Chan, Nucl. Phys. B10:516 (1969).
- [18] M.B. Green, J.H. Schwarz and L. Brink, Nucl. Phys. B198:472 (1982).
- [19] J. Scherk, Nucl. Phys. B31 (1971) 222;
A. Neveu and J. Scherk, Nucl. Phys. B36 (1972) 155;
J. Minahan, Nucl. Phys. B298:36 (1988).
- [20] D.J. Gross, J.A. Harvey, E. Martinec and R. Rohm, Nucl. Phys. B256:253
(1985); Nucl. Phys. B267:75 (1986).
- [21] Z. Bern and D.A. Kosower, Phys. Rev. D38:1888 (1988);
Z. Bern and D. A. Kosower, in proceedings, *Perspectives in String Theory*, eds.
P. Di Vecchia and J. L. Petersen, Copenhagen 1987.
- [22] L. Dixon, J. Harvey, C. Vafa and E. Witten, Nucl. Phys. B261:678 (1985),
B274:285 (1986);
K.S. Narain, Phys. Lett. 169B:41 (1986);
W. Lerche, D. Lust and A.N. Schellekens, Nucl. Phys. B287:477 (1987);

- H. Kawai, D.C. Lewellen and S.-H.H. Tye, Nucl. Phys. B288:1 (1987);
 K.S. Narain, M.H. Sarmadi and C. Vafa, Nucl. Phys. B288:551 (1987);
 I. Antoniadis, C.P. Bachas and C. Kounnas, Nucl. Phys. B289:87 (1987).
- [23] J.H. Schwarz, Phys. Reports 89:223 (1982).
- [24] L. Dixon, V. Kaplunovsky and C. Vafa, Nucl. Phys. B294:43 (1987);
 H. Kawai, D.C. Lewellen and S.-H. H. Tye, Phys. Lett. 191B:63 (1987).
- [25] Z. Bern and D.C. Dunbar, Phys. Rev. Lett. 64:827 (1990); Phys. Lett. 242B:175 (1990).
- [26] R.R. Metsaev and A.A. Tseytlin, Nucl. Phys. B298:109 (1988).
- [27] A.M. Polyakov, Phys. Lett. 103B:207 (1981); 103B:211 (1981).
- [28] M.B. Green, J.H. Schwarz, and E. Witten, *Superstring Theory* (Cambridge University Press) (1987).
- [29] C. Lovelace, Phys. Lett. 34B:500 (1971).
- [30] Z. Bern and D.A. Kosower, Nucl. Phys. B362:389 (1991).
- [31] G. 't Hooft, in Acta Universitatis Wratislavenensis no. 38, 12th Winter School of Theoretical Physics in Karpacz; *Functional and Probabilistic Methods in Quantum Field Theory*, Vol. 1 (1975);
 B.S. DeWitt, in Quantum gravity II, eds. C. Isham, R. Penrose and D. Sciama (Oxford, 1981);
 L.F. Abbott, Nucl. Phys. B185:189 (1981);
 L.F. Abbott, M.T. Grisaru and R.K. Schaefer, Nucl. Phys. B229:372 (1983).
- [32] J.L. Gervais and A. Neveu, Nucl. Phys. B46:381 (1972).
- [33] J.D. Bjorken, Stanford Ph.D. thesis (1958);
 J.D. Bjorken and S.D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, 1965);
 J. Mathews, Phys. Rev. 113:381 (1959);
 S. Coleman and R. Norton, Nuovo Cimento 38:438 (1965);
 C.S. Lam and J.P. Lebrun, Nuovo Cimento 59A:397 (1969);
 C.S. Lam, Nucl. Phys. B397:143 (1993).
- [34] Z. Bern, L. Dixon and D.A. Kosower, Phys. Lett. B302:299,1993; SLAC-PUB-5947, Nucl. Phys. B, to appear;
 R.K. Ellis, W. Giele and E. Yehudai, in progress.
- [35] R.K. Ellis and J.C. Sexton, Nucl. Phys. B269:445 (1986).
- [36] F.A. Berends, W.T. Giele and H. Kuijf, Phys. Lett. 211B:91 (1988).
- [37] M.T. Grisaru, H.N. Pendleton and P. van Nieuwenhuizen, Phys. Rev. D15:996 (1977);
 M.T. Grisaru and H.N. Pendleton, Nucl. Phys. B124:81 (1977);
 S.J. Parke and T. Taylor, Phys. Lett. B157:81 (1985);
 Z. Kunszt, Nucl. Phys. B271:333 (1986);
 M.L. Mangano and S.J. Parke, Phys. Rep. 200:301 (1991).
- [38] Z. Bern, G. Chalmers, L. Dixon, D. Kosower, in preparation.
- [39] S.J. Parke and T.R. Taylor, Phys. Rev. Lett. 56:2459,1986.
- [40] G.D. Mahlon, preprint Fermilab-Pub-93/327-T, hep-ph/9311213.

- [41] K. Roland, Phys. Lett. B289:148 (1992); preprint SISSA/ISAS 131-93-EP;
G. Cristofano, R. Marotta and K. Roland, Nucl. Phys. B392:345 (1993).