SLAC-PUB-6383
December 1993

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Submitted to
Physical Review Letters

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Work supported by Department of Energy contract DE–AC03–76SF00515.

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ABSTRACT

We consider the constraints imposed on meson wave functions by unitarity and by the favored values of the semileptonic decay constants. We show that they require unexpectedly large values of the intrinsic transverse momentum, especially in the case of the B meson. We also obtain a constraint on the heavy quark momentum fraction.

In this report, we show that nontrivial bounds from unitarity can be derived, relating meson decay constants to the transverse momentum distribution of the quark constituents.

A heuristic overview of our procedure is as follows. We form positive-definite integrals containing the two-particle wave function $\psi_{q\bar{q}}(x, k_\perp)$, whose integral corresponds to the meson decay constant f_h, and its square, whose integral is constrained by unitarity. We use the condition of positivity to derive constraints on the behavior of the wave function.

The decay constant is defined by

$$\frac{f_h}{2\sqrt{2N_c}} = \int_0^1 \mathrm{d}x \int^Q \frac{\mathrm{d}^2 k_\perp}{16\pi^3} \quad \psi_{q\bar{q}}(x, k_\perp; Q) + O(Q^{-2}) \quad . \tag{1}$$

The wave function $\psi_{q\bar{q}}$ is weakly dependent on the separation scale Q; we will ignore this dependence, and assume that Q is much larger than a typical intrinsic transverse momentum. We will always assume that the decay constant is real and positive, thus fixing the phase of $\psi_{q\bar{q}}$. For pseudoscalar mesons, the decay constant f_h is entirely independent of Q.

The light-cone wave function must also satisfy

$$\int_0^1 \mathrm{d}x \int^Q \frac{\mathrm{d}^2 k_\perp}{16\pi^3} \quad |\psi_{q\bar{q}}|^2 \leq P_v \ , \tag{2}$$

where $P_v \leq 1$ is the probability to find the meson in its valence state.

Now, consider a region R of $(x, k_\perp)$–space. Define

$$A[R] \;=\; \int_R \frac{\mathrm{d}x\ \mathrm{d}^2k_\perp}{16\pi^3}\ ,$$

$$F[R] \;=\; 2\sqrt{2N_c}\quad \mathrm{Re}\left[\ \int_R \frac{\mathrm{d}x\ \mathrm{d}^2k_\perp}{16\pi^3}\quad \psi_{q\bar{q}}(x, k_\perp)\ \right]\quad .$$

Thus A is the "area" of the region R, and F is the part of the decay constant contributed by the region R. Then we have

$$0 \;\leq\; \int_R \frac{\mathrm{d}x\ \mathrm{d}^2k_\perp}{16\pi^3}\ |\psi_{q\bar{q}}(x, k_\perp) - b|^2 \;\leq\; P_v - \frac{bF[R]}{\sqrt{2N_c}} + b^2 A[R]\quad , \tag{3}$$

so that

$$F[R] \;\leq\; \sqrt{2N_c}\quad \frac{P_v + b^2 A[R]}{b} \tag{4}$$

for all regions R and positive real b. This modest equation turns out to have significant consequences.

Setting $b = \left(P_v/A[R]\right)^{1/2}$, we obtain

$$F[R] \;\leq\; 2\sqrt{2N_c P_v A[R]}\quad ; \tag{5}$$

if we set $P_v = 1$, the contribution to the decay constant from the region R cannot exceed this bound without violating unitarity. More realistic choices of P_v lead to more stringent bounds on $F[R]$.

For example, consider the π meson, whose decay constant is $f_\pi = 133$ MeV. Define

$$\delta\mathcal{M}^2 \;\equiv\; \frac{k_\perp^2}{x} + \frac{k_\perp^2}{1-x} - m_\pi^2 \;=\; \frac{k_\perp^2}{x\bar{x}} - m_\pi^2\quad ,$$

and let $R \equiv \{(x, k_\perp) : \delta\mathcal{M}^2 \leq \mathcal{M}^2_{\rm max}\}$. Then $A[R] = (\mathcal{M}^2_{\rm max} + m_\pi^2)/96\pi^2$, and (with $N_c = 3$) we have

$$F[R] \leq \frac{\sqrt{P_v(\mathcal{M}^2_{\rm max} + m_\pi^2)}}{2\pi} \quad . \tag{6}$$

Thus unitarity requires that half of the pion decay constant must arise from the region in which $\delta\mathcal{M}^2 > 0.15\mathrm{GeV}^2$, and at least 13% from the region where $\delta\mathcal{M}^2 > 0.5\mathrm{GeV}^2$. The contribution to the wave function from regions of sizable intrinsic transverse momentum is thus very substantial.

Brodsky *et al.* [2] have estimated that $P_v(\pi) \simeq 0.25$; using this estimate in eq. (5) leads to the conclusion that half of the pion decay constant arises from the region in which $\delta\mathcal{M}^2 > 0.66$ GeV^2, and 39% from $\delta\mathcal{M}^2 > 1$ GeV^2. The latter virtuality corresponds to $|k_\perp| = \sqrt{x(1-x)} \times 505$ MeV.

Alternatively, we may consider a region $R \equiv \left\{(x, k_\perp) : k_\perp^2 < Q^2\right\}$. Clearly then $A[R] = Q^2/16\pi^2$, and we have $F[R] < Q\sqrt{2N_cP_v}/2\pi$. If $P_v = 0.25$, we obtain $Q \geq 5.13F[R]$, so that for example 26% of the pion decay constant must be contributed by the region where $|k_\perp| > 500$ MeV.

Even more severe constraints can be derived for the B meson, due to the unexpectedly large decay constant $f_B \gtrsim 190$ MeV [3,4] and to the expectation that the heavy b quark should carry the bulk of the longitudinal momentum. In this case, we define

$$\delta\mathcal{M}^2 \equiv \frac{k_\perp^2}{1-x} + \frac{k_\perp^2 + m_b^2}{x} - m_B^2 = \frac{k_\perp^2}{x(1-x)} + \frac{m_b^2}{x} - m_B^2 \quad .$$

Defining R as above, we obtain the bound

$$F[R] \leq \frac{\sqrt{P_v}}{2\pi} \frac{\left(\mathcal{M}^2_{\rm max} + m_B^2 - m_b^2\right)^{3/2}}{(\mathcal{M}^2_{\rm max} + m_B^2)} \quad . \tag{7}$$

Using $m_B = 5.28$ GeV and $m_b = 4.7$ GeV, we find that the region $\delta\mathcal{M}^2 < 4.0$ GeV^2 can support only 100 MeV of the decay constant, and $\delta\mathcal{M}^2 < 6.7$ GeV^2 only 150 MeV. Current lattice estimates tend to cluster around $f_B = 190$ MeV [4], while calculations using heavy-quark symmetry suggest $f_B = 240$ MeV [3]. Thus even the most conservative estimates of the decay constant require the $b\bar{q}$ states to carry a very substantial light-cone virtuality.

A plausible upper bound for $P_v(B)$ is $P_v(B) \leq \sqrt{P_v(\pi)}$; this estimate arises from the assumption that gluons in the meson wavefunction are directly associated with one of the valence quarks, and that gluon radiation from the heavy quark is entirely suppressed. In actuality, we expect that this somewhat overestimates $P_v(B)$, and thus that $P_v(B) = 0.5$ will lead to fairly conservative conclusions.

Inserting $f_B = 190$ MeV and $P_v(B) = 0.5$ into eq. (7), we find that half of f_B must be contributed by the region $\delta\mathcal{M}^2 > 5.9$ GeV^2, and 23% by the region in which $\delta\mathcal{M}^2 > 10$ GeV^2.

The numerically large value of f_B can only be consistent with unitarity if the B wave function in the $q\bar{q}$ state is greatly spread out in momentum space. For example, at $x = 0.9$ the value $\delta\mathcal{M}^2 = 10$ GeV^2 corresponds to $|k_\perp| = 0.97$ GeV, and at $x = 0.8$ to $|k_\perp| = 1.09$ GeV; see Fig. 1. Such large transverse momenta, and sizable values of $1 - x$, must be typical of the B meson.

In place of the constant b of eq. (3), we can insert an arbitrary function $B(x, k_\perp)$. This allows us to obtain unitarity bounds on the contribution to the moment $\langle B \rangle$ from a region R.

The most interesting such constraints arise when we consider the contribution to $\langle Q^2 - k_\perp^2 \rangle$ from the region $k_\perp^2 < Q^2$. With $B \equiv b(Q^2 - k_\perp^2)$, we obtain the bound

$$\mathrm{Re}\Big[\int_{k_\perp^2 < Q^2} \frac{\mathrm{d}^2 k_\perp}{16\pi^2} (Q^2 - k_\perp^2) \psi \Big] \leq \frac{Q^3}{4\pi} \sqrt{\frac{P_v}{3}} \; .$$

Thus the contribution to the integral representing $f_\pi \langle k_\perp^2 \rangle$ from the region $k_\perp^2 < Q^2$ is bounded below by the constraint of unitarity, while that from the region $k_\perp^2 > Q^2$ is greater than the corresponding integral with $k_\perp^2 \to Q^2$. Adding these two bounds, we obtain

$$\langle k_\perp^2 \rangle \geq Q^2 - \frac{Q^3}{2\pi f_\pi} \sqrt{\frac{2 N_c P_v(\pi)}{3}};$$

since this holds for all Q, we obtain

$$\langle k_\perp^2 \rangle \geq \frac{8\pi^2}{9} \frac{f_\pi^2}{N_c P_v} \to \frac{32\pi^2}{27} f_\pi^2 = (455 \text{ MeV})^2;$$

in the final step, we have inserted the favored value $P_v(\pi) = 0.25$.

We can repeat this process for the B, or any other, meson. With the assumption that $P_v(B) = 0.5$, we obtain $\langle k_\perp^2 \rangle \geq (2.4 f_B)^2$. Here the intimate connection between the decay constant and the spread in momentum space is made manifest. Of course, the resulting restrictions on the B meson are rather weak, since the region R in this case includes all values of x.

We can correct this deficiency by using the function $B = b\big(\mathcal{M}_{\max}^2 - \delta\mathcal{M}^2\big)$ to constrain the moment of the virtuality. For nonzero masses, the resulting analytic formulae are quite inconvenient; however, in the phenomenologically interesting region 150 MeV $< f_B/\sqrt{P_v(B)} <$ 400 MeV, the lower bounds lie above the linear bound

$$\langle \delta\mathcal{M}^2 \rangle > (33 \text{ GeV}) \Big(\frac{f_B}{\sqrt{P_v(B)}} - 130 \text{ MeV} \Big).$$

The latter is thus a rigorous bound on the moment associated with the light-cone virtuality. For example, if $f_B = 270\sqrt{P_v(B)}$ MeV, we obtain $\langle \delta\mathcal{M}^2 \rangle > 4.8 \text{ GeV}^2$.

We can repeat the analysis with the π or any other meson; for example, for the pion we obtain

$$\langle \delta\mathcal{M}^2 \rangle \geq \Big(\frac{0.31 \text{ GeV}^2}{P_v(\pi)} - m_\pi^2 \Big) \simeq 1.2 \text{ GeV}^2.$$

Implicit in the above derivations is the assumption that the real part of the tail of the wavefunction has the same sign as the decay constant. At large values of the transverse momentum, this is a good assumption, since the one-gluon exchange kernel whose contribution dominates the wavefunction at large momentum transfer [1] is real and positive.

For example, in the derivation of the lower bound on $\langle k_\perp^2 \rangle$, the value of Q used is roughly $3f_h/\sqrt{P_v} \sim 800$ MeV. While not extremely large, this momentum transfer is sufficient to make the implicit assumption a quite plausible one.

If we make the further assumption that the wave function depends only on the virtuality of the intermediate state, $\psi(x, k_\perp) = \psi(\delta\mathcal{M}^2)$ and the measure of integration over the invariant phase space is

$$\int_{m_b^2 - m_B^2} \frac{(\delta\mathcal{M}^2 + m_B^2 - m_b^2)^2 \, (\delta\mathcal{M}^2 + m_B^2 + 2m_b^2) \, \mathrm{d}\delta\mathcal{M}^2}{96\pi^2(\delta\mathcal{M}^2 + m_B^2)^3} \qquad (8)$$

It is then a simple problem in the calculus of variations to maximize the functional

$$\langle x^1 \rangle \equiv \frac{\int x\psi(x,k_\perp)}{\int \psi(x,k_\perp)} , \tag{9}$$

the first moment of the distribution amplitude, subject to the constraints of eq. (2) and a fixed value of f_B. The extremal function has the form

$$\psi \propto \left(\bar{x^1} - x_0\right) \theta(\bar{x^1} - x_0),$$

where

$$\bar{x^1} \equiv \frac{1}{2}\left(1 + \frac{3m_b^4}{(\delta\mathcal{M}^2 + m_B^2)(\delta\mathcal{M}^2 + m_B^2 + 2m_b^2)}\right)$$

is the average value of x^1 along a curve of constant $\delta\mathcal{M}^2$ [6], and x_0 parametrizes the class of constrained optimal functions. Thus we obtain the rigorous bound for any positive function $\psi(\delta\mathcal{M}^2)$

$$\langle x_b^1 \rangle < 0.84 \qquad \text{for} \qquad f_B = 190\sqrt{P_v(B)}\ \text{MeV} \quad . \tag{10}$$

This should be compared with the estimate $\langle x_b^1 \rangle = 0.90$ obtained in [5], which is (barely) consistent with the estimate of f_B in the same reference, but not with the currently preferred value. Similar constraints can be derived for any choice of $f_B/\sqrt{P_v(B)}$, as shown in Fig. 2. Note that the assumption $\psi > 0$ is crucial. If we choose the value $f_B/\sqrt{P_v(B)} = 270$ MeV, which we believe to be a fairly conservative estimate, we obtain $\langle x^1 \rangle < 0.81$ and consequently $\langle 2x - 1 \rangle < 0.61$. This is a very stringent bound, applicable to a wide class of intuitively reasonable wavefunctions (though it can be circumvented by, for example, the introduction of a widely varying complex phase into the wavefunction).

It is a simple matter to derive similar bounds on other moments of the distribution amplitude; for example, with $f_B/\sqrt{P_v(B)} = 270$ MeV we obtain

$$\langle (2x-1)^2 \rangle < 0.41 \,, \qquad \langle (2x-1)^3 \rangle < 0.35 \,, \qquad \text{and} \qquad \langle (1-x)^{-1} \rangle < 14.2 \,.$$

One might expect that this method could also be used to improve our lower bound on $\langle \delta\mathcal{M}^2 \rangle$; however, it turns out that it serves only to duplicate the bound we have already derived. A little thought shows why; the wavefunction

$$\psi(\delta\mathcal{M}^2) = (\mathcal{M}^2_{\rm max} - \delta\mathcal{M}^2)\, \theta(\mathcal{M}^2_{\rm max} - \delta\mathcal{M}^2)$$

which realizes the bound is the same in both cases.

It must be emphasized that $\langle x^n \rangle$ represents a moment, rather than an expectation value, and that unitarity can provide no constraints on expectation values. However, the amplitudes for exclusive processes are determined by convolutions of wave functions, not of their squares; thus Eq. (10) makes a strong statement about the shape of the wave function $\psi_{B \to b\bar{q}}$.

In sum, currently favored values for the meson decay constant f_B can only be reconciled with unitarity by allowing unexpectedly large values of $k_\perp$, and values of x far from unity, to make sizable contributions to the wave function.

We thank S. Brodsky, R. Akhoury, H. Quinn and M. Neubert for helpful conversations. This work was supported by Department of Energy contract DE–AC03–76SF00515.

References

[1] For a review of light-cone methods, see S. J. Brodsky and G. P. Lepage in *Perturbative Quantum Chromodynamics*, A. H. Mueller, Ed. (World Scientific, New Jersey, 1989), and references therein.

[2] S. J. Brodsky *et al.*, in *Banff 1981, Proceedings, Particles and Fields 2.*

[3] M. Neubert, SLAC–PUB–6263, and references therein; C. W. Bernard et al., Nucl. Phys. **B 30** (Proc. Suppl.), 465 (1993), and references therein. For an early look at expectations for f_B, see A. R. Zhitnitskiĭ, I. R. Zhitnitskiĭ and V. L. Chernyak, *Sov. J. Nucl. Phys.* **38**, 775 (1983) [*Yad. Fiz.* **38**, 1277 (1983)], and M. A. Shifman and A. Voloshin, *Sov. J. Nucl. Phys.* **45**, 242 (1987) [*Yad. Fiz.* **45**, 463 (1987)].

[4] Since this report was composed, estimates of $f_B \sim 190$ MeV have come to be favored by lattice computations. See, *e.g.*, C. W. Bernard *et al.*, UW/PT-93-06 (unpublished).

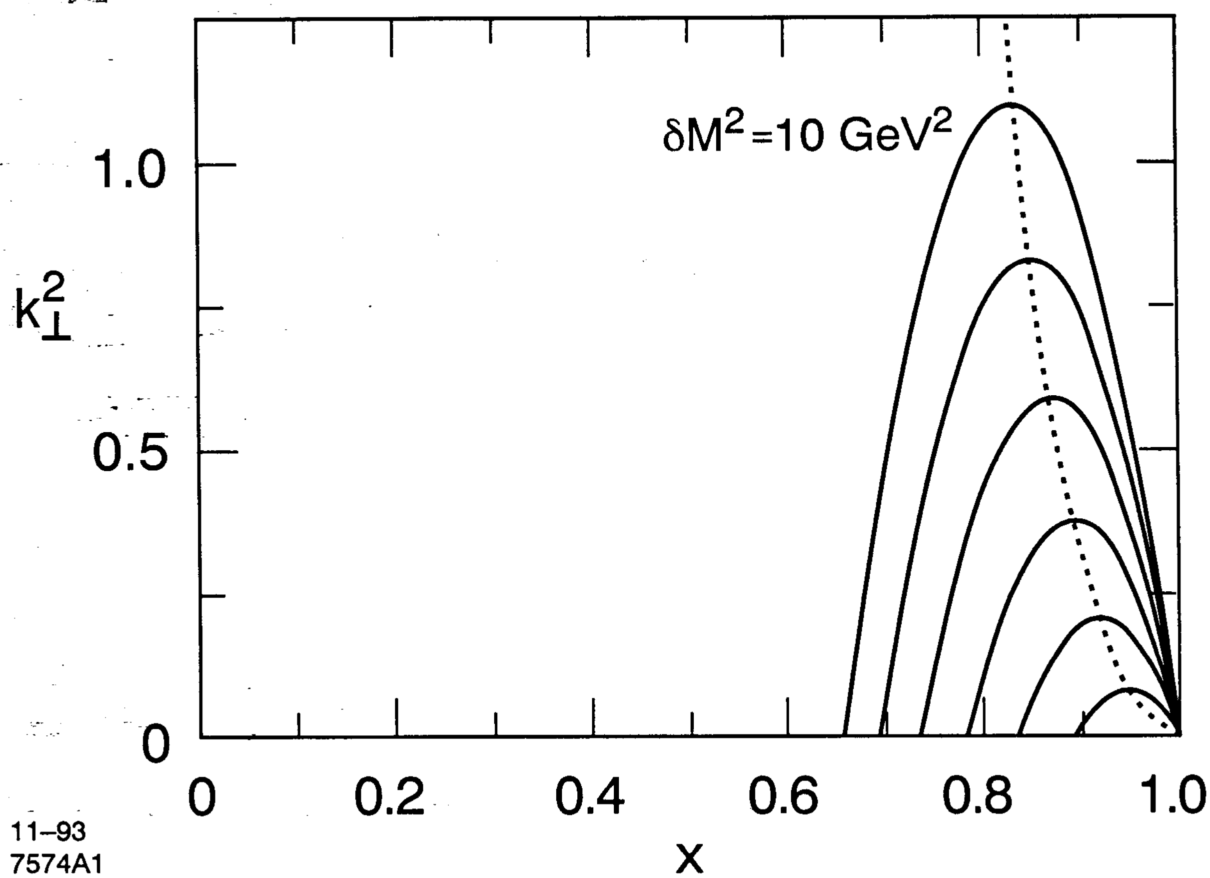
[5] M. A. Shifman and A. Voloshin, in ref. [3].

[6] This paper was originally circulated as SLAC-PUB-6383 (unpublished). Errors in that paper's treatment of the maximization of moments have been corrected in this version.

Figure Captions

Figure 1. Contours of constant $\delta\mathcal{M}^2$ in the $xk_\perp^2$-plane for the B meson. Along the dotted line, the z-component of velocity is the same for the quark and antiquark. Note that as $\delta\mathcal{M}^2$ increases, x tends to decrease.

Figure 2. The excluded region of the moment $\langle x^1 \rangle$ as a function of the parameter $f_B/\sqrt{P_v(B)} > f_B$, which controls the relation between the decay constant and the constraint of unitarity. The assumptions underlying this relation are discussed in the text.



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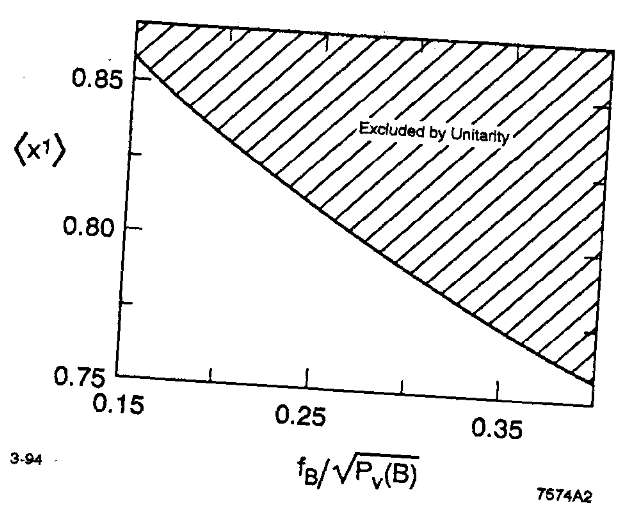

Excluded by Unitarity
0.85
0.80
0.75
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$f_B/\sqrt{P_v(B)}$
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