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NON-GAUSSIAN DENSITY DISTRIBUTION AND ELECTRIC POTENTIAL OF AN OFF-AXIS INJECTED BEAM *

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Nonlinear magnetic fields and chromaticity will cause the vertical off-axis injected beam to filament. In the absence of radiation damping, the vertical beam profile will approach a characteristic stationary "double horn" distribution. In this note we derive the electric potential and a generalized Bassetti-Erskine formula for this stationary distribution. Results from multi-particle tracking illustrate the injection process

KEY WORDS: Beam-Beam, Injection

1 INTRODUCTION

In the present design of the PEP-II a vertical injection scheme has been adopted for the high-energy and the low-energy rings (HER and LER). Due to different nonlinear effects, such as the head-on and the parasitic beam-beam interactions, fringing fields, and higher magnetic multipoles, the tune will depend on the amplitude of the betatron oscillation. As a consequence, an off-centered vertical beam will start to filament into an elongated shape in vertical phase space.

The following list gives a rough estimate of the number of turns until the elongated beam distribution closes to a circular annulus:

A:	LER including nonlinear magnetic elements	8000 turns
B:	linear LER with parasitic crossing	8000 turns
C:	LER with errors, solenoid and fringe fields	1500 turns

The estimated turn number originates: in case A from a normal-form analysis of the one turn map,[1] in case B from Section 4.4.6.1 of the Conceptual Design Report,[2] and in case C from the fast Fourier transform of the tracking data of three different error seeds. These values should be compared to one transverse damping time, which is about 5000 turns.

Another source of filamentation, in addition to nonlinear fields, is chromaticity, which also generates a spread in tune. In this case, however, the original phase

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space distribution will be recovered after a full synchrotron period, since the energy deviation will change sign after half a synchrotron oscillation. This chromatic tune dependence leads to the observed decoherence and recoherence of the barycenter motion in a storage ring.[3] Note, however, that a second-order chromaticity contribution will produce an irreversible distortion of the beam distribution in phase space.

The decoherence of a Gaussian beam due to the beam-beam interaction has been studied recently for the Superconducting Super Collider.[4] It was found that the decoherence time (number of turns) has the form $N_{decoh} = const/\xi$, where ξ is the beam-beam tune shift parameter, and $const \approx 1$.

I attempted to simulate the injection process into the LER by means of multiparticle tracking. One thousand particles have been tracked over 10,000 turns in a LER lattice including orbit, quadrupole, and higher multipole errors. No radiation effects have been considered. The injected beam was assumed to be Gaussian with $\sigma_x = 4$ mm, $\sigma_y = 0.4$ mm, $\sigma_\epsilon = 0.004$, and $\sigma_z = 1$ mm. The vertical barycenter at injection was at $< y >= 20$ mm. Figure 1 displays the vertical beam profiles on turns Nr. 1, 100, 500, 2000, 5000, and 9000. The profile approaches a stationary "double horn" distribution.

Section 2 shows the derivation of the stationary distribution after filamentation and discusses the effect of radiation damping. The electric potential and the electric field of a Gaussian distribution in the horizontal plane and a "double horn" distribution in the vertical plane are derived in Section 3.

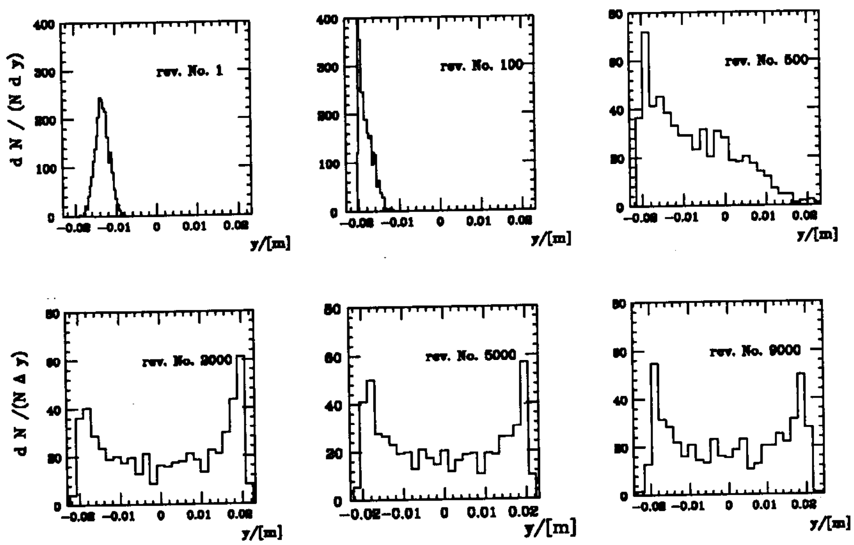

FIGURE 1: Projection of the beam distribution on the vertical axis (1000 particles).

2 PROJECTION OF THE PHASE SPACE DISTRIBUTION FUNCTION

I use action-angle variables to describe an off-center injected beam. At the moment of injection, the beam is assumed to be Gaussian. The time dependent distribution function after off-axis injection is given by [5]

$$\Psi(J,\phi,t) = \frac{1}{2\pi\epsilon_0}\exp\left\{-\frac{1}{\epsilon_0}\left(J + J_0 - 2\sqrt{J_0 J}\cos(\phi - \omega t - \phi_0)\right)\right\}, \qquad (1)$$

where ϵ_0 denotes the injected emittance, and J_0 and ϕ_0 are the coordinates of the injected beam centroid. The frequency ω depends on the action variable in the form of a Taylor expansion:

$$\omega = \omega_0\left(1 - \sum_{n=1}^{\mathrm{N}}\mu_n J^n\right) = \frac{\partial H(J)}{\partial J}. \qquad (2)$$

The coefficients μ_n are evaluated either by means of a Hamilton-Jacobi perturbation technique[7] or by normal-form methods.[1,6] I suppose the Hamiltonian is nondegenerate:

$$\frac{\partial^2 H(J)}{\partial J^2} \neq 0 \quad \text{and} \quad \frac{\partial H(J)}{\partial J} \neq 0. \qquad (3)$$

These conditions exclude more than just linear systems. The distribution function given in Eq. 1 satisfies Liouville's equation[8]

$$\frac{\partial\Psi}{\partial t} = [H,\Psi] \;,$$

where [,] denotes the Poisson bracket. In addition to the action-angle variables, we use the normalized variables (ξ,η),

$$\eta = \sqrt{2J}\cos(\phi) \text{ and } \xi = \sqrt{2J}\sin(\phi),$$

which are related to the measurable coordinates (x,p) by

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}1/\sqrt{\beta} & 0\\ \alpha/\sqrt{\beta} & \sqrt{\beta}\end{pmatrix}\begin{pmatrix}x\\ p\end{pmatrix}, \qquad (4)$$

where α and β are the Twiss parameters. In order to evaluate the density distribution in the variable ξ, we have to integrate the distribution with respect to $d\eta$. For the moment the Taylor expansion in Eq. 2 is truncated at N = 1; hence only sextupole and octupole terms contribute.

To obtain the vertical beam profile we substitute the normalized variables (ξ,η) in Eq. 1 for the action-angle variables and integrate over η:

$$\rho(\xi) = \int_{-\infty}^{\infty}\Psi\,\mathrm{d}\eta = \frac{1}{2\pi\epsilon_0}\exp\left\{-\frac{1}{2\epsilon_0}\left(\xi^2 + \eta_0^2 + \xi_0^2\right)\right\}R(\xi), \qquad (5)$$

with

$$R(\xi) = \int \exp\left\{-\frac{1}{2\epsilon_0}\left(\eta^2 - 2\eta\sqrt{2J_0}\cos(\omega t + \phi_0) - 2\xi\sqrt{2J_0}\sin(\omega t + \phi_0)\right)\right\} \mathrm{d}\eta\,. \tag{6}$$

To evaluate the integral I use the expansion of the exponent into a series of modified Bessel functions,[9]

$$e^{x\cos(\phi)} = \sum_{k=-\infty}^{\infty} I_k(x)e^{ik\phi} \quad \text{and} \quad e^{x\sin(\phi)} = \sum_{k=-\infty}^{\infty} (-i)^k I_k(x)e^{ik\phi}\,. \tag{7}$$

The remaining integral is tabulated[9]

$$R(\xi) = \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \frac{\sqrt{2\pi\epsilon_0}(-i)^{m-2k}}{\sqrt{\alpha_m}} e^{im\varphi + J_0/2\epsilon_0\alpha_m} I_k\left(\frac{J_0}{2\epsilon_0\alpha_m}\right) I_{m-2k}\left(\frac{\sqrt{2J_0}\xi}{\epsilon_0}\right), \tag{8}$$

where

$$\varphi = (1 - \frac{1}{2}\mu_1\xi^2)\omega_0 t + \phi_0, \quad \text{and} \quad \alpha_m = 1 + i\theta m \quad \text{with} \quad \theta = \epsilon_0\mu_1\omega_0 t\,.$$

To evaluate the stationary distribution, I put $t \to \infty$ and recognize that $1/\sqrt{\alpha_m} = 0$ for $m \neq 0$. Thus, in the summation over m, only the contribution for $m = 0$ remains non-zero. The cases $\mu_1 = 0$ or $\omega_0 = 0$ have been excluded by our assumption in Eq. 3. Using Eqs. 5 and 8 we obtain

$$\rho_\infty(\xi) = \frac{1}{\sqrt{2\pi\epsilon_0}} e^{-\frac{1}{2\epsilon_0}(\xi^2 + J_0)} \sum_{k=-\infty}^{\infty} (-1)^k I_k\left(\frac{J_0}{2\epsilon_0}\right) I_{2k}\left(\frac{\sqrt{2J_0}\xi}{\epsilon_0}\right)\,. \tag{9}$$

Using the generating function of the Bessel function in Eq. 7, the stationary distribution function can be represented as an integral:

$$\rho_\infty(\xi) = \frac{1}{\sqrt{2\pi^3\epsilon_0}} \int_0^\pi \exp\left\{-\frac{1}{2\epsilon_0}\left(\xi - \sqrt{2J_0}\cos(\psi)\right)^2\right\} \mathrm{d}\psi\,. \tag{10}$$

Figure 2 shows a typical example of a stationary beam profile after off-axis injection. It was obtained from Eq. 10 and agrees very well with the profiles in Fig. 1 which were generated by multi-particle tracking.

These "double horn" profiles have been obtained from wire scanner measurements in proton storage rings after the beam has been deflected from the closed orbit. Similar density profiles have been measured in the Stanford Linear Collider linac, where the wakefields and chromaticity are the primary causes of the filamentation process.

Up to this point the Hamiltonian has been truncated to second order in J. Provided a stationary distribution exists, the solution in Eq. 9 is valid for a non-truncated Hamiltonian, e.g., $N = \infty$ in Eq. 2. This statement is proven in the appendix.

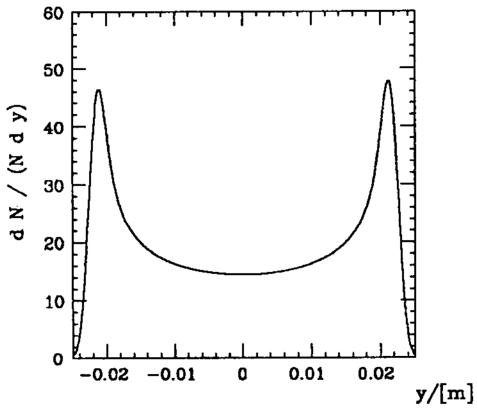

FIGURE 2: Stationary vertical beam profile

It is worth pointing out that the result of the transient profile in Eq. 9 can easily be modified to take the effect of radiation damping into account. The following replacements have to be made:[10]

$$\omega_0\mu t \Longrightarrow \frac{1}{2}\omega_0\mu\tau(e^{2t/\tau}-1),\ \ \epsilon_0 \Longrightarrow \hat{\epsilon}_0(t)=\epsilon_0 e^{-2t/\tau}, \text{and } J_0 \Longrightarrow \hat{J}(t)=J_0 e^{-2t/\tau}, \tag{11}$$

where τ denotes the damping time. Notice that the Liouville Theorem is still valid, and (J,ϕ) are no longer canonical variables.

3 POTENTIAL OF AN OFF-AXIS INJECTED BUNCH AFTER FILAMENTATION

Ions, trapped in the field of the electron beam, may cause significant degradation of machine performance.[11] Another relevant effect at injection is the parasitic beam-beam interaction.[12] The beam-beam kick and the kick received by the ions have to be derived from the beam potential. In this section, I evaluate the beam potential based on the "double horn" distribution. To be specific, the injection is assumed to take place in the vertical plane. I will proceed in a manner very similar to the treatment of a bi-Gaussian distribution.[13] The potential of the beam is given by the Poisson equation

$$\nabla^2 U = -4\pi\rho \ ,$$

where cgs units are used. It was shown in Ref. 13 that the inhomogeneous diffusion equation

$$\nabla^2 U + A^2\frac{\partial U}{\partial t} = -4\pi\rho$$

is considerably easier to solve than the Poisson equation. The Green's function of the non-homogeneous diffusion equation is given by [14]

$$G(x,y,\tau)=\frac{1}{\tau}\exp\{-A^2(x^2+y^2)/4\tau\}u(\tau), \tag{12}$$

where $u(\tau)$ denotes the step function. Now the beam potential can be expressed as

$$U = \int_0^t d\tau \iint \mathrm{d}x_0 \mathrm{d}y_0 \, \rho(x_0, y_0) G(x - x_0, y - y_0, \tau), \tag{13}$$

For the bunch density after filamentation, I use a Gaussian distribution for the horizontal plane and the "double horn" distribution in the vertical plane. With Eqs. 4 and 10 we obtain

$$\rho(x,y) = \frac{N_b e}{2\pi^2 \sigma_x \sigma_{y0}} e^{-\frac{x^2}{2\sigma_x^2}} \int_0^\pi \mathrm{d}\psi \exp\left\{-\frac{1}{2\sigma_{y0}^2}\left(y - \sqrt{2J_0\beta_y}\cos(\psi)\right)^2\right\}, \tag{14}$$

where the factor $N_b e$ takes into account the charge of the bunch, and σ_x and σ_y refer to the transverse beam sizes at injection. Inserting Eqs. 14 and 12 into Eq. 13 and evaluating the integrals over $(\mathrm{d}x_0, \mathrm{d}y_0)$, let $A \to 0$ and find

$$U(x,y) = \frac{N_b e}{\pi} \int_0^\infty \mathrm{d}q \left\{ \int_0^\pi \mathrm{d}\psi \frac{\exp\left(-\frac{x^2}{2\sigma_x^2+q} - \frac{\bar{y}^2(\psi)}{2\sigma_y^2+q}\right) \exp\left(-\frac{\beta_y J_0}{2\sigma_y^2+q}\right) I_0\left(\frac{\beta_y J_0}{2\sigma_y^2+q}\right)}{\sqrt{2\sigma_x^2+q}\sqrt{2\sigma_y^2+q}} \right\}, \tag{15}$$

with

$$\bar{y}(\psi) = y - \sqrt{2\beta_y J_0}\cos(\psi). \tag{16}$$

The second term in Eq. 15 is due to the condition $U(x=0, y=0) = 0$. With the help of Eq. 7, we may express the integral over $d\psi$ in terms of modified Bessel functions:

$$\begin{aligned} U(x,y) &= N_b e \int_0^\infty \mathrm{d}q \left\{ \sum_{k=-\infty}^{\infty} \frac{\exp\left\{-\frac{x^2}{2\sigma_x^2+q} - \frac{y^2}{2\sigma_y^2+q} - \frac{\beta_y J_0}{2\sigma_y^2+q}\right\}}{\sqrt{2\sigma_x^2+q}\sqrt{2\sigma_y^2+q}} (-1)^k I_k\left(\frac{\beta_y J_0}{2\sigma_y^2+q}\right) \right. \\ &\quad \left. \times I_{2k}\left(\frac{\sqrt{2\beta_y J_0}\, y}{2\sigma_y^2+q}\right)\right\} - N_b e \int_0^\infty dq \frac{\exp\left\{-\frac{\beta_y J_0}{2\sigma_y^2+q}\right\}}{\sqrt{2\sigma_x^2+q}\sqrt{2\sigma_y^2+q}} I_0\left(\frac{\beta_y J_0}{2\sigma_y^2+q}\right). \end{aligned}$$

Given the potential, it is straightforward to derive the electric field. For the bi-Gaussian distribution this was done by Bassetti and Erskine.[15] I follow closely their derivation and obtain

$$\begin{aligned} E_x - iE_y &= \frac{-i2N_b e}{\sqrt{2\pi(\sigma_x^2 - \sigma_y^2)}} \int_0^\pi \mathrm{d}\psi \\ &\quad \times \left\{ w\left(\frac{x + i\bar{y}(\psi)}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}}\right) - e^{-\frac{x^2}{2\sigma_x^2} - \frac{\bar{y}^2(\psi)}{2\sigma_y^2}} w\left(\frac{x\frac{\sigma_y}{\sigma_x} + i\bar{y}(\psi)\frac{\sigma_x}{\sigma_y}}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}}\right)\right\}, \end{aligned}$$

where $\bar{y}(\psi)$ is defined in Eq. 16, and $w(z)$ denotes the complex error function.[16] Figure 3 compares the vertical electric field due to the "double horn" distribution with the field due to a centered Gaussian distribution.

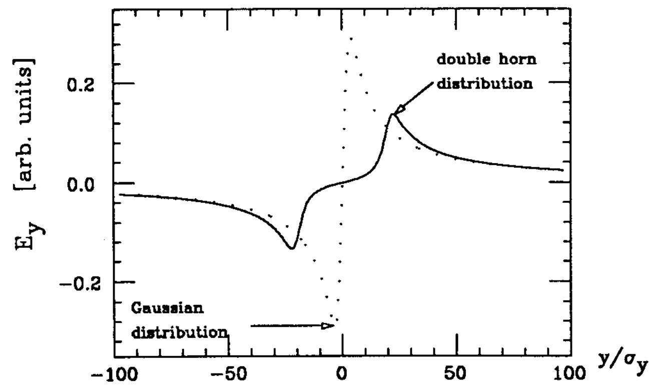

FIGURE 3: Vertical electric field due to different beam profiles.

4 SUMMARY

A simple extension of the Bassetti-Erskine formula for the electric field of an off-axis injected beam after filamentation has been derived. This expression will be useful in simulations of coherent effects at injection.

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APPENDIX

In the case of a linear Hamiltonian, the density profile remains Gaussian, the initial oscillation of the barycenter will never damp out, and a stationary density profile does not exist. Here I suppose the Hamiltonian is nondegenerate, the Taylor expansion in Eq. 2 is not necessarily truncated, and a stationary solution exists. I will show that the stationary density profile is given by Eq. 9.

From the assumptions, the stationary solution must be related to the time average by

$$\rho_\infty(\xi) = \int_{-\infty}^{\infty} \mathrm{d}\eta \Psi_\infty \quad \text{and} \quad \Psi_\infty = \lim_{T\to\infty} \frac{1}{T}\int_0^T \Psi(J,\phi,t)\mathrm{d}t, \tag{17}$$

where Ψ is given by Eq. 1. The integration over dt leads to a series of modified Bessel functions I_k. The subsequent $T \to \infty$ limit eliminates all I_k except of the I_0 contribution

$$\Psi_\infty(\xi,\eta) = \frac{1}{2\pi\epsilon_0} e^{-(J+J_0)/\epsilon_0} I_0\left(\frac{2\sqrt{JJ_0}}{\epsilon_0}\right). \tag{18}$$

Next I replace J by the normalized variables (ξ,η). A special case of Gegenbauer's addition theorem reads [17]

$$I_0(p\sqrt{\xi^2+\eta^2}) = \sum_{k=-\infty}^{\infty} (-1)^k I_k(p\xi) I_k(p\eta), \tag{19}$$

and with Eq. 18,

$$\rho_\infty(\xi) = \frac{1}{2\pi\epsilon_0} \sum_{k=-\infty}^{\infty} (-1)^k I_k\left(\frac{\sqrt{2J_0}\xi}{\epsilon_0}\right) \int_{-\infty}^{\infty} e^{-(2J_0+\xi^2+\eta^2)/(2\epsilon_0)} I_k\left(\frac{\sqrt{2J_0}\eta}{\epsilon_0}\right) d\eta. \tag{20}$$

The subsequent integration over $d\eta$ leads to Eq. 9:

$$\rho_\infty(\xi) = \frac{1}{\sqrt{2\pi\epsilon_0}} e^{-\frac{1}{2\epsilon_0}(\xi^2+J_0)} \sum_{k=-\infty}^{\infty} (-1)^k I_k\left(\frac{J_0}{2\epsilon_0}\right) I_{2k}\left(\frac{\sqrt{2J_0}\xi}{\epsilon_0}\right). \tag{21}$$

It is straightforward to show that the stationary projection of the distribution function is normalized to unity: $\int \rho_\infty(\xi)\mathrm{d}\xi = 1$.