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T/E

On the R and R_τ Ratios at the Five-Loop Level of Perturbative QCD*

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ABSTRACT

We study perturbative QCD at the five-loop level. In particular we consider $R = \sigma_{\text{tot}}(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ and $R_\tau = \Gamma(\tau \to \nu + \text{hadrons})/\Gamma(\tau \to e\nu\overline{\nu})$. We use our method to estimate the five-loop coefficients. As a result, we obtain $\alpha_s(M_Z) = 0.1186(11)$ and $\alpha_s(34\ GeV) = 0.1396(16)$, which are accurate at the 1% level. We also find $R = 3.8350(18)$ which is consistent with R_τ and is accurate to 0.05%.

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Perturbative QCD has been used to describe the strong interaction very successfully, when the energy scale is large enough. This includes the R ratio

$$R = \sigma(e^+e^- \to \text{hadrons}/\sigma(e^+e^- \to \mu^+\mu^-) \tag{1}$$

and also the R_τ ratio

$$R_\tau = \frac{\Gamma(\tau \to \nu + \text{hadrons})}{\Gamma(\tau \to e\nu\overline{\nu})} \tag{2}$$

even though the mass scale M_τ is not very large. Recently Braaten [1] presented a discussion of R_τ and as well, a new quantity, the spin asymmetry parameter A_τ where

$$A_\tau = \frac{R_F - R_B}{R_F + R_B} . \tag{3}$$

R_F and R_B are the "forward" and "backward" components of R_τ

$$R_\tau = R_F + R_B . \tag{4}$$

The lowest order estimates are

$$R_\tau = 3$$

and

$$A_\tau = 1/3 . \tag{5}$$

These estimates are changed by perturbative and non-perturbative corrections as follows:

$$R_F = 2S_{EW}(1 + f_F^{(0)} + f_F^{(2)} + f_F^{(4)} + f_F^{(6)} + \cdots)$$

and

$$R_B = S_{EW}(1 + f_B^{(0)} + f_B^{(2)} + f_B^{(4)} + f_B^{(6)} + \cdots) \tag{6}$$

where

$$S_{EW} = 1.019 \tag{7}$$

is the electroweak correction and the $f_F^{(n)}$ and $f_B^{(n)}$ are proportional to $1/M_\tau^n$ with coefficients that depend logarithmically on M_τ.

The purely perturbative QCD effects from the interactions of massless quarks and gluons for $N_f = 3$ are

$$f_F^{(0)} = \frac{\alpha_s}{\pi} + 5.765\left(\frac{\alpha_s}{\pi}\right)^2 + 34.48\left(\frac{\alpha_s}{\pi}\right)^3 + (d_4^{(3)} + 165.1)\left(\frac{\alpha_s}{\pi}\right)^4 \tag{8}$$

and

$$f_B^{(0)} = \frac{\alpha_s}{\pi} + 4.077\left(\frac{\alpha_s}{\pi}\right)^2 + 10.13\left(\frac{\alpha_s}{\pi}\right)^3 + (d_4^{(3)} - 96.1)\left(\frac{\alpha_s}{\pi}\right)^4 \tag{9}$$

where $\alpha_s = \alpha_s(M_\tau)$ is the running coupling constant of QCD in the $\overline{\mathrm{MS}}$ scheme evaluated at the scale M_τ. The coefficient $d_4^{(3)}$ is the fifth coefficient in the series

$$d_0 = 1,\ d_1 = 1,\ d_2 = 1.64 \text{ and } d_3 = 6.37$$

and has not yet been calculated (perturbative expansion of $-2\pi^2 s(d/ds)\pi^{(1)}(s)$). We will use our estimation method which makes use of Padé approximants to estimate the value of d_4.

From Eq. (9) the Padé Approximant Prediction (PAP) is $d_4^{(3)} = 41$. From the equation below it it is $d_4^{(3)} = 116$. Applying it directly to the d_i series we obtain $d_4^{(3)} = 31$. The average is $d_4^{(3)} = 55$. Finally the PAP for the $(\alpha_s/\pi)^4$ term for R_τ is 133. Thus $d_4^{(3)} = 133 - 78 = 55$, in agreement with the average above.

For further details, see our earlier papers [2]. Thus we take as our value, with conservative error estimates

$$d_4^{(3)} = 55 \begin{array}{c} +60 \\ -24 \end{array} . \tag{10}$$

This is our result for $N_f = 3$.

Our results for

$$f_F^1 = f_F^{(2)} + f_F^{(4)} + f_F^{(6)} = 0.0304$$

and

$$f_B^1 = f_B^{(2)} + f_B^{(4)} + f_B^{(6)} = -0.1082 \tag{11}$$

agree with those of Braaten. The relative contribution to R_τ is

$$\frac{(2f_F^1 + f_B^1)}{3} = -1.58\% . \tag{12}$$

There are various experimental values for R_τ. We use the world-average [3] for B_e and B_μ. From $B_e = 17.76(15)\%$ we obtain

$$R_\tau = 3.658(31) \tag{13}$$

and from $B_\mu = 17.53(19)\%$ we obtain

$$R_\tau = 3.629(24) . \tag{14}$$

The weighted average of Eqs. (13) and (14) is

$$R_\tau = 3.640(19) . \tag{15}$$

From the measured τ lifetime [4], $\tau_\tau = 0.2957(32) \times 10^{-12} S$ one obtains

$$R_\tau = 3.549(39) \ . \tag{16}$$

We take as our value the weighted average of Eqs. (15) and (16)

$$R_\tau = 3.623(17) \tag{17}$$

(see also Ref. [5]).

Now from Eqs. (4), (6), (7), (8), (9), (10), (12) and (17) we obtain our result for $\alpha_s(M_\tau)$,

$$\alpha_s(M_\tau) = 0.3233(89) \tag{18}$$

and using Eq. (3) we obtain

$$A_\tau = 0.413(22) \ . \tag{19}$$

Equation (19) agrees with Braaten's result. Our result for $\alpha_s(M_\tau)$ is somewhat different from Braaten's

$$\alpha_s(M_\tau) = 0.319(17) \ . \tag{20}$$

Note, however, that the error in Eq. (18) is much smaller than the error in Eq. (20). Actually, due to an interpolating error Braaten's result should be

$$\alpha_s(M_\tau) = 0.324(17) \ . \tag{21}$$

We can now obtain $\Lambda^{(3)}$ from the running of α_s,

$$\alpha_s(\mu) = \frac{2\pi}{\beta_0 L}\left\{1 - \frac{\beta_1 \ell n\, 2L}{2\beta_0^2 L} + \frac{1}{4L^2\beta_0^4}\ \left[\beta_1^2 \ell n^2 2L - \beta_1^2 \ell n 2L + \beta_2\beta_0 - \beta_1^2\right]\right\} \tag{22}$$

where β_0, β_1, β_2 are the coefficients of the QCD β function

$$\beta_0 = 11 - \frac{2N_f}{3}$$
$$\beta_1 = 102 - \frac{38N_f}{3} \tag{23}$$
$$\beta_2 = \frac{2857}{2} - \frac{5033}{18} N_f + \frac{325}{54} N_f^2 \; .$$

$L = \ell n \mu / \Lambda$ and N_f is the number of fermions (quarks). For $N_f = 3$ we obtain from Eqs. (18), (22) and (23)

$$\Lambda^{(3)} = 352(17) \; MeV \; . \tag{24}$$

We use the $\overline{\mathrm{MS}}$ scheme throughout this paper.

In order to ensure continuity of α_s as N_f changes, we have derived the following relationships:

$$
\begin{aligned}
\Lambda^{(6)} &= \Lambda^{(5)} \left(\frac{\Lambda^{(5)}}{m_t}\right)^{2/21} \left[\ell n\left(\frac{m_t^2}{\Lambda^{(5)2}}\right)\right]^{-321/3381} \left(\frac{23}{21}\right)^{13/49} \\
\Lambda^{(5)} &= \Lambda^{(6)} \left(\frac{m_t}{\Lambda^{(6)}}\right)^{2/23} \left[\ell n\left(\frac{m_t^2}{\Lambda^{(6)2}}\right)\right]^{321/3703} \left(\frac{21}{23}\right)^{174/529} \\
\Lambda^{(5)} &= \Lambda^{(4)} \left(\frac{\Lambda^{(4)}}{m_b}\right)^{2/23} \left[\ell n\left(\frac{m_b^2}{\Lambda^{(4)2}}\right)\right]^{-963/13225} \left(\frac{25}{23}\right)^{174/529} \\
\Lambda^{(4)} &= \Lambda^{(5)} \left(\frac{m_b}{\Lambda^{(5)}}\right)^{2/25} \left[\ell n\left(\frac{m_b^2}{\Lambda^{(5)2}}\right)\right]^{963/14375} \left(\frac{23}{25}\right)^{231/625} \\
\Lambda^{(4)} &= \Lambda^{(3)} \left(\frac{\Lambda^{(3)}}{m_c}\right)^{2/25} \left[\ell n\left(\frac{m_c^2}{\Lambda^{(3)2}}\right)\right]^{-107/1875} \left(\frac{27}{25}\right)^{231/625} \\
\Lambda^{(3)} &= \Lambda^{(4)} \left(\frac{m_c}{\Lambda^{(4)}}\right)^{2/27} \left[\ell n\left(\frac{m_c^2}{\Lambda^{(4)2}}\right)\right]^{107/2025} \left(\frac{25}{27}\right)^{32/81} \\
\Lambda^{(3)} &= \Lambda^{(2)} \left(\frac{\Lambda^{(2)}}{m_s}\right)^{2/27} \left[\ell n\left(\frac{m_s^2}{\Lambda^{(2)2}}\right)\right]^{-107/2349} \left(\frac{29}{27}\right)^{32/81} \\
\Lambda^{(2)} &= \Lambda^{(3)} \left(\frac{m_s}{\Lambda^{(3)}}\right)^{2/29} \left[\ell n\left(\frac{m_s^2}{\Lambda^{(3)2}}\right)\right]^{107/2523} \left(\frac{27}{29}\right)^{345/841}
\end{aligned}
\tag{25}
$$

These relations differ from those of Marciano [6] by approximately 3%. From Eqs. (25) we obtain

$$\Lambda^{(4)} = 304(16)\ MeV \tag{26}$$

$$\Lambda^{(5)} = 216(13)\ MeV \tag{27}$$

$$\Lambda^{(6)} = 92(6)\ MeV\ . \tag{28}$$

From Eqs. (22) and (27) we obtain

$$\alpha_s(M_Z) = 0.1186(11) \tag{29}$$

whose error is less than 1%! This is consistent with the experimental value from LEP [7] and the latest value from SLD [8]

$$\alpha_s(M_Z) = \begin{cases} 0.120(7) & \text{LEP} \\ 0.118 \pm 0.002(stat) \pm 0.003(sys) \pm 0.010(th) & \text{SLD} \end{cases} . \tag{30}$$

It is clear that a more accurate experimental value is needed. For $\alpha_s(34\ GeV)$ our result is

$$\alpha_s(34\ GeV) = 0.1396(16)\ . \tag{31}$$

From the experimental value for $R = 3\Sigma Q_f^2 r$,

$$r = 1.049(7) \tag{32}$$

one obtains [9]

$$\alpha_s(34\ GeV) = 0.149(21) \tag{33}$$

in agreement with Eq. (31). These results along with α_s evaluated at 5 GeV, 10 GeV, 17.3 GeV, 80.6 GeV and 180 GeV are shown in Table I. It can be seen that all these results are consistent with experiment.

For $N_f = 5$ we have

$$f_F^{(0)} = \left(\frac{\alpha_s}{\pi}\right) + 4.444\left(\frac{\alpha_s}{\pi}\right)^2 + 13.13\left(\frac{\alpha_s}{\pi}\right)^3 + \left(d_4^{(5)} - 7.929\right)\left(\frac{\alpha_s}{\pi}\right)^4 \tag{34}$$

and

$$f_B^{(0)} = \left(\frac{\alpha_s}{\pi}\right) + 3.485\left(\frac{\alpha_s}{\pi}\right)^2 + 1.575\left(\frac{\alpha_s}{\pi}\right)^3 + \left(d_4^{(5)} - 93.14\right)\left(\frac{\alpha_s}{\pi}\right)^4 . \tag{35}$$

We have neglected the term proportional to $(\Sigma Q_i)^2$ as it should be very small.

From Eq. (34) for $f_F^{(0)}$ we obtain $d_4^{(5)} = 35.2$ and from R_τ we get $d_4^{(5)} = 41.8$. From Eq. (35) for $f_B^{(0)}$ we get $d_4^{(5)} = 77.8$ and from the series directly $d_4^{(5)} = 6.47$. We shall be conservative and take as our value

$$d_4^{(5)} = 40 \begin{matrix} +54 \\ -40 \end{matrix} . \tag{36}$$

The R ratio [11] in the $\overline{\text{MS}}$ scheme for $N_f = 5$ is

$$\begin{aligned} R = 3\Sigma Q_f^2 \Bigg[1 + \left(\frac{\alpha_s}{\pi}\right) + 1.411 \left(\frac{\alpha_s}{\pi}\right)^2 - 12.77 \left(\frac{\alpha_s}{\pi}\right)^3 \\ - \frac{1.240(\Sigma Q_f)^2}{3\Sigma Q_f^2} \left(\frac{\alpha_s}{\pi}\right)^3 + R_4 \left(\frac{\alpha_s}{\pi}\right)^4 \Bigg] = 3\Sigma Q_f^2 r \end{aligned} \tag{37}$$

where [12]

$$R_4 = d_4^{(5)} - 89.3 \tag{38}$$

and so

$$R_4 = -49 \begin{matrix} +54 \\ -40 \end{matrix} . \tag{39}$$

Again we neglect the term proportional to $(\Sigma Q_f)^2$, q, since for the case of interest $q = 1/33$ and this term should be negligible.

Now using our result from

$$\alpha_s(34\ GeV) = 0.1396(16) \tag{40}$$

we obtain

$$r = 1.0459(5) \tag{41}$$

and

$$R = 3.8350(18) \ . \tag{42}$$

These results are accurate at the 0.05% level!

For 31.6 GeV we obtain

$$\alpha_s(31.6\ GeV) = 0.1415(16) \tag{43}$$

and, hence,

$$r(31.6\ GeV) = 1.0465(6) \ . \tag{44}$$

This should be compared to the experimental result [14]

$$r(31.6\ GeV) = 1.0527(50) \ . \tag{45}$$

In conclusion, we have shown how one can use our Padé Approximant Prediction (PAP) Method to estimate R_τ at the five-loop level of PQCD. This estimate has then been used to obtain more accurate predictions for $\alpha_s(\mu)$ for various values of μ. The agreement with experiment is excellent!

We have also used our result for $\alpha_s(34\ GeV)$ to obtain the R ratio accurate to 0.05%. It should be emphasized that once we have fixed $\alpha_s(M_\tau)$ all the results in this paper are determined and have been obtained with no adjustable parameters. Now we need to improve the accuracy of the experimental values!

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TABLE I

Experimental and Predicted Values for $\alpha_s(\mu)$

$\mu(GeV)$	Theoretical Values for $\alpha_s(\mu)$	Experimental Values for $\alpha_s(\mu)$
1.7769	0.3233(89)	input
17.3	0.1591(20)	0.18(5) Ref. 10
31.6	0.1415(16)	0.160(16) Ref. 14
34.0	0.1396(16)	0.148(22) Ref. 9
80.6	0.1209(11)	0.123(25) Ref. 13
91.173	0.1186(11)	0.120(7) Ref. 7
		0.118 ± 0.002 (stat) ± 0.003 (sys) ± 0.010 (th) Ref. 8
180.0	0.1076(9)	—