

Nonperturbative Contributions to the Inclusive Rare Decays $B \rightarrow X_s \gamma$ and $B \rightarrow X_s \ell^+ \ell^-$ *

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Abstract

We discuss nonperturbative contributions to the inclusive rare B decays $B \rightarrow X_s \gamma$ and $B \rightarrow X_s \ell^+ \ell^-$. We employ an operator product expansion and the heavy quark effective theory to compute the leading corrections to the decay rate found in the free quark decay model, which is exact in the limit $m_b \rightarrow \infty$. These corrections are of relative order $1/m_b^2$, and may be parameterised in terms of two low-energy parameters. We also discuss the corrections to other observables, such as the average photon energy in $B \rightarrow X_s \gamma$ and the lepton invariant mass spectrum in $B \rightarrow X_s \ell^+ \ell^-$.

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1. Introduction

The rare decays of B mesons have never been of greater interest, both experimentally and theoretically. The first observation of a decay mediated by the quark transition $b \rightarrow s$ recently has been reported by the CLEO Collaboration [1], who found a branching fraction for the process $B \rightarrow K^* \gamma$ of $(4.5 \pm 1.9 \pm 0.9) \times 10^{-5}$. Such transitions are typically induced by the exchange of virtual heavy quanta, the effects of which appear at low energies as local operators multiplied by small coefficients. It is hoped that the detection of these suppressed interactions in the guise of rare B decays may provide a direct window to physics at much higher scales.

In order for such a hope to be realised, however, it is necessary to connect the quark-level operators which are generated perturbatively to the hadronic transitions which are actually observed. This involves the consideration of nonperturbative hadronic matrix elements, which typically are incalculable. One common approach to this problem is to consider inclusive rates such as $B \rightarrow X_s$ rather than individual exclusive channels, and to model the inclusive transition by the decay of a free bottom quark to a free strange quark. It is hoped that for the b quark mass m_b sufficiently large, the operator mediating $b \rightarrow s$ acts over distances short compared to the scales of confinement and strong QCD interactions, and the approximation is a good one.

The issue of how good this approximation really is originally was addressed by Chay, Georgi and Grinstein [2]. Using the tools of the heavy quark effective theory (HQET), they showed that the free quark model is in fact the first term in a controlled expansion in $1/m_b$, and hence is arbitrarily accurate as $m_b \rightarrow \infty$. In addition, they demonstrated that there are no contributions to the rate at subleading ($1/m_b$) order, and that any corrections could only come in at order $1/m_b^2$ or higher.

In this paper, we extend the work of Chay *et al.* to compute the leading corrections to free quark decay, for the inclusive processes $B \rightarrow X_s \gamma$ and $B \rightarrow X_s \ell^+ \ell^-$. While the $1/m_b^2$ corrections here are not particularly large, it is important to know their size if the free quark decay model is to be trusted. We also believe that our computation is a very nontrivial application of HQET in a somewhat unfamiliar regime, and is hence quite interesting in its own right.

Finally, we note that work which overlaps with ours has been performed recently, in a somewhat different formalism, by Bigi *et al.* [3].

2. The Operator Product Expansion and Matrix Elements in HQET

In this section we will discuss our procedure in general terms, to elucidate the structure of the expansion before besieging the reader with particular details. We are interested in the rare decays of b quarks, such as $b \rightarrow s\gamma$ or $b \rightarrow se^+e^-$, which are mediated at low energies by local operators of the form

$$\mathcal{O}P(\phi) = \bar{s}\Gamma b P(\phi). \quad (2.1)$$

Here $P(\phi)$ is meant to stand for some function of perturbatively interacting fields such as leptons or a photon, and Γ is a general Dirac structure. Interactions such as (2.1) are typically induced at high energies by the exchange of virtual W bosons, top quarks, or new exotic quanta. At low energies they appear in the effective Hamiltonian as local operators, with coefficients which may be computed using renormalisation group techniques. We will take the presence of such operators simply as given; our interest will be in the evaluation of their hadronic matrix elements. We note that operators of the form (2.1) are not the only relevant ones which will appear at low energies; for example, we will typically find four-quark operators as well. For these, the techniques which we will present below will only be appropriate when the invariant mass of the intermediate $q\bar{q}$ pair is far from any quarkonium resonances. We will return to this issue in our discussion of the decay $b \rightarrow se^+e^-$.

For now, however, we restrict ourselves to operators with the structure (2.1). They induce quark level transitions of the form $b \rightarrow s$. However, since the quarks are confined, what is observed is the decay $B \rightarrow X_s$, in which a B meson decays to an arbitrary hadronic state X_s with strangeness $S = -1$. (Decays from the lowest lying bottom baryon, Λ_b , are also possible.) Hence we need to compute matrix elements of the form

$$\langle X_s \cdots | \mathcal{O}P(\phi) | B \rangle, \quad (2.2)$$

where the ellipses denote the additional perturbatively interacting fields which couple to $P(\phi)$. Unfortunately, exclusive hadronic matrix elements such as (2.2) are governed by nonperturbative strong interactions and are typically incalculable. At best, $SU(3)$ and heavy quark symmetries may be used to relate the form factors which appear in one such matrix element to those which appear in another [4]. But computations from first principles are not at this point possible.

Instead of considering the exclusive modes individually, then, we will sum over all possible strange final states X_s . As has been shown by Chay *et al.* [2], the inclusive decay

rate may in fact be calculated reliably. Previous computations of the inclusive rate have relied on the free quark decay model, in which the sum over exclusive decays is modeled by the decay of an on-shell bottom quark to an on-shell strange quark. For $m_b \rightarrow \infty$ this is justified by arguing that the decay is essentially a short distance process, which occurs on time scales much shorter than those which govern the eventual hadronisation of the final state. This argument can be made precise within a controlled expansion in inverse powers of the bottom quark mass m_b [2], and we will be able to compute the leading corrections to this limit.

Squaring the matrix element (2.2) and summing over X_s , we find a differential decay rate of the form

$$d\Gamma = \frac{1}{2M_B} \sum_{X_s} d[\text{P.S.}] (2\pi)^4 \delta^{(4)}(P_B - P_X - q) \langle B | i\mathcal{O}^\dagger P(\phi)^\dagger | X_s \cdots \rangle \langle X_s \cdots | i\mathcal{O} P(\phi) | B \rangle. \quad (2.3)$$

Here P_B and P_X are the momenta of the initial B and final X_s systems, and $q = P_B - P_X$ is the momentum transferred to the other decay products. The symbol $d[\text{P.S.}]$ denotes an appropriate phase space differential. The part of $d\Gamma$ which involves the fields $P(\phi)$ may be calculated perturbatively. We then find that $d\Gamma$ is equal to the product of known factors times an expression $W(q)$ which involves only the quark and gluon fields:

$$W(q) = \sum_{X_s} (2\pi)^4 \delta^{(4)}(P_B - P_X - q) \langle B | \mathcal{O}^\dagger | X_s \rangle \langle X_s | \mathcal{O} | B \rangle. \quad (2.4)$$

Here the sum over X_s includes the hadronic phase space integral. The treatment of this nonperturbative expression is the subject of the rest of this section.

We begin by noting that $W(q)$, being essentially a total decay rate, is related by the optical theorem to the discontinuity in a forward scattering amplitude. That is, we may write

$$W(q) = 2\text{Im} T(q), \quad (2.5)$$

where an example of the time-ordered product

$$T(q) = \langle B | T\{\mathcal{O}^\dagger, \mathcal{O}\} | B \rangle \quad (2.6)$$

is shown in fig. 1.

Now we come to a crucial observation [2]. The sum over X_s in eq. (2.4) includes hadronic states with a large range of invariant masses, $M_K^2 \leq P_X^2 \leq M_B^2$. The energy

which flows into the hadronic system X_s scales with m_b as the bottom mass increases, and in the limit $m_b \rightarrow \infty$ is typically much larger than the energy scale Λ_{QCD} which characterizes the strong interactions. Hence, in all but a corner of the Dalitz plot, in which $P_X^2 \approx m_s^2$, the strange quark in fig. 1 is far from its mass shell. In position space, this means that the points at which \mathcal{O} and \mathcal{O}^\dagger act must be very near each other on the scale of nonperturbative QCD, and it is appropriate to perform an operator product expansion of the time-ordered product in eq. (2.6). This operator product expansion may be computed perturbatively in $\alpha_s(m_b)$. It will be valid over almost all of the Dalitz plot, failing only in the region where P_X^2 is small. In the large m_b limit, the fractional contribution of this bad region to the total phase space integral is negligible, and our calculation of the inclusive decay rate based on this expansion will be reliable. Our approach, then, will be to perform a systematic expansion in inverse powers of m_b , of which the leading term will be the result in the $m_b \rightarrow \infty$ limit of the theory [2]. However, we will also be able to compute the leading corrections to this limit, using the tools of the heavy quark effective theory.

In this section we will discuss the form of the operator product expansion, and how to take the hadronic matrix elements of the operators which come out of it. When we apply this formalism in the following sections, the expressions which we derive sometimes will be quite lengthy. Here we will concentrate only on the structure of the procedure. In general, then, the time-ordered product (2.6) may be expanded in a series of local operators suppressed by powers of the mass of the bottom quark,

$$T\{\mathcal{O}^\dagger, \mathcal{O}\} \stackrel{\text{OPE}}{=} \frac{1}{m_b} \left[\mathcal{O}_0 + \frac{1}{2m_b} \mathcal{O}_1 + \frac{1}{4m_b^2} \mathcal{O}_2 + \dots \right]. \quad (2.7)$$

The operator \mathcal{O}_n is an operator of dimension $3 + n$, with n derivatives.

At this point, it is useful to introduce the heavy quark effective theory (HQET) [5], an effective theory of QCD in which the mass of the b quark is taken to infinity. This effective theory implements on the lagrangian level the new “spin-flavor” symmetry of QCD which arises in this limit [6]. Both the mass and the spin of the b quark decouple from the soft bound state dynamics of the hadron of which it is a part; so far as the light degrees of freedom are concerned, the heavy quark is nothing but a static, point-like source of color. The exchange of soft gluons with the light degrees of freedom leave the b quark always almost on shell. Thus we can write its four-momentum p_b^μ as the sum of its “on-shell” momentum $m_b v^\mu$ and a “residual momentum” k^μ , such that the components of k^μ are

always small compared to m_b . It is then convenient to replace the usual quark field $b(x)$ by a new two-component field $h(x)$ with fixed four-velocity v^μ ,

$$h(x) = e^{im_b v \cdot x} P_+ b(x), \quad (2.8)$$

where $P_+ = \frac{1}{2}(1 + \not{v})$ projects onto the quark, rather than antiquark, degrees of freedom. This effective field has the property that a derivative acting on $h(x)$ yields the residual momentum k^μ rather than the full momentum p_b^μ . An expansion in terms of D_μ/m_b then becomes sensible. Expanding in powers of $1/m_b$, we may invert (2.8) to find

$$b(x) = e^{-im_b v \cdot x} [1 + i\not{D}/2m_b + \dots] h(x). \quad (2.9)$$

Inserting this into the usual QCD lagrangian $\bar{b} i\not{D} b$, we find the effective lagrangian for HQET [5],

$$\mathcal{L} = \bar{h} v \cdot iD h + \delta\mathcal{L} \quad (2.10)$$

where the correction terms [7]

$$\delta\mathcal{L} = \frac{1}{2m_b} \bar{h} (iD)^2 h - \frac{1}{2m_b} Z_1(\mu) \bar{h} (v \cdot iD)^2 h + \frac{1}{2m_b} Z_2(\mu) \bar{h} s^{\mu\nu} G_{\mu\nu} h + O(1/4m_b^2) \quad (2.11)$$

are treated as perturbations to the $m_b \rightarrow \infty$ limit. Here the gluon field strength is defined by $G_{\mu\nu} = [iD_\mu, iD_\nu]$, and $s^{\mu\nu} = -\frac{i}{2}\sigma^{\mu\nu}$. The renormalisation constants are given by

$$\begin{aligned} Z_1(\mu) &= 3 \left(\frac{\alpha_s(m_b)}{\alpha_s(\mu)} \right)^{8/25} - 2, \\ Z_2(\mu) &= \left(\frac{\alpha_s(m_b)}{\alpha_s(\mu)} \right)^{9/25} \end{aligned} \quad (2.12)$$

above the charm threshold.

Because the operator product expansion (2.7) is an expansion in D_μ/m_b , we must express the operators \mathcal{O}_n in terms of the HQET field $h(x)$ rather than the full fields $b(x)$. However, as we shall see, it turns out to be convenient to leave the *leading* operator in terms of $b(x)$, and to expand the rest in $h(x)$. The operators \mathcal{O}_n which appear in the expansion (2.7) then take the form

$$\begin{aligned} \mathcal{O}_0 &= \bar{b} \Gamma b, \\ \mathcal{O}_1 &= \bar{h} \Gamma iD_\mu h, \\ \mathcal{O}_2 &= \bar{h} \Gamma iD_\mu iD_\nu h, \end{aligned} \quad (2.13)$$

and so forth. In each case, Γ denotes an arbitrary Dirac structure, in which we also absorb all dependence on the external momentum q , as well as on any other variables. We will keep operators in the expansion with up to two derivatives.

We now turn to the evaluation of the forward matrix elements of the operators \mathcal{O}_n between B meson states. At leading order, we need matrix elements of the form

$$\langle B | \bar{b} \Gamma b | B \rangle, \quad (2.14)$$

which is nonzero only for $\Gamma = 1$ or $\Gamma = \gamma^\mu$. In the second case, the conservation of the b-number current in QCD yields the matrix element normalised absolutely,

$$\langle B | \bar{b} \gamma^\mu b | B \rangle = 2P_B^\mu. \quad (2.15)$$

This, of course, is why we left \mathcal{O}_0 in terms of the field $b(x)$ in eq. (2.13). As for the scalar current, it may be rewritten in terms of the vector current plus higher dimension operators of the form of \mathcal{O}_2 [3],

$$\bar{b} b = v_\mu \bar{b} \gamma^\mu b + \frac{1}{2m_b^2} \bar{h} [(iD)^2 - (v \cdot iD)^2 + s^{\mu\nu} G_{\mu\nu}] h + \dots \quad (2.16)$$

This identity may easily be proven by using eq. (2.9) to expand both sides in terms of the effective field h . It is only meaningful when the four-velocity v^μ of the b field is fixed. The correction term in eq. (2.16) may be absorbed into \mathcal{O}_2 . Hence, the leading term in the expansion of $T(q)$ may be evaluated unambiguously, using eq. (2.15). In fact, the leading term is precisely the free quark decay model result, which becomes exact in the limit $m_b \rightarrow \infty$ [2]. The subleading operators \mathcal{O}_n in the operator product expansion (2.7) will provide systematically the corrections for finite b quark mass.

The evaluation of the matrix elements of the higher dimension operators \mathcal{O}_1 and \mathcal{O}_2 involves the equation of motion of the effective theory [8]. This is given by the lowest order lagrangian,

$$v \cdot iDh = 0. \quad (2.17)$$

Since the external states are characterized only by their four-velocity v^μ , Lorentz invariance severely restricts the forward matrix elements of operators of the form (2.13). For the operator \mathcal{O}_1 of dimension four, we find

$$\langle M | \mathcal{O}_1 | M \rangle = \langle M | \bar{h} \Gamma iD_\mu h | M \rangle = \langle M | \bar{h} \Gamma v_\mu v \cdot iDh | M \rangle. \quad (2.18)$$

However, this is now the matrix element of an operator which vanishes by the equation of motion (2.17). Politzer [9] has shown that all such matrix elements vanish identically; his proof is outlined in the Appendix. Since \mathcal{O}_1 is the only possible source of corrections of order $1/m_b$ to the lowest order result, we see that the leading corrections to the free quark decay model are actually of second order in the heavy quark expansion. As first pointed out by Chay *et al.*, this is a most surprising result, since *exclusive* decay modes all presumably receive corrections already at order $1/m_b$. Somehow these individual contributions must cancel in the inclusive rate.

The dimension five operators do give nonvanishing contributions, of order $1/m_b^2$. However, their forward matrix elements have a very simple parameterisation [10]. The symmetries of the effective theory may be used to write the matrix element as an ordinary Dirac trace,

$$\langle M | \mathcal{O}_2 | N \rangle = \langle M | \bar{h} \Gamma iD_\mu iD_\nu h | M \rangle = M_B \text{Tr} \{ \Gamma P_+ \psi_{\mu\nu} P_+ \}, \quad (2.19)$$

where

$$\psi_{\mu\nu} = \frac{1}{3} \lambda_1 (g_{\mu\nu} - v_\mu v_\nu) + \frac{1}{2} \lambda_2 i \sigma_{\mu\nu}. \quad (2.20)$$

The mass parameters λ_1 and λ_2 are defined in terms of certain expectation values in the effective theory,

$$\begin{aligned} \langle M^{(*)} | \bar{h} (iD)^2 h | M^{(*)} \rangle &= 2M_B \lambda_1, \\ \langle M^{(*)} | \bar{h} s^{\mu\nu} G_{\mu\nu} h | M^{(*)} \rangle &= 2M_B d_{M^{(*)}} \lambda_2(\mu), \end{aligned} \quad (2.21)$$

where $d_M = 3$ and $d_{M^*} = -1$. The μ -dependence of λ_2 cancels that of the renormalisation constant $Z_2(\mu)$ (2.12). We note that $Z_2(m_b) = 1$; hence from this point on we will drop it and by λ_2 mean $\lambda_2(m_b)$.

The role which these parameters play in the effective theory is revealed when one expands the masses of the heavy pseudoscalar and vector mesons in powers of $1/m_b$:

$$\begin{aligned} M_B &= m_b + \bar{\Lambda} - \frac{1}{2m_b} (\lambda_1 + 3\lambda_2) + \dots, \\ M_{B^*} &= m_b + \bar{\Lambda} - \frac{1}{2m_b} (\lambda_1 - \lambda_2) + \dots. \end{aligned} \quad (2.22)$$

In this expansion, the term $\bar{\Lambda}$ represents the energy of the light degrees of freedom in the meson. We see that λ_1 and λ_2 are higher order effects of the finite b quark mass; λ_1 is essentially a ‘‘Fermi motion’’ effect, while λ_2 , the leading spin symmetry-violating

correction, arises from the hyperfine chromomagnetic interactions. From (2.22) we have the well-known relation

$$\lambda_2 = \frac{m_b}{8}(M_{B^*} - M_B) \approx \frac{1}{4}(M_{B^*}^2 - M_B^2) = 0.12 \text{ GeV}^2, \quad (2.23)$$

where we are neglecting higher-order corrections in $1/m_b$. There have been attempts to extract both λ_1 and λ_2 from QCD sum rules by computing the matrix elements (2.21), with the results [11]

$$\begin{aligned} 0 &\leq \lambda_1 \leq 1 \text{ GeV}^2, \\ \lambda_2(1 \text{ GeV}) &= 0.12 \pm 0.02 \text{ GeV}^2. \end{aligned} \quad (2.24)$$

The parameter λ_2 is much better determined in this approach and agrees nicely with the experimental B - B^* mass splitting (2.23).

Finally, there is one other source of corrections of order $1/m_b^2$, namely time-ordered products of \mathcal{O}_1 with the correction $\delta\mathcal{L}$ to the effective lagrangian (2.10). As discussed in the Appendix, these arise because the states $|M\rangle$ in the effective theory differ at order $1/m_b$ from those $|B\rangle$ of QCD. The difference is compensated for at each order by computing matrix elements of the form [8][10]

$$\begin{aligned} &i \int dx \langle M | T \{ \bar{h} \Gamma h, \delta\mathcal{L}(x) \} | M \rangle, \\ &i \int dx \langle M | T \{ \bar{h} \Gamma iD_\mu h, \delta\mathcal{L}(x) \} | M \rangle, \end{aligned} \quad (2.25)$$

and so on. We did not encounter the time-ordered product $\delta\mathcal{L}$ with the dimension three operator \mathcal{O}_0 above, because we were able to compute its matrix element (2.15) directly in full QCD. For the operators of dimension four, since we are working in the effective theory, we must evaluate eq. (2.25). First, we use the fact that the external states depend only on the four-velocity v^μ to write the analogue of eq. (2.18):

$$i \int dx \langle M | T \{ \bar{h} \Gamma iD_\mu h, \delta\mathcal{L}(x) \} | M \rangle = i \int dx \langle M | T \{ \bar{h} \Gamma v_\mu v \cdot iDh, \delta\mathcal{L}(x) \} | M \rangle. \quad (2.26)$$

We may now apply the identity derived in the Appendix, which exploits the fact that the operator which appears on the right-hand side of eq. (2.26) vanishes by the equation of motion of the effective theory. We then obtain the matrix element of a local current,

$$i \int dx \langle M | T \{ \bar{h} \Gamma iD_\mu h, \delta\mathcal{L}(x) \} | M \rangle = -v_\mu \cdot \frac{1}{2m_b} \langle M | \bar{h} \Gamma P_+ F(D) h | M \rangle, \quad (2.27)$$

where $F(D) = (iD)^2 + s^{\mu\nu}G_{\mu\nu}$. This matrix element may then be evaluated in terms of λ_1 and λ_2 , using eq. (2.19).

The result of this long and involved procedure is an expression for the nonperturbative hadronic quantity $T(q)$, of the form

$$T(q) = T_0(q^2, v \cdot q) + \frac{1}{4m_b^2}T_2(q^2, v \cdot q) + \dots \quad (2.28)$$

The expansion is a series in $1/m_b$ and $\alpha_s(m_b)$, and the ellipses in (2.28) denote higher order terms in both small parameters. (The radiative corrections to T_0 have been computed previously [12]; we will not include those to T_2 .) We now take the imaginary part of $T(q)$ to recover $W(q)$, multiply by the perturbative part of the matrix element which couples to $P(\phi)$, and compute the inclusive differential width $d\Gamma$. This may then be integrated to give the total width, Γ , or other smoothly weighted distributions. As we have mentioned, the contribution of T_0 will be precisely that of the free quark decay model [2]. The leading corrections to the $m_b \rightarrow \infty$ limit are of relative order $1/m_b^2$ and encoded in T_2 ; they are expressible entirely in terms of the mass parameters λ_1 and λ_2 . We will now apply this procedure, and compute these corrections, for two interesting examples.

3. Application to Rare B Decays

We will consider inclusive decays of the form $B \rightarrow X_s\gamma$ and $B \rightarrow X_s\ell^+\ell^-$, where $\ell = e$ or μ is a light lepton. They are governed by the effective Hamiltonian density

$$\mathcal{H}_{\text{eff}} = -\frac{4G_F}{\sqrt{2}}V_{tb}V_{ts}^* \sum_j c_j(\mu)O_j(\mu), \quad (3.1)$$

where the sum is over the truncated set of local operators

$$\begin{aligned} O_1 &= \bar{s}_\alpha \gamma^\mu P_L b_\alpha \bar{c}_\beta \gamma_\mu P_L c_\beta \\ O_2 &= \bar{s}_\alpha \gamma^\mu P_L b_\beta \bar{c}_\beta \gamma_\mu P_L c_\alpha \\ O_7 &= \frac{e}{16\pi^2} m_b \bar{s}_\alpha \sigma^{\mu\nu} P_R b_\alpha F_{\mu\nu} \\ O_8 &= \frac{e^2}{16\pi^2} \bar{s}_\alpha \gamma^\mu P_L b_\alpha \bar{e} \gamma_\mu e \\ O_9 &= \frac{e^2}{16\pi^2} \bar{s}_\alpha \gamma^\mu P_L b_\alpha \bar{e} \gamma_\mu \gamma_5 e, \end{aligned} \quad (3.2)$$

Here $P_L = \frac{1}{2}(1 - \gamma_5)$ and $P_R = \frac{1}{2}(1 + \gamma_5)$ are helicity projection operators, $F_{\mu\nu}$ is the photon field strength and α, β are colour indices. We have included in O_8 and O_9 only the

coupling to the electron current; the coupling to the muon is analogous. There are also additional operators, such as $\bar{s}_\alpha \gamma^\mu P_L b_\alpha [\bar{u}_\beta \gamma_\mu P_L u_\beta + \dots + \bar{b}_\beta \gamma_\mu P_L b_\beta]$, which contribute to these decays, but their coefficients are small and we shall neglect them. The coefficients $c_j(m_b)$ have been calculated in leading logarithmic approximation, both in the standard model and in certain minimal extensions, and are presented in refs. [13]–[19]

We now apply the procedure of the previous section to compute the rates for inclusive decays mediated by the operators (3.2). The first step is to construct the operator product expansion (2.7), which takes the form

$$T\{\bar{b}\Gamma_1 s, \bar{s}\Gamma_2 b\} \stackrel{\text{OPE}}{=} \frac{1}{m_b} \left[\mathcal{O}_0 + \frac{1}{2m_b} \mathcal{O}_1 + \frac{1}{4m_b^2} \mathcal{O}_2 + \dots \right]. \quad (3.3)$$

For now, we will allow Γ_1 and Γ_2 to be arbitrary Dirac matrices. To compute the terms in this series, we must expand the diagrams in fig. 1 in powers of $1/m_b$. Fixing the four-velocity of the external b quark to be v^μ , we may expand its momentum as $p_b^\mu = m_b v^\mu + k^\mu$. Then the graph in fig. 1(a) gives

$$\begin{aligned} i\mathcal{M} &= -i\bar{u}_b \frac{\Gamma_1 (m_b \not{p} - \not{q} + \not{k} + m_s) \Gamma_2}{(m_b v - q + k)^2 - m_s^2 + i\epsilon} u_b \\ &= -\frac{i}{m_b x} \bar{u}_b \Gamma_1 (\not{p} - \not{q} + \hat{m}_s) \Gamma_2 u_b \\ &\quad - \frac{i}{m_b^2} \bar{u}_b \left(\frac{1}{x} \Gamma_1 \not{k} \Gamma_2 - \frac{2}{x^2} \Gamma_1 (\not{p} - \not{q} + \hat{m}_s) \Gamma_2 (v - \hat{q})^\alpha k_\alpha \right) u_b \\ &\quad + O(1/m_b^3) \end{aligned} \quad (3.4)$$

where $\hat{q} = q/m_b$, $\hat{m}_s = m_s/m_b$, and

$$x = 1 - 2v \cdot \hat{q} + \hat{q}^2 - \hat{m}_s^2 + i\epsilon \quad (3.5)$$

contains the pole corresponding to an on-shell strange quark, near the end of the physical cut. The spinor u_b which appears is the ordinary on-shell b quark spinor of QCD. From the matrix element (3.4), we may deduce the first two terms in the operator product expansion (3.3):

$$\begin{aligned} \mathcal{O}_0 &= \frac{1}{x} \bar{b} \Gamma_1 (\not{p} - \not{q} + \hat{m}_s) \Gamma_2 b, \\ \mathcal{O}_1 &= \frac{2}{x} \bar{h} \Gamma_1 \gamma^\alpha \Gamma_2 iD_\alpha h - \frac{4}{x^2} (v - \hat{q})^\alpha \bar{h} \Gamma_1 (\not{p} - \not{q} + \hat{m}_s) \Gamma_2 iD_\alpha h. \end{aligned} \quad (3.6)$$

Note that while we have left the leading operator in terms of the four-component fields $b(x)$, we have expanded \mathcal{O}_1 in terms of the two-component effective fields $h(x)$. The reason for this choice was discussed in the previous section.

To obtain \mathcal{O}_2 we must also expand the one-gluon graph in fig. 1(b), in order to identify the contribution from the gluon field strength $G_{\mu\nu} = [iD_\mu, iD_\nu]$. Additional contributions to \mathcal{O}_2 arise when the full QCD fields $b(x)$ in \mathcal{O}_1 are replaced by the effective theory fields $h(x)$ via the relation (2.9). Equivalently, one may expand the spinors u_b in the matrix element (3.4) in terms of the two-component spinors u_h of HQET,

$$u_b = \left[1 + \frac{\not{k}}{2m_b} + O\left(\frac{1}{m_b^2}\right) \right] u_h, \quad (3.7)$$

and check that the result may be made covariant. Finally, there will be corrections at order $1/m_b^2$ if the leading operator \mathcal{O}_0 contains a scalar current $\bar{b}b$, because of the expansion (2.16). These are still contained in \mathcal{O}_0 and are *not* included in \mathcal{O}_2 . A straightforward calculation then yields

$$\begin{aligned} \mathcal{O}_2 = & \frac{16}{x^3} (v - \hat{q})^\alpha (v - \hat{q})^\beta \bar{h} \Gamma_1 (\not{p} - \not{q} + \hat{m}_s) \Gamma_2 iD_\alpha iD_\beta h \\ & - \frac{4}{x^2} \bar{h} \Gamma_1 (\not{p} - \not{q} + \hat{m}_s) \Gamma_2 (iD)^2 h \\ & - \frac{4}{x^2} (v - \hat{q})^\beta \bar{h} \Gamma_1 \gamma^\alpha \Gamma_2 (iD_\alpha iD_\beta + iD_\beta iD_\alpha) h \\ & + \frac{2}{x^2} \hat{m}_s \bar{h} \Gamma_1 i\sigma_{\alpha\beta} \Gamma_2 G^{\alpha\beta} h \\ & - \frac{2}{x^2} i\epsilon^{\mu\lambda\alpha\beta} (v - \hat{q})_\lambda \bar{h} \Gamma_1 \gamma_\mu \gamma_5 \Gamma_2 G_{\alpha\beta} h \\ & + \frac{2}{x} \bar{h} (\gamma^\beta \Gamma_1 \gamma^\alpha \Gamma_2 + \Gamma_1 \gamma^\beta \Gamma_2 \gamma^\alpha) iD_\beta iD_\alpha h \\ & - \frac{4}{x^2} (v - \hat{q})^\alpha \bar{h} \gamma^\beta \Gamma_1 (\not{p} - \not{q} + \hat{m}_s) \Gamma_2 iD_\beta iD_\alpha h \\ & - \frac{4}{x^2} (v - \hat{q})^\alpha \bar{h} \Gamma_1 (\not{p} - \not{q} + \hat{m}_s) \Gamma_2 \gamma^\beta iD_\alpha iD_\beta h. \end{aligned} \quad (3.8)$$

To continue any further, we must specify the Dirac structures Γ_1 and Γ_2 .

3.1. $B \rightarrow X_s \gamma$

For the transition $B \rightarrow X_s \gamma$, only the operator O_7 from (3.2) contributes and the operator product expansion simplifies considerably. We now contract the terms in the time ordered product (3.3) with the external photon fields and take the matrix element between B mesons to construct the hadronic object $T(q)$ defined in eq. (2.6). Because the decay is to an on-shell photon, $q^2 = 0$ is fixed, and $T(q)$ becomes a function only of the

scaled photon energy $v \cdot \hat{q} = E_\gamma/m_b$. Including the matrix element of the photon field, we find

$$\begin{aligned} \tilde{T}(v \cdot \hat{q}) &\equiv i^2 \langle B | T \{ \bar{b} \sigma^{\mu\nu} s, \bar{s} \sigma^{\rho\tau} b \} | B \rangle \cdot \sum_{\epsilon=1,2} \langle \gamma(\hat{q}, \epsilon) | F_{\mu\nu} F_{\rho\sigma} | \gamma(\hat{q}, \epsilon) \rangle \\ &= -16 M_B m_b (v \cdot \hat{q})^2 \left[\frac{1}{x} - \frac{\lambda_1}{2m_b^2} \left(\frac{5 - 6v \cdot \hat{q}}{3x^3} \right) + \frac{\lambda_2}{2m_b^2} \cdot \frac{3}{x^2} \right]. \end{aligned} \quad (3.9)$$

The sum is over the transverse polarizations of the photon, and there is a factor of i from each insertion of the effective Hamiltonian (3.1). We neglect contributions of order \hat{m}_s^2 . Note that we distinguish between m_b , the bottom *quark* mass which arises in the operator product expansion, and M_B , the *B meson* mass which arises from the relativistic normalisation of the states (and therefore drops out of the final expression). The inclusive rate for $B \rightarrow X_s \gamma$ is then given by

$$\begin{aligned} \Gamma_{B \rightarrow X_s \gamma} &= \frac{\alpha G_F^2}{8\pi^3} \frac{m_b^2}{M_B} |V_{tb} V_{ts}^*|^2 |c_7(m_b)|^2 \text{Im} \int \frac{d^3 k}{(2\pi)^3 2E_\gamma} \tilde{T}(v \cdot \hat{q}) \\ &= \frac{\alpha G_F^2}{32\pi^5} \frac{m_b^4}{M_B} |V_{tb} V_{ts}^*|^2 |c_7(m_b)|^2 \oint z \tilde{T}(z) dz, \end{aligned} \quad (3.10)$$

where $z = v \cdot \hat{q}$, and the contour integral is taken around the pole at $x = 0$. It is straightforward to evaluate this integral, and we find

$$\Gamma_{B \rightarrow X_s \gamma} = \frac{\alpha G_F^2}{16\pi^4} m_b^5 |V_{tb} V_{ts}^*|^2 |c_7(m_b)|^2 \left[1 + \frac{1}{2m_b^2} (\lambda_1 - 9\lambda_2) \right]. \quad (3.11)$$

This expression for the total rate agrees with the result of ref. [3]. The first term is just what one would obtain in the free quark decay model.

We may consider using the same method to compute certain features of the photon energy spectrum. Of course, the precise shape of the spectrum is not available to us, in particular its behaviour near the endpoint of maximum E_γ . This is true at *any* order in the $1/m_b$ expansion, because in this region the strange quark approaches its mass shell and the operator product expansion breaks down. For example, the fact that the true endpoint of the photon energy spectrum is found not at $E_\gamma = m_b/2$ but rather at $(M_B^2 - M_K^2)/2M_B$ is entirely unavailable to us in this formalism.

It is instructive, however, to generalise eq. (3.10) to calculate the nonperturbative contributions to the moments of the energy spectrum. For example, the deviation of the average photon energy from that of the free quark decay model is

$$\begin{aligned} \langle E_\gamma \rangle &= \frac{1}{\Gamma_{B \rightarrow X_s \gamma}} \frac{\alpha G_F^2}{32\pi^5} \frac{m_b^4}{M_B} |V_{tb} V_{ts}^*|^2 |c_7(m_b)|^2 \oint (m_b z) z \tilde{T}(z) dz \\ &= \frac{m_b}{2} \left(1 - \frac{\lambda_1 + 3\lambda_2}{2m_b^2} \right). \end{aligned} \quad (3.12)$$

This result has an interesting structure, if we compare it to the $1/m_b$ expansion of the B meson mass (2.22). Naïvely, we might have expected $\langle E_\gamma \rangle$ to be shifted from the free quark value $m_b/2$ to half the physical meson mass $M_B/2$. However, that is not what we find; only the order $1/m_b^2$ terms contribute. The reason is that the correction $\bar{\Lambda}$ to M_B in eq. (2.22) is the contribution to the meson mass of the light antiquark and the other light degrees of freedom, which in this formalism are mere spectators to the decay. Since they are present in the final state X_s as well as in the initial state, they do not represent additional energy available to the photon. By contrast, the higher-order mass corrections proportional to λ_1 and λ_2 arise from terms in the effective Hamiltonian of the b quark, representing its “Fermi motion” and its chromomagnetic interaction with the soft hadronic surroundings. These bound state shifts in the b quark energy are then reflected in the average photon energy $\langle E_\gamma \rangle$.

We could also generalise eq. (3.12) to higher moments of the photon spectrum. However, there arises an additional complication if we insist that the moments we compute be experimentally meaningful quantities. This is because they are constructed by convolving a power of the photon energy with the measured energy spectrum, but this spectrum is only related to our computation once our result (3.9) has been smeared over typical hadronic scales. That is, $\tilde{T}(z)$ should be replaced by the smoothed quantity

$$\tilde{T}_f(z) = \int dz' f(z - z') \tilde{T}(z'), \quad (3.13)$$

where $f(x)$ is some smearing function of width $\delta = \Delta E/m_b$. If we take $f(x)$ to be a Gaussian distribution, $f(x) = \exp(-x^2/\delta^2)/\sqrt{\pi\delta^2}$, we can calculate the moments analytically. Keeping terms of order δ^2 , for the n 'th moment we find

$$\begin{aligned} \Gamma^{(n)} &= \frac{\alpha G_F^2}{32\pi^5} \frac{m_b^4}{M_B} |V_{tb}V_{ts}^*|^2 |c_7(m_b)|^2 \oint (m_b z)^n z \tilde{T}_f(z) dz \\ &= \left(\frac{m_b}{2}\right)^n \left[1 + 2n(n-1) \frac{(\Delta E)^2}{2m_b^2} - \frac{n(n+2)}{3} \frac{\lambda_1}{2m_b^2} - 3n \frac{\lambda_2}{2m_b^2} \right] \Gamma_{B \rightarrow X_s \gamma}. \end{aligned} \quad (3.14)$$

The total rate ($n = 0$) and average energy ($n = 1$) which we have already presented are unaffected by this procedure, but the same is not true for the moments with $n \geq 2$. Note that the effect of the smearing is proportional to δ^2 rather than δ , and so is formally of the same order as the nonperturbative corrections we have been considering. However, in order for our inclusive predictions to be meaningful, the resolution ΔE with which the photon energy spectrum is measured actually must be much greater than Λ_{QCD} , so that

many exclusive states are always summed over. Hence, it is in fact this resolution, rather than the nonperturbative effects, which will dominate the corrections to the moments $\Gamma^{(n)}$. In addition, real gluon emission will broaden the energy spectrum over the entire allowed phase space $0 < E_\gamma < m_b/2$, which will affect substantially the shape of the experimentally measured spectrum [12].

3.2. $B \rightarrow X_s \ell^+ \ell^-$

The transition $B \rightarrow X_s \ell^+ \ell^-$ receives contributions from the complete set of operators in (3.2). In particular, unlike the decay $B \rightarrow X_s \gamma$, the four-quark operators O_1 and O_2 have non-vanishing matrix elements. In order for our treatment of the four-quark operators to be valid, it is crucial that the invariant mass of the lepton pair not be near any resonances in the charm system such as the ψ , so that strong final state interaction corrections will be small. In this case we can treat the contributions from O_1 and O_2 as effectively local on the scale of hadronic interactions.

It is convenient to separate the total rate for $B \rightarrow X_s \ell^+ \ell^-$ into two terms, corresponding to the decay to left- and right-handed leptons

$$d\Gamma_{B \rightarrow X_s \ell^+ \ell^-} \propto W_L^{\mu\nu}(\hat{q}^2, v \cdot \hat{q}) L_{\mu\nu}^L + W_R^{\mu\nu}(\hat{q}^2, v \cdot \hat{q}) L_{\mu\nu}^R, \quad (3.15)$$

where the lepton tensors are given by

$$L^{L(R)\mu\nu} = p_+^\mu p_-^\nu + p_+^\nu p_-^\mu - g^{\mu\nu} p_+ \cdot p_- \pm i\epsilon^{\mu\nu\sigma\rho} p_{+\sigma} p_{-\rho}. \quad (3.16)$$

Here p_+ and p_- are respectively the four-momenta of the ℓ^+ and ℓ^- . Since we are restricting ourselves to $\ell = e$ or μ , we neglect the masses of the leptons.

The two Lorentz structures which arise from the effective Hamiltonian (3.2) are $\gamma_\mu(1 - \gamma_5)$ and $\sigma_{\mu\nu}(1 + \gamma_5)\hat{q}_\nu$. Hence it is convenient to write

$$\begin{aligned} \Gamma_2^{L(R)} &= \frac{1}{2}(1 - \gamma_5)\gamma_\mu \left[A^{L(R)} - B^{L(R)} \hat{q}/\hat{s} \right], \\ \Gamma_1^{L(R)} &= \Gamma_2^{L(R)\dagger}, \end{aligned} \quad (3.17)$$

where $\hat{s} = \hat{q}^2$. The A 's and B 's are then combinations of the coefficients c_1, \dots, c_9 :

$$\begin{aligned} A^L &= c_8(m_b) - c_9(m_b) + [3c_1(m_b) + c_2(m_b)] g(m_c/m_b, \hat{s}), \\ A^R &= c_8(m_b) + c_9(m_b) + [3c_1(m_b) + c_2(m_b)] g(m_c/m_b, \hat{s}), \\ B^L &= B^R = -2c_7(m_b), \end{aligned} \quad (3.18)$$

where the function $g(m_c/m_b, \hat{s})$ multiplying c_1 and c_2 arises from taking the one-loop matrix elements of O_1 and O_2 and has the form

$$g(z, \hat{s}) = -\frac{4}{9} \log z^2 + \frac{8}{27} + \frac{16}{9} \frac{z^2}{\hat{s}} - \frac{2}{9} \sqrt{1 - \frac{4z^2}{\hat{s}}} \left(2 + \frac{4z^2}{\hat{s}} \right) \times \log \left(\frac{\sqrt{1 - 4z^2/\hat{s}} + 1 + i\epsilon}{\sqrt{1 - 4z^2/\hat{s}} - 1 + i\epsilon} \right). \quad (3.19)$$

Integrating over the lepton phase space, the total decay rate is given by

$$\begin{aligned} \Gamma_{B \rightarrow X_s \ell^+ \ell^-} &= \frac{1}{2M_B} \int \frac{d^3 p_+}{(2\pi)^3 2E_+} \frac{d^3 p_-}{(2\pi)^3 2E_-} (W_L^{\mu\nu} L_{\mu\nu}^L + W_R^{\mu\nu} L_{\mu\nu}^R) \\ &= \frac{m_b^2}{128\pi^4 M_B} \int d\hat{s} d\hat{E}_1 \text{Im} \oint dv \cdot \hat{q} (T_L^{\mu\nu} L_{\mu\nu}^L + T_R^{\mu\nu} L_{\mu\nu}^R). \end{aligned} \quad (3.20)$$

We must next perform the contour integral in the $v \cdot \hat{q}$ plane and then the \hat{E}_1 integral to obtain the differential decay width. Since the calculation is quite tedious and the intermediate expressions extremely lengthy, we present only the final result:

$$\begin{aligned} \frac{d\Gamma_{B \rightarrow X_s \ell^+ \ell^-}}{d\hat{s}} &= \frac{G_F^2 \alpha^2}{256\pi^5} m_b^5 |V_{tb} V_{ts}^*|^2 (1 - \hat{s}) \\ &\times \sum_{i=L,R} \left\{ \frac{1}{6} (1 - \hat{s})(1 + 2\hat{s}) |A^i|^2 + \frac{1}{6} (1 - \hat{s})(1 + 2/\hat{s}) |B^i|^2 \right. \\ &\quad - (1 - \hat{s}) \text{Re}(B^{*i} A^i) \\ &\quad + \frac{\lambda_1}{2m_b^2} \left[\left(-\frac{1}{3} \hat{s}^2 + \frac{1}{2} \hat{s} + \frac{5}{6} \right) |A^i|^2 - \frac{1}{6} (1 + \hat{s}) |B^i|^2 \right. \\ &\quad \left. \left. + \left(\hat{s} - \frac{5}{3} \right) \text{Re}(B^{*i} A^i) \right] \right. \\ &\quad + \frac{\lambda_2}{2m_b^2} \left[\left(-5\hat{s}^2 + \frac{15}{2} \hat{s} + \frac{1}{2} \right) |A^i|^2 - \frac{5}{2} (1 + \hat{s}) |B^i|^2 \right. \\ &\quad \left. \left. + (7\hat{s} - 5) \text{Re}(B^{*i} A^i) \right] \right\}. \end{aligned} \quad (3.21)$$

The summation is over the two chirality states of the leptons.

The leading term in eq. (3.21) reproduces the free quark decay model result obtained in refs. [14][18], while the subsequent terms are the leading non-perturbative contributions to the decay rate. It is interesting to note that unlike the parton level result, which has a characteristic $1/\hat{s}$ behaviour at small \hat{s} from the one-photon intermediate state, the non-perturbative corrections approach a finite constant value as $\hat{s} \rightarrow 0$. The differential spectrum for the invariant mass of the lepton pair is plotted in fig. 2. We have chosen

a top quark mass of $m_t = 150 \text{ GeV}$, along with $m_b = 4.5 \text{ GeV}$, $\alpha_s(m_W) = 0.12$ and $\alpha_s(m_b) = 0.21$, to generate the spectrum. The free quark decay model result ($\lambda_1 = \lambda_2 = 0$) is presented along with the spectrum for $\lambda_1 = 0.5 \text{ GeV}^2$ and $\lambda_2 = 0.12 \text{ GeV}^2$. We have normalised the width for this decay to that for semileptonic B decay (which includes the nonperturbative corrections given in ref. [20]). The modification to the $B \rightarrow X_s \ell^+ \ell^-$ rate is reasonably large and tends to enhance the overall rate for high mass lepton pairs by order 10%.

4. Summary and Conclusions

Because of the necessary cuts to remove backgrounds, the full spectrum from a decay such as $B \rightarrow X_s \gamma$ and $B \rightarrow X_s \ell^+ \ell^-$ is not available in an accelerator experiment. It is therefore important to understand well the shapes of these spectra if one is to relate the observed branching fractions to fundamental parameters of the electroweak theory. This is particularly true for the high photon energy and high invariant lepton mass regions of the Dalitz plot. Modifications to the simplest model, that of free quark decay, arise from strong interactions that can be classified heuristically as perturbative and non-perturbative corrections.

The perturbative corrections arising from gluon bremsstrahlung and one-loop effects for $B \rightarrow X_s \gamma$ have been computed previously [12]. It is to the non-perturbative corrections that we have addressed ourselves in this paper. We have detailed the formalism for treating the semi-hadronic inclusive decays of mesons containing a single heavy quark. Upon summing over all hadronic final states, one may express the rate for a given process in terms of a time-ordered product of quark bilinears. This time-ordered product is then expanded in a series of local operators, the matrix elements of which either are known or may be parameterised simply. Heavy quark symmetries and the heavy quark effective theory play a key role in the analysis.

We have applied these tools to the rare decays $B \rightarrow X_s \gamma$ and $B \rightarrow X_s \ell^+ \ell^-$. The leading non-perturbative corrections to the free quark decay model, of relative order $1/m_b^2$, may be expressed entirely in terms of two low-energy parameters. One of these is determined from the splitting between the heavy pseudoscalar and vector mesons; a model-dependent estimate of the other comes from QCD sum rules. In addition to the total rates, we have computed the correction to the average photon energy in $B \rightarrow X_s \gamma$ and found the shift to

be small. The correction to the spectrum for $B \rightarrow X_s l^+ l^-$ is larger and for high invariant mass lepton pairs is at about the 10% level.

Finally, we note that there has been considerable recent work in which a similar formalism has been applied to semileptonic b decays [20].

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Appendix A. Matrix Elements and the Equation of Motion

In this appendix we derive an identity for the matrix element of a time-ordered product of two operators, where one of the operators vanishes by the equation of motion of the theory. This will be a generalisation of a proof by Politzer [9] that matrix elements of single operators which vanish by the equation of motion themselves vanish. We will derive our result within the context of the heavy quark effective theory (HQET), because this is the application which we have in mind, but with obvious modifications our result is completely general.

We begin by recalling how such time-ordered products arise within HQET. In this effective theory, the heavy quark part of the lagrangian takes the form [5][7]

$$\mathcal{L} = \bar{h}v \cdot iDh + \frac{1}{2m_b} \bar{h}F(D)h + \dots, \quad (\text{A.1})$$

where

$$F(D) = (iD)^2 - Z_1(\mu)(v \cdot iD)^2 + Z_2(\mu)s^{\mu\nu}G_{\mu\nu}, \quad (\text{A.2})$$

and the ellipses denote terms of higher order in the $1/m_b$ expansion. Here the gluon field strength is defined by $G_{\mu\nu} = [iD_\mu, iD_\nu]$, and $s^{\mu\nu} = -\frac{i}{2}\sigma^{\mu\nu}$. The renormalisation constants $Z_1(\mu)$ and $Z_2(\mu)$ are given in Section 2. The equation of motion in HQET is derived from the leading term in the lagrangian (A.1), and is simply

$$v \cdot iDh = 0. \quad (\text{A.3})$$

Instead of being included in the equation of motion, the corrections to \mathcal{L} in eq. (A.1) are treated as perturbations. They reappear in the following way: because the states $|M\rangle$ of HQET are defined by the truncated equation of motion (A.3), they differ from those $|B\rangle$ of full QCD. The states $|M\rangle$ have the significant advantage that, unlike $|B\rangle$, they are independent of the heavy quark mass m_b , and so have simple transformations under the spin-flavor symmetries of the effective theory. The difference between $|M\rangle$ and the physical states $|B\rangle$ is then compensated by including in the matrix elements of effective operators additional time-ordered products with the subleading terms in \mathcal{L} [8][10]. That is, if we have an effective operator $\bar{h}'C(D)h$ whose matrix element we require between eigenstates $|B\rangle$ of full QCD, then we must write

$$\begin{aligned} \langle B(p') | \bar{h}'C(D)h | B(p) \rangle &= \langle M(v') | \bar{h}'C(D)h | M(v) \rangle \\ &+ \frac{1}{2m_b} i \int dx \langle M(v') | T \{ \bar{h}'C(D)h, \bar{h}F(D)h(x) \} | M(v) \rangle + \dots \end{aligned} \quad (\text{A.4})$$

We have shown the expansion up to order $1/m_b$ explicitly; the ellipses denote terms of higher order which may be included if more accuracy is needed. We consider here the general case in which the initial and final heavy quarks have different four-velocities. The field h creates a heavy quark with velocity v^μ , while h' creates one with velocity v'^μ . There is a separate effective lagrangian (A.1) for each of these fields, but for simplicity we will include the $1/m_b$ corrections only for the field h . The time-ordered products in (A.4) are new nonperturbative matrix elements which must be evaluated if one wishes to use the effective theory beyond leading order.

We will be concerned with a special case of eq. (A.4), in which the operator $\bar{h}'C(D)h$ vanishes by the equation of motion (A.3) of the effective theory.¹ That is, $C(D)$ takes the particular form

$$C(D) = A(D)v \cdot iD, \quad (\text{A.5})$$

where $A(D)$ may include an arbitrary Dirac structure. Politzer has shown that matrix elements of such operators, such as would appear in the first term in eq. (A.4), vanish [9]:

$$\langle M(v') | \bar{h}'A(D)v \cdot iDh | M(v) \rangle = 0. \quad (\text{A.6})$$

¹ We are grateful to A. Manohar for discussions of this point. See also ref. [20].

Note that it is the effective theory states $|M\rangle$ which appear here. The purpose of this appendix is to generalise this argument to prove a similar identity for the time-ordered product appearing in the second term of (A.4), namely, that

$$\begin{aligned} i \int dx \langle M(v') | T \{ \bar{h}' A(D) v \cdot iDh, \bar{h} F(D) h(x) \} | M(v) \rangle \\ = - \langle M(v') | \bar{h}' A(D) P_+ F(D) h | M(v) \rangle, \end{aligned} \quad (\text{A.7})$$

where $P_+ = \frac{1}{2}(1 + \not{v})$. For the computation of this paper, we will apply this identity in the case that $A(D)$ actually contains no derivatives, and at zero recoil; then the matrix element on the right-hand side of (A.7) is of the simple form

$$\langle M | \bar{h} \Gamma iD_\mu iD_\nu h | M \rangle, \quad (\text{A.8})$$

and can be evaluated in terms of the constants λ_1 and λ_2 as in Section 2.

Before proving the identity (A.7), however, we note its relation to the result we would obtain by taking a different approach. Instead of introducing the states $|M\rangle$, which are eigenstates of the lowest order effective lagrangian, we could chose to work always in terms of the full states $|B\rangle$ of QCD. This would be undesirable, in that it would reintroduce the mass-dependence which it is the goal of HQET to remove, thereby obscuring the spin-flavor symmetries of the heavy quark limit. However, if we do so, the equation of motion is given by the full Lagrangian (A.1), taking the form

$$v \cdot iDh = -\frac{1}{2m_b} P_+ F(D) h + \dots \quad (\text{A.9})$$

This equation of motion may be applied directly to matrix elements between the states $|B\rangle$. We then find for the matrix element (A.4) the relation

$$\langle B(p') | \bar{h}' C(D) h | B(p) \rangle = -\frac{1}{2m_b} \langle B(p') | \bar{h}' A(D) P_+ F(D) h | B(p) \rangle. \quad (\text{A.10})$$

Inserting eqs. (A.6) and (A.7) into eq. (A.4), and noting that the states $|M\rangle$ and $|B\rangle$ differ only at order $1/m_b$, we see that this is the same result that we find working entirely within the effective theory. The proof which we now present may be seen as verifying the consistency of the effective theory approach. It is both an application and a generalisation of the proof of Politzer.

We begin by using the LSZ reduction formula to write the desired matrix element in terms of a vacuum expectation value:

$$\begin{aligned}
& i \int dx \langle M(v') | T \{ \bar{h}' A(\mathbb{D}) v \cdot i\mathbb{D}h(0), \bar{h}F(\mathbb{D})h(x) \} | M(v) \rangle \\
&= i \int dx \int dz dz' e^{ik \cdot z} e^{ik' \cdot z'} v \cdot i\mathbb{D}_z v' \cdot i\mathbb{D}_{z'} \\
&\quad \cdot \langle 0 | T \{ \bar{q}\Gamma' h'(z'), \bar{h}' A(\mathbb{D}) v \cdot i\mathbb{D}h(0), \bar{h}F(\mathbb{D})h(x), \bar{h}\Gamma q(z) \} | 0 \rangle.
\end{aligned} \tag{A.11}$$

Here the operators $\bar{h}\Gamma q$ and $\bar{q}\Gamma' h$ interpolate the initial and final meson states, respectively. (Of course, the proof is valid for any external heavy hadrons, not just mesons.) Note that in HQET, the one-particle poles are projected out by the differential operator $v \cdot i\mathbb{D}$ rather than by $[(i\mathbb{D})^2 - M_B^2]$ as in full QCD.

We now write the generating functional for Green's functions of this theory:

$$\exp(iW) = \int [dh][d\bar{h}][dh'][d\bar{h}'][dA_\mu] \exp \left\{ i \int dy [\mathcal{L}_0 + S_J + S_L + S_M + \dots] \right\}. \tag{A.12}$$

Here

$$\mathcal{L}_0 = \bar{h}v \cdot i\mathbb{D}h + \bar{h}'v' \cdot i\mathbb{D}h' \tag{A.13}$$

is the lagrangian of the effective theory, and we have included explicitly a variety of relevant source terms:

$$\begin{aligned}
S_J &= J \bar{h}' A(\mathbb{D}) v \cdot i\mathbb{D}h, \\
S_L &= L \bar{h}F(\mathbb{D})h, \\
S_M &= K \bar{h}\Gamma q + K' \bar{q}\Gamma' h'.
\end{aligned} \tag{A.14}$$

The ellipses denote sources for the fermions and gauge fields, and gauge-fixing terms which will play no role in the analysis. With these definitions, then, we have

$$\begin{aligned}
& \frac{\delta}{\delta iK'(z')} \frac{\delta}{\delta iJ(0)} \frac{\delta}{\delta iL(x)} \frac{\delta}{\delta iK(z)} \exp(iW) \Big|_{\text{sources}=0} \\
&= \langle 0 | T \{ \bar{q}\Gamma' h'(z'), \bar{h}' A(\mathbb{D}) v \cdot i\mathbb{D}h(0), \bar{h}F(\mathbb{D})h(x), \bar{h}\Gamma q(z) \} | 0 \rangle.
\end{aligned} \tag{A.15}$$

We now perform a shift of the integration variable \bar{h} ,

$$\bar{h} = \bar{h}^* - J \bar{h}' A(\mathbb{D}) P_+, \tag{A.16}$$

insert it into the generating functional (A.12), and drop terms of order J^2 . We then obtain shifts in some of the expressions (A.13) and (A.14):

$$\begin{aligned}
\mathcal{L}_0 &= \bar{h}^* v \cdot iDh - J \bar{h}' A(D)v \cdot iDh + \bar{h}' v' \cdot iDh', \\
S_J &= J \bar{h}' A(D)v \cdot iDh, \\
S_L &= L \bar{h}^* F(D)h - LJ \bar{h}' A(D)P_+ F(D)h, \\
S_M &= K \bar{h}^* \Gamma_q - KJ \bar{h}' A(D)P_+ \Gamma_q + K' \bar{q} \Gamma' h'.
\end{aligned} \tag{A.17}$$

Note that the original source term S_J cancels against the shift in \mathcal{L}_0 in (A.17), but new terms appear in S_L and S_M . Replacing the dummy variable \bar{h}^* by \bar{h} , we recover the generating functional (A.12), but with the source S_J changed,

$$S_J \rightarrow -LJ \bar{h}' A(D)P_+ F(D)h - KJ \bar{h}' A(D)P_+ \Gamma_q. \tag{A.18}$$

We now repeat the derivative in eq. (A.15). When we set the sources to zero, we see that a derivative with respect to L or K must come with a derivative with respect to J to give a nonzero contribution. We then find

$$\begin{aligned}
&\langle 0 | T \{ \bar{q} \Gamma' h'(z'), \bar{h}' A(D)v \cdot iDh(0), \bar{h} F(D)h(x), \bar{h} \Gamma_q(z) \} | 0 \rangle \\
&= i\delta(z) \langle 0 | T \{ \bar{q} \Gamma' h'(z'), \bar{h} A(D)h(x), \bar{h}' A(D)P_+ \Gamma_q(0) \} | 0 \rangle \\
&\quad + i\delta(x) \langle 0 | T \{ \bar{q} \Gamma' h'(z'), \bar{h}' A(D)P_+ F(D)h(0), \bar{h} \Gamma_q(z) \} | 0 \rangle.
\end{aligned} \tag{A.19}$$

Finally, we must perform the integral ($i \int dx \int dz dz' e^{ik \cdot z} e^{ik' \cdot z'} v \cdot iD_z v' \cdot iD_{z'}$) to recover the matrix element (A.11). In this integral, the first term on the right-hand side of eq. (A.19) vanishes for an on-shell state with $v \cdot k = 0$, because the integral over z is trivial and there is no longer a one-particle pole to pick out. The second term, however, yields an S -matrix element in the usual way,

$$\begin{aligned}
&i \int dx i\delta(x) \int dz dz' e^{ik \cdot z} e^{ik' \cdot z'} v \cdot iD_z v' \cdot iD_{z'} \\
&\quad \cdot \langle 0 | T \{ \bar{q} \Gamma' h'(z'), \bar{h}' A(D)P_+ F(D)h(0), \bar{h} \Gamma_q(z) \} | 0 \rangle \\
&= -\langle M(v') | \bar{h}' A(D)P_+ F(D)h(0) | M(v) \rangle.
\end{aligned} \tag{A.20}$$

Thus we obtain the desired identity,

$$\begin{aligned}
&i \int dx \langle M(v') | T \{ \bar{h}' A(D)v \cdot iDh, \bar{h} F(D)h(x) \} | M(v) \rangle \\
&= -\langle M(v') | \bar{h}' A(D)P_+ F(D)h | M(v) \rangle.
\end{aligned} \tag{A.21}$$

Note that the term in $F(D)$ proportional to $(v \cdot iD)^2$ will not contribute here, since this matrix element is of the form (A.6) and hence vanishes by the equation of motion.

A few additional comments are in order. First, Politzer's result (A.6) for the matrix elements of an operator which vanishes by the equation of motion follows from an identical derivation, but with the derivative $\delta/\delta iL$ omitted. In this case the second term of eq. (A.19) does not appear, and we obtain zero instead of the right-hand side of eq. (A.20). (We stress that the intermediate result (A.19), which is the key to both proofs, is derived by Politzer in full generality.) Second, it is clear how this result is to be generalised to the time-ordered product of an arbitrary number of operators. Essentially, we obtain a term on the right-hand side for each contraction of $\bar{h}' A(D)v \cdot iDh$ with an operator insertion $\bar{h}G(D)h$, where $G(D)$ is any function of covariant derivatives. More than one contraction may be required to give a nonzero result; for example, an operator of the form $\bar{h}' A(D)(v \cdot iD)^n h$ will have a nonvanishing matrix element only when included in a time-ordered product with n other operators such as $\bar{h}F(D)h$. Finally, we reiterate that while for concreteness we have framed our derivation within the heavy quark effective theory, it is in fact completely general.

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Figure Captions

- Fig. 1. Feynman diagrams contributing to the time-ordered product $T\{\bar{b}\Gamma_1 s, \bar{s}\Gamma_2 b\}$.
- Fig. 2. Invariant mass spectrum for $B \rightarrow X_s \ell^+ \ell^-$. The solid line corresponds to the parton model, while the dashed line corresponds to $\lambda_1 = 0.5 \text{ GeV}^2$ and $\lambda_2 = 0.12 \text{ GeV}^2$. The cusp at $\hat{s} = (2m_c/m_b)^2$ corresponds to the charm threshold. Near this point our estimate of the nonperturbative corrections is not valid due to resonance effects.

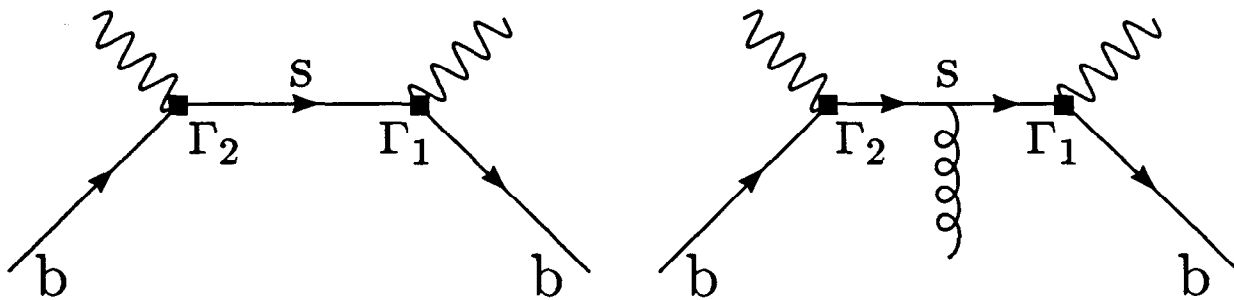


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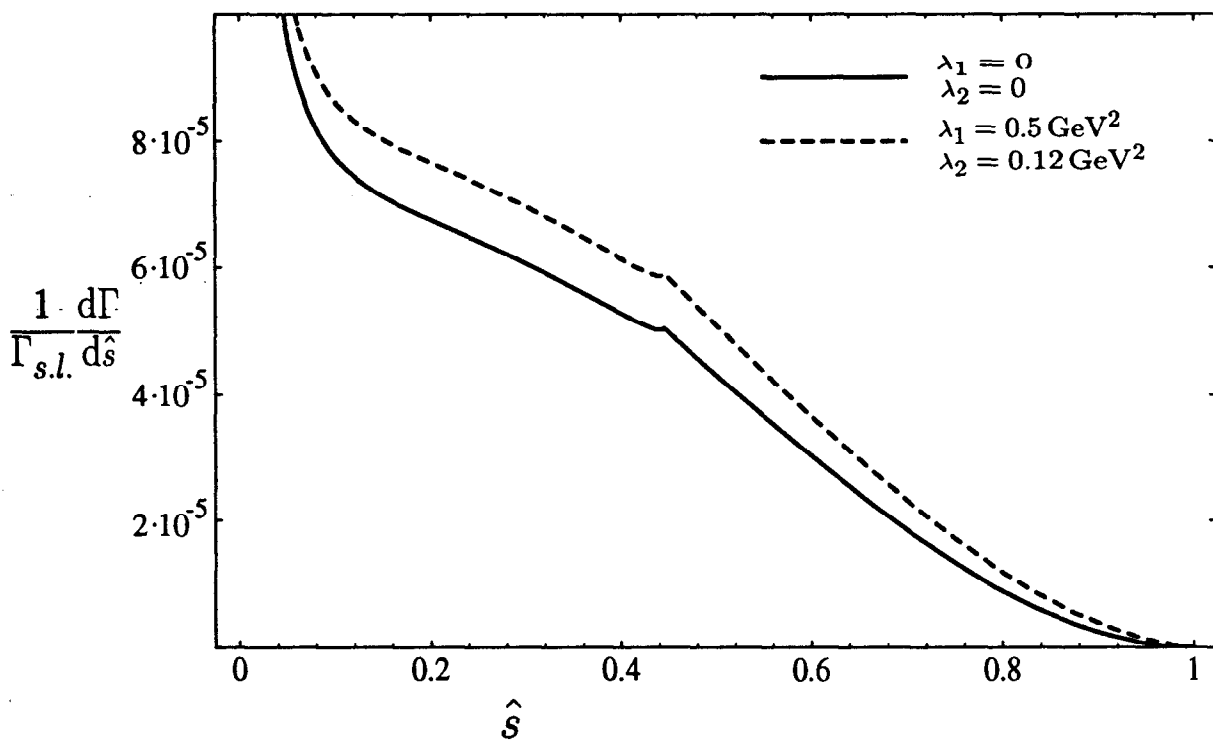


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