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AXIAL NEAR-SYMMETRY, ANGULAR MOMENTUM, AND POLAR EXCITATION*

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ABSTRACT

The mechanism of the excitation of the Earth's polar instability in the absence of external torques is explored. The conventional perturbation scheme used to simplify the Liouville equation has oversimplified the excitation physics. The Earth becomes axially near-symmetrical and slightly triaxial during polar excitation, which induces additional change in the moment of inertia. Polar excitation is due to mass redistribution in a part of the Earth to appear as a relative angular momentum involving both motion and rotation, while the rest of the Earth is only in rotation. Relative angular momentum arising due to mass redistribution consists of two terms; one is due to motion, and the other involves the products of inertia induced by the motion. Motion excites a wobble of the rotation axis around the principal axis. The products of inertia force the "instantaneous figure axis" to shift away from the principal axis to initiate secular polar shift, while the rotation axis continues to wobble around the "instantaneous figure axis" at its new position. Wobble can thus be excited alone by motion, but secular polar shift comes always along with a wobble and can only be excited by the products of inertia. The "residual" products of inertia induced by motion and the products of inertia arising due to axial near-symmetry constitute continued polar excitation. During continued polar excitation, the "instantaneous figure axis" around which the rotation axis wobbles will no longer be the principal axis. The principal axis will no longer be symmetrical, with its new location to be determined. New definition of relative

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angular momentum facilitates a simpler and physically more justified way to determine the angular momenta of different parts of the Earth, such as the atmosphere, oceans, outer core, earthquake and tectonic movements, or even meteorite impact, that involve motion. A meteorite impact is insignificant for polar excitation if its mass is not great enough to induce substantial products of inertia in the Earth. The relative angular momentum of the atmosphere does not have the physical inconsistencies as that of the conventional angular momentum function. True polar wandering in geologic history is expected to be small. The recent secular polar shift is unlikely attributable to the Earth's viscoplastic response to the Pleistocene deglaciation.

1. INTRODUCTION

The rotation of the Earth is very complex. The pull of the Moon and the Sun and other planets on the Earth's equatorial bulge causes the precession of the Earth's rotation axis in space. The complex interplay of the orbits of the Sun and the Moon is associated with oscillations of shorter periods, the forced nutation (Munk and MacDonald, 1960, p. 6-7). Free nutation or "Eulerian nutation" of the Earth's rotation axis is what the Chandler wobble is referred to. There are yet other irregularities in the Earth's rotation, such as secular polar shift, annual wobble, and changes in the length of day. Recent geodetic observations have also detected rapid components in polar motion and changes in the length of day (Eubanks et al, 1988; Salstein and Rosen, 1989; Rosen et al, 1990; Hide and Dickey, 1991). This paper concerns some fundamentals in the modeling of polar motion. Polar motion here refers to the Earth's rotation instability excited in the absence of external torques, including primarily the Chandler wobble and secular polar shift, or polar wandering in geological history.

The Liouville equation (Munk and MacDonald, 1960, p. 9-10; Lambeck, 1980, p. 33-36; Moritz and Mueller, 1988, p. 122-124) is a generalized Eulerian equation of motion which allows particles in a rotating system to move among themselves. The rotation described by the Liouville equation is therefore no longer rigid because of its tolerance of motion or mass redistribution within the system. A complete solution of the equation is extremely difficult; all inquiries concerning rotation irregularities take the form of special solutions to the equation through a perturbation scheme which simplifies the equation. The most widely used perturbation scheme in the study of polar instability is from Munk and MacDonald (1960, p.38-39). In this scheme the angular momentum is separated into that due to matter distribution and that due to motion relative to a rotating frame with its major axis nearly parallel to the rotation axis. First-order perturbation is then introduced into both the rotation velocity and the moment of inertia in the matter part of the angular momentum. With such a scheme, the Liouville equation reduces to simple first-order linear differential equations which neatly separate the two equatorial components of the rotation from its axial component (Munk and MacDonald, 1960, p. 38, eqs. 6.1.2 and 6.1.3). The scheme explains the separation of the rotation axis from the instantaneous figure axis through polar excitation. However, as Pan (1975, 1982) points out, the scheme has oversimplified the excitation physics. In the Liouville equation, only the rotation is an unknown, and as we shall see, the changes in the moment of inertia can actually be, instead of introduced as a mathematical perturbation, derived physically in connection to motion to become an integral part of the mass redistribution that excites the rotation perturbation or polar motion.

In the absence of external torques, polar excitation must be due to mass redistribution within the Earth which perturbs the angular momentum. A mass redistribution will involve both motion (mass transportation) and change in the moment of inertia (mass redistribution). Motion excites a separation of the rotation axis from the instantaneous figure axis, while change in the moment of inertia will force the instantaneous figure axis to shift away from its original (symmetrical) position to a slightly asymmetrical or near-symmetrical position. The shift of the instantaneous figure axis will then induce additional change in the moment of inertia (Pan, 1975, 1982, 1985). A polar excitation will, therefore, involve motion and two forms of change in the moment of inertia, that due to mass redistribution and that due to the shift of the instantaneous figure axis to a near-symmetrical position. The perturbation to the moment of inertia introduced in the Munk and

MacDonald scheme does not reflect such two physically distinct forms of change in the moment of inertia. On the other hand, the relative angular momentum in the scheme is quite general, implying motion in the whole system relative to a rotating reference frame. Such a relative angular momentum can only excite a separation of the rotation axis from the instantaneous figure axis, but not a change in the moment of inertia to force the shift of the instantaneous figure axis. The relative angular momentum is, therefore, not physically linked to the matter perturbation introduced in the scheme.

As Munk and MacDonald (1960, p. 14 and p. 38) themselves have pointed out, their scheme is not valid for polar wandering. In the scheme, the location of the major axis of the reference frame, or the reference axis, is not unique and is also physically uncertain, with its approximation deteriorating, as Pan (1982) points out, with order twice the amplitude of the Chandler wobble. The reference axis will eventually disassociate itself from polar motion as the rotation axis drifts away to pursue after the instantaneous figure axis for polar stability. Such a reference frame is, therefore, adequate only for the description of the damping of the wobble or the products of inertia about the *wobbling rotation axis*, but not the damping of the products of inertia that excited polar motion.

This paper gives a more detailed analysis of the polar excitation mechanism. The Earth's axial near-symmetry and slight triaxiality will be explored first. The reference frame is chosen to be geocentric and physically located in the Earth, with its major axis aligned with the *axis of reference* (Munk and MacDonald, 1960, p. 5) around which the rotation axis revolves. Such a reference frame is valid both for wobble and for polar wandering. First-order perturbation is introduced only into the Earth's rotation, while the change in the moment of inertia will be linked physically to the motion that carries out mass redistribution. With such a perturbation scheme, polar excitation arising due to motion in a part of the Earth, such as the fluctuation of atmosphere, ocean currents, or flows in the liquid outer core, becomes mathematically simpler to treat and physically more justified. It has also been found that secular polar shift always comes along with a wobble of the rotation axis around an "instantaneous figure axis" which is no longer the principal axis. The present physical location of the principal axis in the Earth is yet to be determined.

2. AXIAL NEAR-SYMMETRY

For a body of revolution or an asymmetric body rotating free of external torques, a stable rotation can be reached about the axis of either maximum or minimum moment of inertia. However, this statement is true only if the rotating body is perfectly rigid. For a non-rigid body or a body rotating with energy dissipation, a stable rotation can be reached only about the axis of maximum moment of inertia, or the principal axis (Thomson, 1963, p. 212-214). For a rheological heavenly body like the Earth, a stable rotation can be reached, therefore, only about the principal axis in a minimum energy configuration. A separation of the rotation axis from the principal axis, as has been observed for the Earth, implies that the rotation is in an unstable state of a higher rotational energy configuration. This is polar excitation. Here, however, we shall put aside polar excitation for the moment, and look first at the geometry that comes along with it.

Figure 1 illustrates diagrammatically the polar excitation geometry based on the Munk and MacDonald scheme (Munk and MacDonald, 1960, p. 41 and 44, Fig. 6.1; Pan, 1985, Fig. 4). In the figure, the initial figure axis is the Earth's *original* principal axis which would be symmetrical if the Earth were biaxial. The instantaneous figure axis is the axis around which the rotation axis wobbles; it is also called the excitation axis by Munk and MacDonald (1960, p. 41). Whereas, the deformation axis is the axis symmetrical to the Earth's equatorial bulge.

In a physical body, there is only one coordinate system, the axes of the principal moments of inertia or the *principal axes,* referring to which the inertia tensor is diagonal. Therefore, looking at the geometry alone, we can easily see that if the initial figure axis in Figure 1 is the major axis of the coordinate system referring to which the inertia tensor is diagonal, then, referring to a coordinate system with the excitation axis as its major axis the inertia tensor will not be diagonal. Geometrically, this means that if the initial figure axis is symmetrical, then the excitation axis, no matter how close it is to the initial figure axis, will not be a symmetrical axis. Otherwise, the Earth would have to be either perfectly spherical or totally fluid. This is the axial near-symmetry of the Earth.

The axial near-symmetry can be expressed in an analytical form. Let, as shown in Fig. 2, the (a,b,c) system be the Earth's original or initial principal axes referring to which the inertia tensor *was* diagonal, with the major axis, the c-axis, aligned with the initial figure axis in Fig. 1. Let the (x,y,z) system be the Earth's axes of instantaneous moments of inertia, with its major axis, the z-axis, aligned with the excitation axis in Fig. 1. Let the angle pair (θ, ϕ) be the axial near-symmetry, where θ is the deviation angle between the c- and z-axes, and ϕ is the azimuth angle between corresponding equatorial axes of the two systems. Let $A < B < C$ be the Earth's original principal moments of inertia about the (a,b,c) system, then the moments and products of inertia about the (x,y,z) system are (Pan, 1975, 1982, 1983, 1985):

$$
\begin{aligned}
I_x &= A\cos^2\phi + B\sin^2\phi,\\
I_y &= \left(A\sin^2\phi + B\cos^2\phi\right)\cos^2\theta + C\sin^2\theta,\\
I_z &= \left(A\sin^2\phi + B\cos^2\phi\right)\sin^2\theta + C\cos^2\theta,\\
I_{xy} &= (B-A)\sin\phi\cos\phi\cos\theta,\\
I_{xz} &= (B-A)\sin\phi\cos\phi\sin\theta, \text{ and}\\
\mathrm{I}_{yz} &= \left(C - A\sin^2\phi - B\cos^2\phi\right)\sin\theta\cos\theta.
\end{aligned} \tag{1}
$$

From Eq. 1 and Fig. 1, we can see that, during polar excitation, the Earth becomes axially near-symmetrical and slightly triaxial even if it was originally biaxial. The products of inertia would disappear only if the Earth were either a perfect sphere, or totally fluid such that it could adjust immediately to its new rotation. Equation 1 also shows that, after the axial near-symmetry is accounted for, the excitation axis in Fig. 1 is no longer a generalization of the "principal axis" or "axis of figure" as Munk and MacDonald (1960, p. 41, footnote) have suggested. In this paper, the excitation axis or the quoted "instantaneous figure axis" is, therefore, not equivalent to the principal axis as it is in the conventional sense.

Using presently available values for the Earth's principal moments of inertia, $A = 8.0108\times10^{44}$ g-cm^2, $B = 8.0110\times10^{44}$ g-cm^2, $C = 8.0372\times10^{44}$ g-cm^2, and for the axial near-symmetry, $\theta = 0.8^\circ = 0.014$ rad (Pines and Shaham, 1973), and $\phi = 27.3^\circ$, 40° (Pan, 1975), as well as arbitrary values 0° and 45°, Pan (1982) calculates from Eq. 1 the Earth's moments and products of inertia about the (x,y,z) system (listed in Table 1). From the table we can see that the moments of inertia are about the same magnitude as that of the principal moments of inertia, while the products of inertia are 4 to 6 orders smaller than the moments of inertia. These values show that the assumption in the Munk and MacDonald scheme that the Earth's moments of inertia after perturbation are the same as those prior to perturbation is valid as a first-order approximation. What the scheme has ignored are the products of inertia that come along with the axial near-symmetry. These products of inertia are of comparable order of magnitude as, or even greater than, the products of inertia that may be induced by mass redistribution in the Earth. For instance, the magnitude of I_{yz}, listed in Table 1, is an order of magnitude greater than that induced by the two major Pleistocene ice sheets, Laurentide and Fennoscandia, which is about 3.5×10^{39} g-cm^2, based on a total mass of 2.4×10^{22} g for the two ice sheets (Sabadini et al, 1982).

The above calculation is, however, from a geological history point of view, to demonstrate that the products of inertia arising from the axial near-symmetry are not negligible for polar excitation. The values listed in Table 1, therefore, may not accurately represent the instantaneous moments and products of inertia of the present Earth. More accurate determination of the Earth's instantaneous moments and products of inertia depends on a more accurate location of the Earth's principal axes, and consequently the magnitude of the axial near-symmetry (θ, ϕ). The (a,b,c) system in Fig. 2 is the Earth's original or initial principal axes which, after polar excitation, is no longer the Earth's principal axes, while the (x,y,z) system is obviously not the principal axes. The location of the Earth's principal axes after polar excitation is, therefore, yet to be determined, and we will discuss it further later on. More critical is the designation of a reference frame in the Earth.

Munk and MacDonald (1960, p. 10-14) point out that, because of the Earth's non-rigidity, it is unlikely to find a truly body-fixed frame in the Earth, and reference frames such as the *Tisserand* and the *principal axes* were the obvious choices for mathematical simplicity. They also point out that the geographic frame may be chosen with its major axis aligned with either "the axis of figure" or with the rotation axis, and they have chosen the latter to conform with their treatment of the change in the moment of inertia as a first-order perturbation and the excitation axis as the principal axis. Chao (1984) further examines the different reference frames, including Smith's (1977) invariant frame, and concludes that all the theoretical frames are physically or observationally approximate. In this paper, the (x,y,z) system in Fig. 2 is the obvious choice for a reference frame because it accounts for the axial near-symmetry. In order to make this frame physically more meaningful, let it be geocentric and the z-axis be aligned with the *axis of reference* (Munk and MacDonald, 1960, p. 5) or the *geographic axis* (Smith, 1977) around which the rotation axis revolves. On the other hand, the y-axis is chosen to be in the direction of secular polar shift, while the x-axis is perpendicular to y- and z-axes in the right-handed system. Such a reference frame is physically always associated with polar motion, including polar wandering. The *axis of reference* can be geodetically determined, and if the direction of secular polar shift can also be determined to certain accuracy, this reference frame may also be adopted for shorter-term observation. However, one should note that this reference frame is not the conventional "principal axes" frame, since, as has previously been noted, the *axis of reference* or the z-axis of the frame is not the Earth's principal axis.

3. ANGULAR MOMENTUM

The angular momentum in the Munk and MacDonald scheme (1960, p. 9, eqs. 3.1.3, 3.1.4 and 3.1.5) is quite general. It is in fact based on a system of particles moving among themselves, and then converted into that of a continuous system, about a reference frame rotating relative to an instantaneously coinciding inertial frame fixed in space. Munk and MacDonald (1960, p. 38) assume that the perturbation to the matter part of the angular momentum is negligibly small in comparison with the Earth's original angular momentum. This assumption is sound because polar excitation is unlikely to involve mass redistribution in the whole Earth. However, in the scheme, the motion part of the angular momentum is in a general form for motion in the whole system relative to a rotating frame, and is independent of the matter perturbation to the angular momentum. The scheme has, therefore, overlooked the physical link between the matter and motion parts of the angular momentum during polar excitation. A matter perturbation to the angular momentum would require a concurring motion or transportation to carry it out.

In this paper, the Earth's angular momentum is derived by separating the Earth into two parts based on the involvement of the part in motion or mass redistribution. Let **I** be the moment of inertia of the part of the Earth *free* of motion, ω be the rotation velocity, and **h** be the angular momentum of the part of the Earth *involved* in motion; then the Earth's total angular momentum is

$$\mathbf{H} = \mathbf{I} \cdot \omega + \mathbf{h} \tag{2}$$

In Eq. 2, **I** can be only slightly different from the Earth's original inertia tensor, since the motion or mass redistribution that excites a polar motion is expected to involve only a minor part of the Earth. **I** is not different from that of a rigid body either, since it involves no motion, only rotation (Becker, 1954, p. 273-274; Goldstein, 1980, p. 190-192; Thomson, 1963, p. 102-104). No generality is lost taking Eq. 1 as this inertia tensor, with the previously defined (x,y,z) system the reference frame rotating relative to an instantaneously coinciding inertial frame fixed in space. Whereas, **h** can be derived by allowing motion only in the part of the Earth that contributes to polar excitation, about the same (x,y,z) system rotating relative to an instantaneously coinciding inertial frame fixed in space. **h** will then involve both motion and rotation and consist of two terms, that due to motion and that due to the moments and products of inertia induced by the motion (Pan, 1975, 1982). With Ω as the Earth's average rotation speed, we have the three components of **h**:

$$h_x = p_x - \Omega \Delta I_{xz}, \quad h_y = p_y - \Omega \Delta I_{yz}, \quad \text{and} \quad h_z = p_z - \Omega \Delta I_z \tag{3}$$

where (p_x, p_y, p_z) arises from the motion (u_x, u_y, u_z) relative to the (x,y,z) frame. This term is the same as the relative angular momentum defined by Munk and MacDonald (1960, p. 9, eq. 3.1.5), except that the motion no longer involves the whole Earth but only the part of the Earth that excites polar motion; i. e.,

$$p_x = \int (u_z y - u_y z) dM, \quad p_y = \int (u_x z - u_z x) dM, \quad \text{and} \quad p_z = \int (u_y x - u_x y) dM, \tag{4}$$

where M is the mass of the part of the Earth involved in motion. On the other hand, in the second term,

$$\Delta I_z = \int (x^2 + y^2) dM, \quad \Delta I_{xz} = \int xz dM, \quad \text{and} \quad \Delta I_{yz} = \int yz dM \tag{5}$$

are the moment and products of inertia induced by the redistribution of mass M. The second term appears because **h** refers to a frame rotating relative to an instantaneously coinciding inertial frame fixed in space. According to Moritz and Mueller (1988, p. 117-119), for a geocentric system, the angular momentum equation referring to the (spatially moving) origin of a body frame has the same form as if it referred to the (spatially fixed) origin of an inertial frame. We may, therefore, still call **h** the relative angular momentum. *Redistributional* angular momentum may be a good substitution, while Moritz and Mueller (1988, p. 114) also call its old definition the *deformational* angular momentum.

The Earth's angular momentum defined in Eqs. 2–5 not only accounts for the axial near-symmetry, but also links motion with the change in the moments and products of inertia it induces as an integral part of the mass redistribution. This definition of the Earth's angular momentum facilitates a simpler and physically more justified way to determine the angular momenta of the different parts of the Earth, such as the atmosphere, oceans, outer core, and even tectonic movements in the lithosphere and asthenosphere that involve motion. The Earth's total angular momentum can be determined through individually defining **h** for each part of the Earth that involves motion, and then adding these together with the rest of the Earth as a rigid or non-rigid (elastic or viscoplastic) body only in rotation. Angular momentum arising from transit motions, such as earthquakes or meteorite impact, can also be determined using Eq. 3. In such cases, the first term of the relative angular momentum will behave like a Dirac function (Munk and MacDonald, 1960, p. 46-47, right of fig. 6.2; Chao, 1984), since it will disappear as soon as the motion ceases. Whereas, the second term will behave like a Heaviside step function (Munk and MacDonald, 1960, p. 46-47, left of fig. 6.2; Chao, 1984), since the moments and products of inertia induced by the motion will not disappear with the cessation of the motion. The moments and products of inertia in the second term can thus be called the "residual" moments and products of inertia.

With the above manipulation, the perturbation to the Earth's angular momentum in a polar excitation becomes physically responsive to the mass redistribution that causes the perturbation. Though the reference frame chosen is not truly a body-fixed frame, it is unique and physically located in the Earth. Pan (1975, 1978, 1982, 1985) uses this scheme to simplify the Liouville equation for the study of polar motion and related dynamics. It has been found that the Chandler wobble has multiple frequencies and is quasi-permanent (Pan, 1982), while the gyroscopic effect or gyricity dominates geodynamics in an axially near-symmetrical Earth (Pan, 1993).

4. POLAR EXCITATION AND PRINCIPAL AXIS

Substituting Eqs. 1 and 3 into Eq. 2 and then into the Liouville equation, and using a linearization process similar to that of Munk and MacDonald (1960, p. 36) to neglect higher order terms, we obtain, with ($\cdot$) designating *d/dt* relative to the (x,y,z) frame, the polar excitation function free of external torques (Pan, 1975, eq. 6; 1982, eq. 7):

$$
\begin{aligned}
\Psi_x &= \frac{1}{(I_z - I_x)\Omega^2}\left[(I_{xz} + I_{xz})\Omega^2 + \left(\dot{I}_{yz} + \dot{I}_{yz}\right) - p_x - \dot{p}_y\right], \\
\Psi_y &= \frac{1}{(I_z - I_y)\Omega^2}\left[-(I_{yz} + I_{yz})\Omega^2 + \left(\dot{I}_{xz} + \dot{I}_{xz}\right) + p_y - \dot{p}_x\right], \quad \text{and} \\
\Psi_z &= \frac{1}{I_z\Omega^2}\left[\left(\dot{I}_z + \dot{I}_z\right) + \dot{p}_z\right],
\end{aligned}
\tag{6}
$$

where $\dot{I}_z$, $\dot{I}_{xz}$, and $\dot{I}_{yz}$ are the rates of change of the corresponding moment and products of inertia in Eq. 1; $\Delta\dot{I}_z$, $\Delta\dot{I}_{xz}$, and $\Delta\dot{I}_{yz}$ are the rates of change of ΔI_z, ΔI_{xz}, and ΔI_{yz}; and $\dot{p}_x$ $\dot{p}_y$, and $\dot{p}_z$ are those of p_x, p_y, and p_z. Equation 6 is slightly different from the excitation function derived by Munk and MacDonald (1960, p. 38, eq. 6.1.5; Lambeck, 1980, p. 35, eq. 3.2.7a; Moritz and Mueller, 1988, p. 301, eq. 5-82) due to the inclusion of the axial near-symmetry as well as the integration of the perturbing moments and products of inertia into the relative angular momentum (Pan, 1975, 1982). If the Earth's axial near-symmetry and triaxiality are negligibly small, Eq. 6 will approach the excitation function of Munk and MacDonald, except for a sign difference arising from the different approach to perturbation.

Equation 6 can be used to demonstrate that in a rotationally unstable Earth the excitation axis (Munk and MacDonald, 1960, p. 41) or the geographic axis (Smith, 1977) around which the rotation axis revolves is not the principal axis. From the definition of the principal axis we know that the excitation axis in Fig. 1 or the z-axis in Fig. 2 can be the principal axis only if the products of inertia about the (x,y,z) frame vanish. This condition can be met only when the *non-motion* part of the x- and y-components of the excitation function in Eq. 6 becomes zero; polar excitation will then behave like a Dirac function (Munk and MacDonald, 1960, p. 46-47). This can occur only when the first term of the relative angular momentum in Eq. 3 prevails over the second term. An extreme case is when the motion induces no products of inertia, and thus no axial near-symmetry, in the Earth. In that case the excitation axis or instantaneous figure axis will stay at the same position, the initial figure axis in Fig. 1, while the rotation axis will shift away and wobble around. However, in general a relative angular momentum will involve both motion and consequent mass redistribution; i. e., it will involve both terms in Eq. 3, though one may prevail over the other in due course of their occurance. Motion may prevail at the incipience of the excitation. At that moment, the rotation axis will shift away from the initial figure axis and wobble around. Afterward motion may diminish or even vanish, but the "residual" products of inertia may not disappear with the cessation of the motion. The products of inertia will then force the excitation axis or "instantaneous figure axis" to shift away from the initial figure axis to a new position around which the rotation axis will continue to wobble.

The above can be demonstrated analytically. From Eq. 6 we see that the excitation axis can be the principal axis only if

$$
\begin{aligned}
&p_x + \frac{\dot{p}_y}{\Omega} - \left(\dot{I}_{yz} + \dot{I}_{yz}\right) \geq \left(I_{xz} + I_{xz}\right)\Omega, \quad \text{and} \\
&p_y - \frac{\dot{p}_x}{\Omega} + \left(\dot{I}_{xz} + \dot{I}_{xz}\right) \geq \left(I_{yz} + I_{yz}\right)\Omega.
\end{aligned}
\tag{7}
$$

That is, the excitation axis can become the principal axis only when the relative angular momentum components arising due to motion and its rate of change as well as the *initial* rate of change of the products of inertia are greater than or equal to those due to the products of inertia. However, Eq. 7 holds only when the products of inertia are negligibly small. It fails if the products of inertia appear to dominate. An extreme case is one in which motion ceases completely after the incipient polar excitation. The velocity and acceleration terms in Eq. 6 become zero (though rate of change of the products of inertia may continue in a different magnitude). The x- and y-components of the excitation function will then become

$$\Psi_x = \frac{I_{xz} + \Delta I_{xz}}{I_z - I_x} + \frac{\dot{I}_{xz} + \Delta \dot{I}_{xz}}{(I_z - I_x)\Omega}, \text{ and}$$
$$\Psi_y = -\frac{I_{yz} + \Delta I_{yz}}{I_z - I_y} + \frac{\dot{I}_{yz} + \Delta \dot{I}_{yz}}{(I_z - I_y)\Omega}. \tag{8}$$

Equation 8 shows that the "residual" products of inertia and the products of inertia arising from axial near-symmetry do not cancel each other to nullify the excitation function, but instead add to each other to constitute continued polar excitation. The same relationship also holds between their rates of change. This demonstrates that, during polar excitation due only to the products of inertia, the excitation axis or "instantaneous figure axis" around which the rotation axis wobbles is neither the principal axis nor a symmetrical axis to the Earth. The Munk and MacDonald scheme has overlooked this slight deviation of the "instantaneous figure axis" from the principal axis during polar excitation, and thus ignored the change in the moment of inertia that will come along with the shift of the "instantaneous figure axis" away from its original (symmetrical) position.

Figure 3 presents a diagrammatic depiction or geometric interpretation of the mechanism of polar excitation in general. The figure is a plot on a plane perpendicular to the initial figure axis in Fig. 1. Point c is the pole of the c-axis in Fig. 2, and ca and cb are the projections of the a- and b-axis, respectively. z is the pole of the z-axis in Fig. 2, and zx and zy are the projections of the x- and y-axis, respectively. The circular curve is the trace of the rotation pole ω. At the incipient moment of polar excitation, while motion prevails over the products of inertia it induces, the rotation pole ω will suddenly shift away from the initial figure pole c to revolve around. As polar excitation continues, the "residual" products of inertia accumulate. The "instantaneous figure pole" will then be forced to shift away from the initial figure pole c to the excitation pole z. The axial near-symmetry or the deviation angular distance cz will appear as soon as the "instantaneous figure pole" shifts away from pole c towards pole z. The rotation pole ω will then immediately revolve around the "instantaneous figure pole" at its new position z.

5. LOCATION OF THE PRINCIPAL AXIS

We have so far demonstrated that prior to polar excitation the initial figure axis was the Earth's principal axis. During polar excitation the excitation axis or "instantaneous figure axis" is the principal axis only if motion prevails over the products of inertia. Whereas, during continued excitation due only to the products of inertia, the excitation axis is the axis around which the rotation axis revolves but is no longer the principal axis, and the initial figure axis is also no longer the principal axis because of the appearance of the "residual" moments and products of inertia. Then where is the principal axis while polar excitation is due solely to the products of inertia? Moritz and Mueller (1988, p. 125-126) show the determination of the principal axis in the conventional way. Here we look at the problem a little differently.

Let the (a',b',c') system be the principal axes of an Earth in continued polar excitation, and (θ', ϕ') be the axial near-symmetry between the (a',b',c') system and the (x,y,z) system. At the incipience of continued polar excitation, we may assume that the excitation axis or "instantaneous figure axis" has not drifted very far away from the initial figure axis, and the principal axis stays along the polar path (Pan, 1975, 1978, 1985). Then we can let $\phi' = \phi$ and $\theta' = \theta - \delta$, where δ is a small angle. The relation between the (a',b',c') system and the (a,b,c) system is

$$\begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} = \begin{bmatrix} \cos^2\phi + \sin^2\phi\cos\delta & -\cos\phi\sin\phi(1-\cos\delta) & -\sin\phi\sin\delta \\ -\cos\phi\sin\phi(1-\cos\delta) & \sin^2\phi + \cos^2\phi\cos\delta & -\cos\phi\sin\delta \\ \sin\phi\cos\delta & \cos\phi\sin\delta & \cos\delta \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{9}$$

Here it should be noted that the (a,b,c) system was rigidly fixed to the Earth at the incipience of polar excitation and is now no longer the Earth's principal axes.

Equation 9 says that during continued polar excitation, the Earth's principal axis is different from the initial figure axis by a small angle δ. From Eq. 8 and Pan (1975), this angle is,

$$\delta = \sqrt{\left(\frac{\Delta I_{xz}}{I_z - I_x}\right)^2 + \left(\frac{\Delta I_{yz}}{I_z - I_y}\right)^2}. \qquad (10)$$

Equation 10 is, however, only valid at or around the incipient moment of continued polar excitation. Around that time the excitation axis or "instantaneous figure axis" is still close to the initial figure axis. As is depicted in Fig. 4, if the excitation pole z has wandered a considerable distance from the initial figure pole c, the angular distance cc' or the angular difference δ becomes large. The principal axis will then no longer be in the proximity of the initial figure axis. Pan (1985) intuitivly guesses that during such a time the principal axis will be in the close proximity of, but not identical to, the deformation axis. This is because the deformation axis will always migrate with the principal axis via rheological deformation to pursue after the "instantaneous figure axis", while the "instantaneous figure axis" pursues after the rotation axis.

6. DISCUSSION

From the above we learn that the wobble of the rotation axis around the principal axis can be excited alone if the perturbing relative angular momentum is due only to motion, while a relative angular momentum involving products of inertia will excite both wobble and secular polar shift. That is, there can be a wobble with no secular polar shift, but secular polar shift will always come along with a wobble. The wobble will be damped whenever the rotation axis reaches the "instantaneous figure axis." However, secular polar shift will stop only when the "instantaneous figure axis" has reached the principal axis; i. e., only when the "residual" products of inertia and the products of inertia arising due to axial near-symmetry are both damped. Damping of either wobble or secular polar shift, therefore, does not mean a stable rotation of minimum energy configuration has been reached. A stable rotation can be reached only when the wobble and the secular polar shift have both been damped; i. e., only when the rotation axis has been completely aligned with the principal axis. Yet, if polar excitation recurs before the Earth reaches a stable rotation, the deviation of the "instantaneous figure axis" from the principal axis may enlarge (Pan, 1975, 1985). Therefore, as long as there is repeated polar excitation, the "instantaneous figure axis", as well as the rotation axis, can hardly reach the principal axis.

The above conclusions concern some recent observations. Alvarez and Asaro (1990) contend that a giant asteroid or comet about 10 km in diameter may have struck the Earth some 65 million years ago. This asteroid or comet would have weighed about 1.73×10^{18} g , and would have been able to excite a wobble about 1.15×10^{-5} if it had hit vertically at 45° latitude with a speed of 10^{6} cm/sec, which is slightly greater than the present Chandler wobble. However, the excitation is due primarily to the mass of the asteroid. If the mass of the asteroid were one ton (10^{8} g), then polar excitation would only be on the order of 10^{-16}, a negligible value. This implies that the impact of a meteorite is insignificant for polar excitation if its mass is not great enough to induce substantial products of inertia in the Earth. However, the above calculation also depends on the assumption of the magnitude of the rate of change of the products of inertia. If this rate of change were as great as that which would arise from the impact speed, then the impact of a giant asteroid 10 km in diameter could excite a polar motion with a magnitude on the order of degrees, or could even tip the Earth's obliquity (Dones and Tremaine, 1993).

Polar motion components of meteorological origin are now routinely determined through the angular momentum function of the atmosphere (Hide and Dickey, 1991). The angular momentum function of the atmosphere (Barnes et al, 1983; Moritz and Mueller, 1988, p. 300-303; Bell et al, 1991) is in fact nothing other than the normalized relative angular momentum in Eq. 3, but Eq. 3 has a better physical justification. The z-component of the angular momentum function of the atmosphere has a physical character different from that of the x- and y-components, and was derived by arbitrarily setting it to the negative of the z-component of the Munk and MacDonald excitation function. No such physical inconsistencies exist in the relative angular momentum components in Eq. 3, which are all of the same physical character. The z-component is naturally positive. The angular momenta of oceans and the outer core can also be similarly determined using Eq. 3. For instance, the angular momentum of the outer core can be calculated from Eq. 3, as well as from a modification of the conventional models (Smith, 1977; Wahr, 1981a, 1981b, 1982, 1983; Sasao et al, 1980; Moritz and Mueller, 1988, p. 148-179 and 263-279) to include both motion and rotation.

The fluctuations of atmosphere, currents in the oceans, and flows in the outer core involve both motion and the products of inertia, but the products of inertia will fluctuate with the motion or even vanish with the cessation of the motion. Earthquakes and tectonic movements, rapid or slow, will, however, induce a relative angular momentum involving motion and the "residual" products of inertia. In the study of the seismic excitation of the Chandler wobble (Mansinha and Smylie, 1967; Dahlen, 1971, 1973; Smylie and Mansinha, 1971; Israel et al, 1973; Mansinha et al, 1979), concentration was only on the calculation of the products of inertia that can be induced by an earthquake displacement, and the motion part of the relative angular momentum was considered to be negligibly small. Nevertheless, as implied by Eqs. 6, 7 and 8, one of the major contributions to polar excitation is from the rate of change of the products of inertia. If the magnitude of the rate of change of the products of inertia is comparable to that arising from the motion speed, as was mentioned in the asteroid impact case, its contribution to polar excitation will then be quite significant. Pan (1975, 1982) discusses this rate of change under the rate of change of the excitation function, and demonstrates that even if this rate of change has the same magnitude as that arising from secular polar shift or plate motion, it will still contribute notably to polar excitation.

Jurdy and van der Voo (1974, 1975) and Jurdy (1981) observe that true polar wandering since the Late Cretaceous is small. This is compatible with the conclusions reached in this paper, since the deviation of the excitation axis or "instantaneous figure axis" from the principal axis during polar excitation can never be large. However, the suggestion that the present secular polar shift may be attributable to the viscoplastic response of the Earth to the Pleistocene deglaciation (Nakiboglu and Lambeck, 1980; Sabadini and Peltier, 1981) needs further discussion. The main point of argument is that secular polar shift always comes along with a wobble of the rotation axis around the "instantaneous figure axis" which is no longer the principal axis. A calculation based on the rate of change of the products of inertia given by Nakiboglu and Lambeck (1980) does not show that the postglacial rebound would be able to excite a significant wobble comparable to the present Chandler wobble. Secular polar shift represents the Earth's attempt to eliminate the deviation between the "instantaneous figure axis" and the principal axis through damping of the products of inertia. The postglacial rebound is more likely to be a part of the Earth's effort to damp the products of inertia through secular polar shift, rather than being the cause of secular polar shift. Polar instability is more likely to be excited by a relative angular momentum that will involve redistribution of a great amount of mass. Pan (1963, 1968, 1970, 1971, 1972, 1975, 1983) discusses the episodic polar excitation between geologic epochs due to explosive release of excess thermal energy accumulated in the interior through a gigantic failure along some weak zones of the lithosphere. Such a polar excitation would be able to reach a magnitude as great as 10^{-2} (Pan, 1975). Pines and Shaham (1973) suggest that some violent events in geologic history, such as the escape of the Moon from the Earth or the impact of a giant meteorite, might have excited polar instability. However, polar excitation due to the impact of a giant meteorite has been shown to be fairly insignificant, of the order of 10^{-5} at most, dependent primarily upon the mass of the meteorite.

7. CONCLUDING SUMMARY

(1) Rather than introducing it as a first-order mathematical perturbation, the perturbing moments and products of inertia in the Liouville equation can be physically linked to motion as an integral part of the mass redistribution that causes the angular momentum perturbation. (2) In the absence of external torques, polar excitation is due to mass redistribution in a part of the Earth which perturbs the angular momentum. This makes the Earth become axially near-symmetrical and slightly triaxial. Polar excitation involves motion and two forms of change in the moment of inertia, arising respectively from mass redistribution and axial near-symmetry. (3) A reference

frame that is physically located in the Earth with its major axis aligned with the *axis of reference* or *geographic axis* is valid both for wobble and for secular polar shift or polar wandering. (4) The Earth's total angular momentum can be derived from the involvement of its different parts in mass redistribution. The part of the Earth in mass redistribution will give a relative angular momentum involving both motion and rotation, while the rest of the Earth can be treated as a rigid or non-rigid (elastic or viscoplastic) body only in rotation. This definition of the angular momentum facilitates a simpler and physically more justified way to determine the angular momenta of the different parts of the Earth, such as the atmosphere, oceans, outer core, earthquakes and tectonic movements, or even meteorite impact, that involve motion. (5) A relative angular momentum consists of two terms; one due to motion, and the other involving the products of inertia induced by the motion. Motion excites a wobble of the rotation axis around the principal axis. The products of inertia force the excitation axis or "instantaneous figure axis" to shift away from the principal axis, while the rotation axis wobbles around the "instantaneous figure axis" at its new position. (6) The products of inertia arising due to axial near-symmetry will join with the "residual" products of inertia induced by motion to make up continued polar excitation. During continued polar excitation, the excitation axis or "instantaneous figure axis" is no longer the principal axis. The principal axis is no longer symmetrical, with its new location to be determined. (7) Polar motion consists primarily of wobble and secular polar shift. Wobble is due to the separation of the rotation axis from the instantaneous figure axis, while secular polar shift arises due to the shift of the "instantaneous figure axis" away from the principal axis. Wobble can be excited alone if the perturbing relative angular momentum is due only to motion. In this case the instantaneous figure axis around which the rotation axis wobbles will remain the principal axis. Secular polar shift always comes along with a wobble and can only be excited by the products of inertia. In this case the "instantaneous figure axis" around which the rotation axis wobbles will no longer be the principal axis. (8) The impact of a giant meteorite is insignificant for polar excitation if its mass is not great enough to induce substantial products of inertia in the Earth. The angular momentum of the atmosphere can be determined directly as a relative angular momentum, which has a better physical justification than the conventional angular momentum function. The rate of change of the moments and products of inertia can contribute notably to polar excitation. True polar wandering in geologic history is expected to be small, because the deviation of the excitation axis or "instantaneous figure axis" from the principal axis in a polar excitation can never be large. The present secular polar shift is not likely to be attributable to the viscoplastic response to the Pleistocene deglaciation, because secular polar shift always comes along with a wobble of the rotation axis around the "instantaneous figure axis," but calculation does not show that the viscoplastic response to deglaciation is able to excite a wobble as is presently observed.

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LIST OF FIGURES

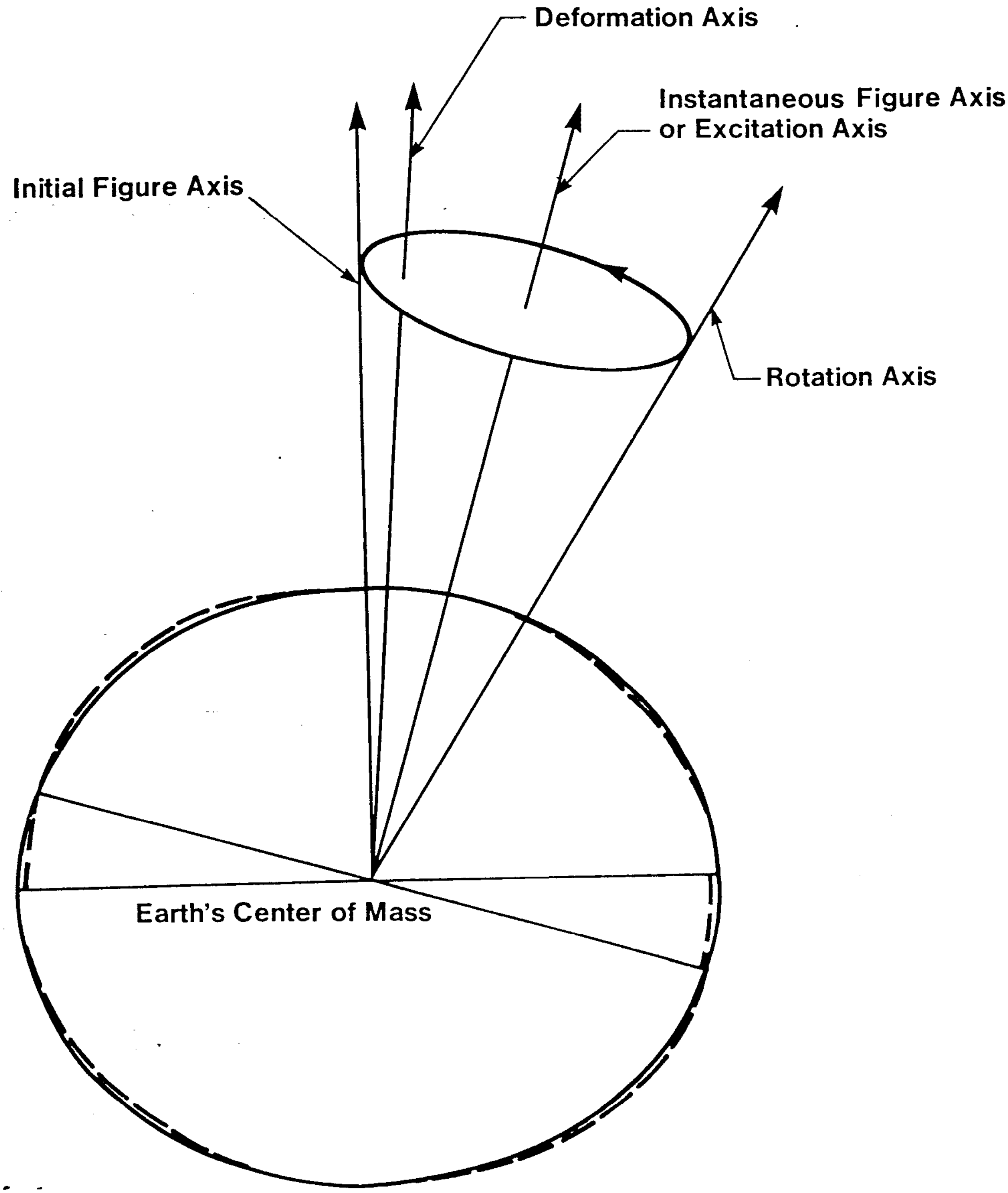
Deformation Axis
Instantaneous Figure Axis
or Excitation Axis
Initial Figure Axis
Rotation Axis
Earth's Center of Mass

Figure 1

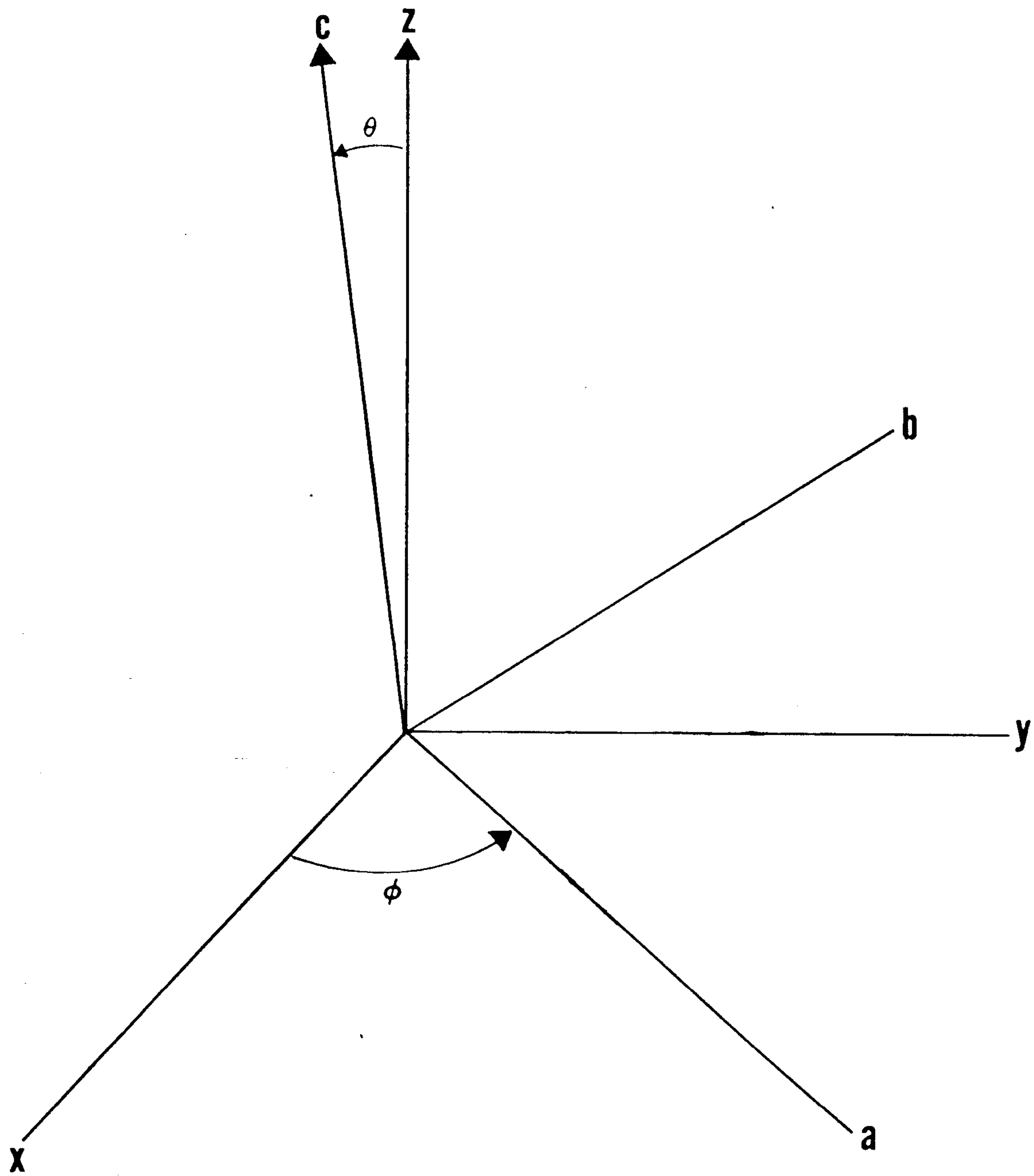

Figure 2

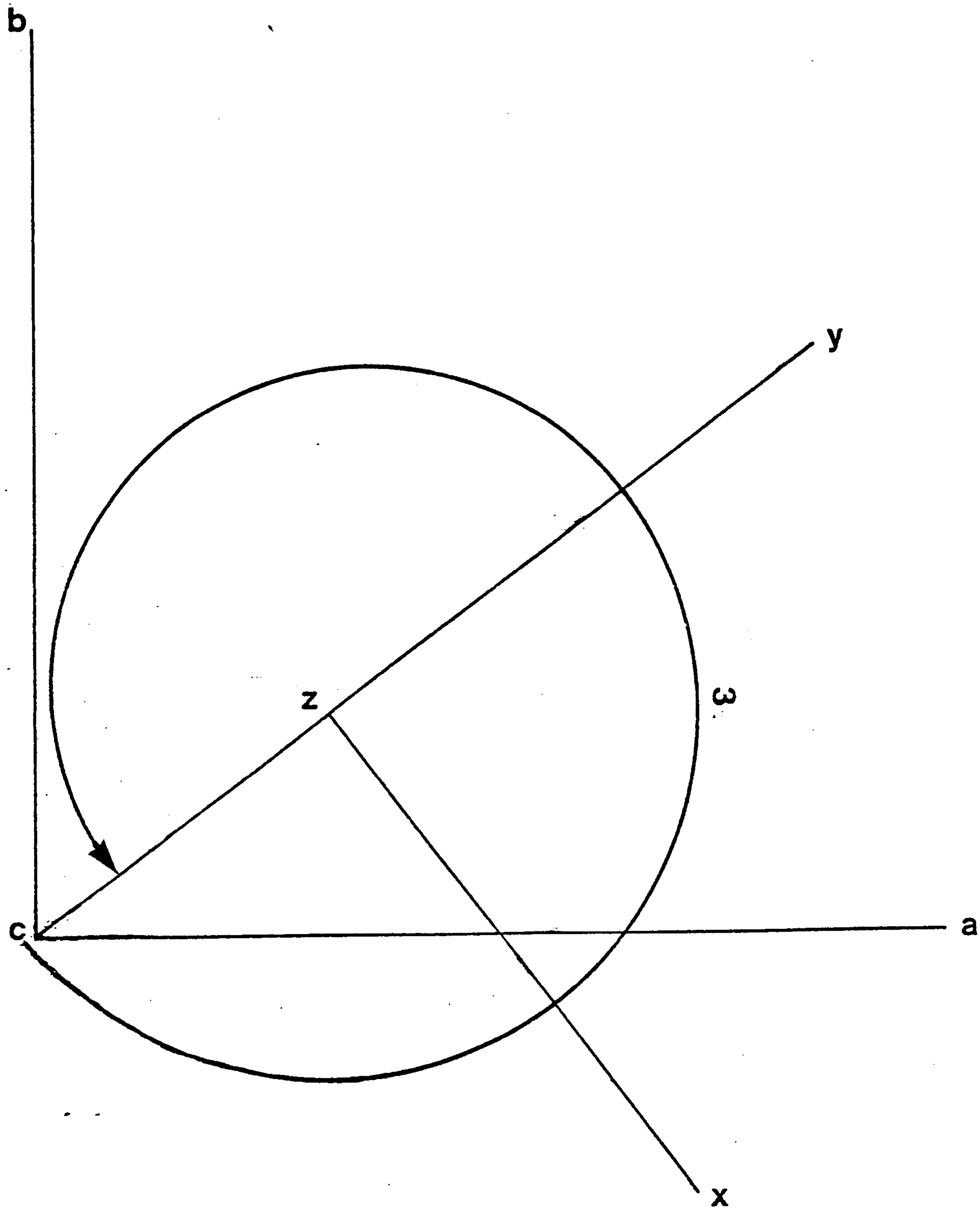

Figure 3

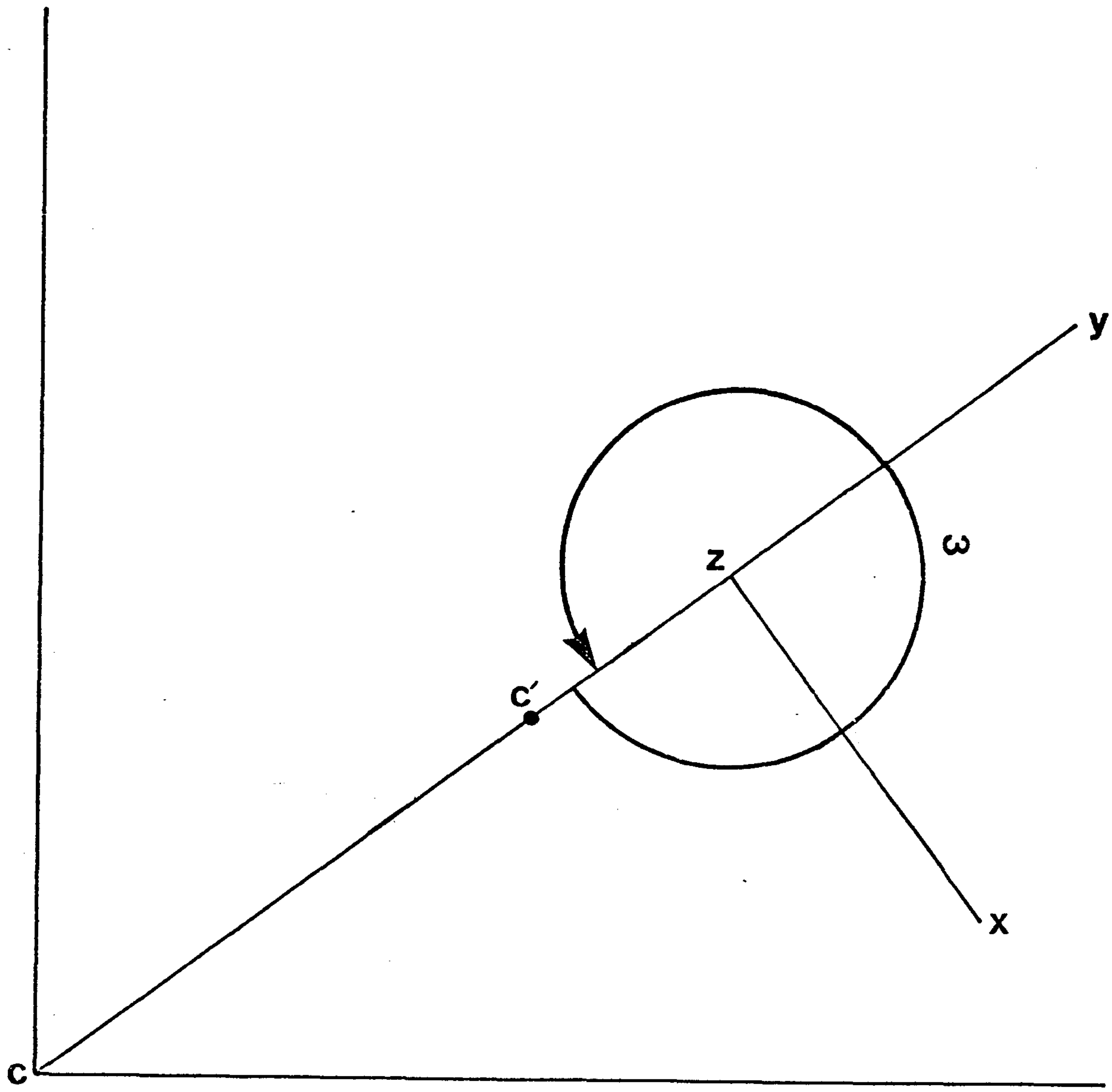

Figure 4