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DESIGN OF A WINDOWLESS DIGITAL FILTER USING FFT ALGORITHM*

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ABSTRACT

The Gibbs phenomenon appears in a discrete Fourier transform due to incomplete periodicity; i. e., the waveform has not reached a full cycle within its period. Data flipping furnishes a complete periodic cycle to the waveform and thus suppresses the Gibbs phenomenon. This facilitates the design of a digital filter using fast Fourier transform without windowing. The filter does lowpass, bandpass, highpass, bandstop, notch, or single-frequency-pass simply by manipulating the band limits. The filter can be affected by spectral resolution and the slope discontinuity at the end data points. The reduction of such effects and an alternative design are discussed.

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I. INTRODUCTION

Digital filters are designed through approximations [1]. A practical filter differs from an ideal filter in that it has loss characteristics. The loss characteristics of a filter can be reduced by increasing the order of the transfer function at the expense of computing time, but cannot be totally eliminated. It is, therefore, not possible to design an ideal digital filter through approximations. Another approach to designing a digital filter is by using the discrete Fourier transform (DFT) [1] [2] [3], but it is plagued by the Gibbs phenomenon, an inherited property of the DFT arising from the truncation of the Fourier series or the finite length of the data with discontinuities at the end points. The conventional way to treat the Gibbs phenomenon is either to introduce a transition between passbands and stopbands, or to apply windows to taper the data to zero at the end points. The introduction of transition is a rudimentary way to avoid discontinuities in the frequency response, while windowing will distort the waveform or induce spectral leakage. The design of an ideal digital filter through DFT can be achieved, therefore, only if the Gibbs phenomenon can be suppressed without involving a window. This report deals with a simple technique which will fulfill this requirement.

II. TECHNIQUE

The technique for suppressing the Gibbs phenomenon in a DFT without windowing is called *data flipping* [4]. It eliminates the discontinuities at the end points of the data with no weighting of the data points in between. The technique is quite simple. The data is reflected across the last point to create a new data set twice the length of the original, with the original last data point at the center of mirror images. Such a flipping of the data creates an even function, but will not affect the spectral content of the original data [2] [4] [5]. The even function is then input into a fast Fourier transform (FFT), and the unwanted bands are zeroed out from the spectra. The "passed" spectral bands are then inversely transformed back to the time domain. Filtering is complete. After filtering, only the original length of the data needs to be preserved.

III. FUNDAMENTAL

In a DFT, the input data of length N are implicitly treated as a periodic waveform with period equal to N (modulo N). The Gibbs phenomenon appears due to the discontinuities at the end points of the data. These discontinuities are equivalent to the multiplication of the data by a rectangular function, which, once transformed into the frequency domain, becomes the convolution of the spectra with a sinc function and thus produces the Gibbs phenomenon. In the DFT, the Gibbs phenomenon is therefore always associated with windowing [3]. It has been observed, however, that the discontinuities at, or the difference between, the end data points actually indicate that the waveform has not reached a complete cycle within its period (the length of the data). That is, the period ends before the waveform has completed its full cycle, the periodicity in the DFT is thus incomplete. The real task of the conventional windows is, therefore, to force the end data points to become equal in order for the waveform to reach a complete DFT periodicity. This implies that tapering or weighting of the data to zero at the end points in a window is not really necessary, if the end points are equal to give the waveform a complete periodicity. Data flipping is not "windowing" in the conventional sense. It creates an even function of modulo 2N, which will not only make the end data points become equal to give the waveform a complete periodicity, but will also keep the spectral content of the original data from distortion or leakage. The even function has no discontinuities at the end points to make it equivalent to multiplication by a rectangular function, but instead becomes an impulse train of complete periodicity. Once transformed into the frequency domain, its spectra are no longer convoluted with a sinc function, but are still a periodic impulse train [3]; the Gibbs phenomenon thus will not appear in the spectra. It has also been found that, as long as the spectra of a waveform with complete periodicity are no longer smeared by the Gibbs phenomenon, the zeroing-out of unwanted spectral bands in the frequency domain will not consist of multiplication by a rectangular function. That is, the inverse discrete Fourier transform (IDFT) of the "passed" spectra will not become a convolution with a sinc function to produce the Gibbs phenomenon in the time domain, but is a periodic impulse train excluding unwanted spectral components. Because of the disassociation of the even function from windowing, spectral leakage will disappear. The end-effect is the only artifact that needs further reduction.

After the waveform has reached a complete periodicity in the time domain, the zeroing-out of spectral bands in the frequency domain, which no longer consists of multiplication by a rectangular function, needs some more explanation. In the time domain, a data point represents a point of the periodic waveform at a particular time, which is continuous to the next data points of the same waveform cycle; i. e., all the successive data points belong to *one* periodic cycle. The discontinuities at, or the difference between, the end data points indicate that the waveform has not reached a complete cycle. Whereas, in the frequency domain, if not smeared by the Gibbs phenomenon, each point represents the spectral component at a particular frequency, which is independent of the spectral components of adjunct frequencies. Periodicity is then only a repetition of the (magnitude) differences between individual spectral components at successive frequencies. The zeroing-out of spectral components at particular frequencies is therefore equivalent to the substitution of spectral components of zero magnitude at these frequencies. This will not induce an inconsistency between the cycle and length of the periodicity that can be considered as equivalent to multiplication by a rectangular function. Yet, a further question may arise: Why does the Gibbs phenomenon still appear in the time domain in the IDFT of a partly zeroing-out spectra if the spectra are from a waveform with an incomplete periodicity? This is because in the spectra of such a waveform, the Gibbs phenomenon will not only create sidelobes, but will also smear the mainlobe [3] [4]. A simple zeroing-out of unwanted bands from such spectra *cannot* clean up all the smearing. The "leftover" Gibbs phenomenon therefore will be carried over to the time domain.

It has been proven that the DFT of a periodic impulse train is also a periodic impulse train [3]. Here we may, therefore, start directly from the DFT of a discrete even function, instead of involving the sampling theorem and the unit impulse function. Let $r(n)$, $n = 0, 1, 2, ..., N-1$, be the data set to be filtered. Then, data flipping will create a new data set $r_e(n)$ such that,

$$r_e(n) = \begin{cases} r(n) & n = 0,1,2,\ldots \mathrm{N}-1 \\ r(2\mathrm{N}-1-n) & n = \mathrm{N},\mathrm{N}+1,\ldots 2\mathrm{N}-1. \end{cases} \quad (1)$$

This defines an even function with an even number of data points and equal end points, and the equal data points $r_e(\mathrm{N}-1)$ and $r_e(\mathrm{N})$ are the center of mirror images.

Let $R(k)$, $k = 0, 1, 2, \ldots, (\mathrm{N}-1)/2$, be the spectra of $r(n)$. Then the spectra of $r_e(n)$ can be represented as,

$$R_e(k') = \frac{1}{2}\left[R(k') + e^{j\frac{2\pi k'}{\mathrm{N}}} R^*(k')\right] \qquad k' = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \frac{(\mathrm{N}-1)}{2} \quad , \quad (2)$$

where $\mathrm{j} = \sqrt{-1}$; $R(k')$ is identical to $R(k)$ except that $k' = k/2$; and $R^*(k')$ is the conjugate of $R(k')$. The even symmetry of $r_e(n)$ gives $R_e(k') = e^{\frac{2\pi k'}{\mathrm{N}}} R_e^*(k')$ [4]; the transform is therefore not real. The spectral content of the original data is not affected by the data flipping, but the spectral resolution is doubled; i.e., $k' = k/2 = 0, 1/2, 1, 3/2, \ldots, (\mathrm{N}-1)/2$. Since the even function is not a window but has a full waveform cycle within its period, it becomes equivalent to a periodic impulse train; the Gibbs phenomenon thus will not appear in $R_e(k')$. This allows the zeroing-out of unwanted spectral bands in $R_e(k')$ with no spectral leakage or smearing from the Gibbs phenomenon. However, note that $R_e(k')$ is not the original spectra $R(k)$ but is a combination of $R(k')$ and its phase-shifting conjugate $e^{j\frac{2\pi k'}{\mathrm{N}}} R^*(k')$. Note also that $R(k')$ is not exactly identical to $R(k)$ since its spectral resolution has been doubled. Because of the phase-shifting interference between $R(k')$ and $e^{j\frac{2\pi k'}{\mathrm{N}}} R^*(k')$ when they are added together, $R_e(k')$ will exhibit an oscillation which is not in the original spectra $R(k)$. This oscillation or phase-shifting interference does not have a connection with the Gibbs phenomenon, or an influence on the original spectra.

After the even function has been transformed into the frequency domain with the Gibbs phenomenon suppressed, filtering is simply a zeroing-out of the unwanted bands and then an IDFT to get back to the time domain. Let $k_l \le k' \le k_h$ be the frequency band limits of the filter; a zeroing-out of the spectral components at frequencies outside these limits is now no longer equivalent to multiplication by a rectangular function, but instead is a substitution of zero spectral components at those frequencies. That is, the spectral components outside the frequency band $k_l \le k' \le k_h$, $k' = 0$, 1/2, 1, 3/2,...,(N-1)/2, are now all of zero magnitude. Then, the frequency sampling theorem ensures that the filtered output $r_o(n)$ is uniquely determined by the spectral content within the band limits [1], [3]:

$$r_o(n) = \sum_{n'=0}^{2N-1} r_e(n')\left[\frac{1}{2N}\sum_{k'=k_l}^{k_h} e^{j\frac{2\pi(n-n')k'}{N}}\right], \quad n = 0,1,2,\ldots N-1 \quad , \tag{3}$$

where $\frac{1}{2N}\sum_{k'=k_l}^{k_h} e^{j\frac{2\pi(n-n')k'}{N}}$ acts as a filter convoluting with the input, and for the output only the original length N is preserved.

Equations (1), (2), and (3) demonstrate that through data flipping we can achieve our goal of forcing the waveform to reach a complete periodicity without affecting the spectral content of the data; the Gibbs phenomenon is thus suppressed without waveform distortion, spectral leakage, or transition between passbands and stopbands. This technique facilitates the design of a digital filter using the FFT algorithms. More detailed derivations of the equations can be found in the Appendix.

IV. DESIGN

After the Gibbs phenomenon has been suppressed, filtering with FFT becomes straightforward. This involves only a FFT of the flipped data into the frequency domain, zeroing out the unwanted spectral bands, and then an IFFT back to the time domain. Using this simple technique, we can do a lowpass, bandpass, highpass, bandstop, notch, or single-frequency-pass filtering simply by manipulating the frequency band limits.

Let k_q be the Nyquist frequency. Filtering is then only a matter of changing the frequency band limits; i. e.,

$$\begin{array}{ll} \text{Lowpass} & 0 = k_l < k_h < k_q \\ \text{Bandpass} & 0 < k_l < k_h < k_q \\ \text{Highpass} & 0 < k_l < k_h \geq k_q \\ \text{Bandstop} & 0 < k_l > k_h < k_q \\ \text{Notch} & 0 < k_l = k_h < k_q. \end{array} \tag{4}$$

With a proper choice of the band limits k_l and k_h, a single-frequency-pass filtering can also be made.

This digital filter has been in practical use for some time, it has outperformed the Betterworth, Chebyshev, and elliptic filters which are implemented in the same system [7]. The data flipping technique has also been applied successfully to suppress the ringing effect in the frequency domain for an accurate measurement of attenuation and Q from spectral ratios [4].

V. DISCUSSION

The Spectral Resolution. We know that the frequency sampling interval $\Delta k'$ is determined by the time sampling interval Δn and the original data length N; i. e., $\Delta k' = 1/(2N\Delta n)$. If either Δn or N is small, then k' may not be small. It is therefore very likely that no spectral points are exactly at the frequencies of the band limits requested by the user. The filter will then be forced to search for the frequencies closest to the band limits. This will cause a shift of the passbands or stopbands requested by the user. Such a shift of the band limits may be reduced by flipping the data more than once to further refine the spectral resolution, at the expense of computing time. That is, $\Delta k' = 1/(2^m N\Delta n)$ for $m = 1, 2, \ldots$ This filter does not need a transition between passbands and stopbands, and the band limits k_l and k_h are the exact edges of the passbands or stopbands. However, the spectra of signals and noises are usually close together, sometimes even indistinguishable. This raises a problem in band separation. An accurate or appropriate choice of band limits to ensure passing of only and all desired spectral bands is critical for the performance of the filter.

The End-effects. Data flipping can make the waveform complete a periodic cycle to suppress the Gibbs phenomenon, but can not ensure that the cycle is smooth at the end points with continuous slope like within the waveform. That is, although the waveform is now in a complete cycle, its slope at the end points may not be continuous. This produces end-effects. The slope discontinuity at the last point of the original data is not much of a problem. Data flipping converts this point into a local peak or trough in the middle of the waveform, and its dual existence as mirror images ensures a zero slope there for the continuity of the waveform between the two symmetrical portions of the even function. The only concern is that if the cusp at this point is too sharp, it may introduce a frequency component which will affect the band limits selection and thus induce the end-effect or leakage. The slope discontinuity at the first data point can be similarly treated by adding identical points to the end of the even function to also ensure a zero slope there. This may make the even function no longer symmetrical, but will not affect the completeness of the waveform periodicity, although the period has been elongated and the spectral resolution further refined. Adding more points to the end of the even function will in fact reduce further the end-effect at the first output data point. This is a very useful characteristic. It allows the filter to accept any FFT algorithms, particularly those which require that the data length be a power of two. However, one should note that the period of the even function is not a full period or multiple periods of a sinusoid. For a sinusoid, adding identical points to the end of its period makes its waveform discontinuous after a full cycle, and thus also induces the Gibbs phenomenon. At the first data point or the last point of the even function there is yet another minor problem. Theoretically, the last point of the even function is not truly the last point of a complete periodic cycle, but is the beginning (the first point) of the *next* cycle. This means that the last point of the even function is redundant for a complete periodicity, which will also induce a minor end-effect. The simplest way to treat this end-effect is to omit the last point from the even function. However, if the data is long and the length of the even function needs to be an even number, then omitting two or more points may even be more effective. The omission of the last point will not affect the adding of points identical to the new last point on the end. This reduces the end-effects from slope discontinuity, but then the end-effect from the redundant last point will become invisible. Moreover, since the waveform is a combination of all spectral components in the data, the end-effects also include those arising from noises, if the separation between the frequency bands of the signals and noises is not clear cut. Therefore, choosing appropriate band limits, such as by refining the spectral resolution to leave out as many unwanted components as possible, will greatly reduce the contamination of the end-effects from noises.

Alternative Design. There is a way to design the filter without a redundant last point in the even function but still keep the length even. This involves making $r_e(n)$ an even function with the last original point as the *sole* center of mirror images, and then omitting the last point from this even function; i. e.,

$$r_e(n) = \begin{cases} r(n), & n = 0,1,2,\ldots N-2 \\ r(2N-2-n) & n = N-1, N, \ldots 2N-3 \end{cases} \qquad (5)$$

Then, we have,

$$R_e(k') = \frac{1}{2}\left[R_l(k') + R_r^*(k')\right], \quad k' = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \frac{(N-2)}{2} \quad , \qquad (6)$$

where $R_l(k')$ and $R_r^*(k')$ are not identical to $R(k')$ and $R^*(k')$; $R_l(k')$ is the spectra of $r(n)$ with no last point, while $R_r^*(k')$ is the spectra of $r(n)$ with no first point. This design will help to reduce the minor end-effect at the first output point, but unfortunately will also increase the end-effect at the last output point, since in this design the center of the mirror images is no longer even, and thus cannot ensure a zero slope there. The derivations of (6) is shown in the Appendix.

FFT Application. This filter can take any FFT algorithms. Those which require that the data length be a power of two are particularly useful. In using such algorithms, the blanks between the data length and a power of two need to be filled in. Since the end points of the even function do not need to be zero, we can simply fill these blanks with the first data point or the last point of the even function. Filling in the blanks with the first data point will supress the Gibbs phenomenon at the first output points even if data flipping is not applied to the input data. On the other hand, if the last point, or last few points, are omitted from the even function for the sake of reducing of the minor end-effect, then we can fill in these blanks with the last point not omitted.

Figure 1 is a sample plot which shows the performance of this filter. The *diamond line* in the figure is the input data; the *solid line* is the desired output; the *dotted line* is the output of the filter; the *plus line* is the output without data flipping; and the *cross line* is the output with a Hanning window applied to the data. The data length is 100 points, and the FFT is a power of two.

VI. CONCLUSIONS

Data flipping is effective for the suppression of the Gibbs phenomenon in the DFT of data of finite length with discontinuities at the end points. The Gibbs phenomenon appears in a DFT due to the incomplete periodicity of the waveform of the input data. Data flipping creates an even function, which gives the waveform a complete periodicity without windowing but with a double spectral resolution. The Gibbs phenomenon is thus suppressed, as are the artifacts associated with windowing, such as waveform distortion or spectral leakage. A simple data flipping can not completely eliminate the end-effects. The end-effects arise due to the slope discontinuity at the end data points, but also contain contamination from noises. They can be greatly reduced by adding or, in a minor case, omitting points at the end of the even function, as well as by making a more accurate separation between the frequency bands of signals and noises. The incorporation of the data flipping technique into an FFT algorithm facilitates the design of a digital filter which does lowpass, bandpass, highpass, bandstop, notch, or single-frequency-pass simply by manipulating the frequency band limits. The filter does not need a transition between passbands and stopbands, but an accurate choice of the frequency band limits will greatly enhance its performance. The filter accepts any FFT algorithms. Those which need data length to be a power of two are particularly useful. The filter can also be designed in a slightly different way. However, the performance of the alternative design is inferior to the principal design.

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APPENDIX

DERIVATION OF FUNDAMENTAL EQUATIONS

The DFT or spectra $R_e(k)$ of the even function $r_e(n)$ in (1) in the text are,

$$\begin{aligned} R_e(k) &= \frac{1}{2N}\sum_{n=0}^{2N-1} r_e(n) e^{-j\frac{2\pi nk}{2N}} \\ &= \frac{1}{2N}\sum_{n=0}^{N-1} r(n) e^{-j\frac{2\pi nk}{2N}} + \frac{1}{2N}\sum_{n=N}^{2N-1} r(2N-1-n) e^{-j\frac{2\pi nk}{2N}} \quad , \end{aligned} \tag{A1}$$

where $j = \sqrt{-1}$. Then, letting $k' = k/2$, and substituting $n' = 2N-1-n$ into the second term of (A1), we have,

$$R_e(k) = \frac{1}{2}\left[R(k') + e^{j\frac{2\pi k'}{N}} R^*(k') \right] \qquad k' = 0,1,2,\ldots,\frac{(N-1)}{2} \quad . \tag{2}$$

On the other hand, from (1), we have, $r_e(n) = r_e(2N-1-n)$, $n = 0, 1, 2,\ldots,2N-1$; then,

$$\begin{aligned} R_e(k) &= \frac{1}{2N}\sum_{n=0}^{2N-1} r_e(n) e^{-j\frac{2\pi nk}{2N}} = \frac{1}{2N}\sum_{n=0}^{2N-1} r_e(2N-1-n) e^{-j\frac{2\pi nk}{2N}} \\ &= \frac{1}{2N}\sum_{n=0}^{2N-1} r_e(n') e^{-j\frac{2\pi(2N-1-n')k}{2N}} = \frac{1}{2N}\sum_{n=0}^{2N-1} r_e(n') e^{j\frac{2\pi n'k}{2N}} e^{j\frac{2\pi k}{2N}} \\ &= e^{j\frac{2\pi k}{2N}} R_e^*(k) \qquad\qquad k = 0,1,2,\ldots,N-1 \quad . \end{aligned} \tag{A2}$$

Now substituting $k' = k/2$, we have,

$$R_e(k') = e^{j\frac{2\pi k'}{N}} R_e^*(k') \qquad k' = 0,\frac{1}{2},1,\frac{3}{2},\ldots,\frac{(N-1)}{2} \quad . \tag{A3}$$

After zeroing out the unwanted spectral bands outside of $k_l \leq k' \leq k_h$, the IDFT of $R_e(k')$ is,

$$
\begin{aligned}
r_o(n) &= \sum_{k'=k_l}^{k_h} R_e(k') e^{j\frac{2\pi nk'}{N}} = \sum_{k'=k_l}^{k_h} \left[\frac{1}{2N} \sum_{n'=0}^{2N-1} r_e(n') e^{-j\frac{2\pi n'k'}{N}} \right] e^{j\frac{2\pi nk'}{N}} \\
&= \sum_{n'=0}^{2N-1} r_e(n') \left[\frac{1}{2N} \sum_{k'=k_l}^{k_h} e^{j\frac{2\pi(n-n')k'}{N}} \right] \qquad n = 0,1,2,\ldots,N-1 \quad .
\end{aligned}
\tag{3}
$$

From (5) we can derive at (6) in the text as follows,

$$
\begin{aligned}
R_e(k) &= \frac{1}{2(N-1)} \sum_{n=0}^{2N-3} r_e(n) e^{-j\frac{2\pi nk}{2(N-1)}} \\
&= \frac{1}{2(N-1)} \left[\sum_{n=0}^{N-2} r(n) e^{-j\frac{2\pi nk}{2(N-1)}} + \sum_{n=N-1}^{2N-3} r(2N-2-n) e^{-j\frac{2\pi nk}{2(N-1)}} \right] \\
&= \frac{1}{2} \left[\frac{1}{N-1} \sum_{n=0}^{N-2} r(n) e^{-j\frac{2\pi nk'}{N-1}} + \frac{1}{N-1} \sum_{n'=1}^{N-1} r(n') e^{j\frac{2\pi n'k'}{N-1}} \right] \\
&= \frac{1}{2} \left[R_l(k') + R_r^*(k') \right] \qquad k' = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \frac{(N-2)}{2} \quad .
\end{aligned}
\tag{6}
$$

Figures

Figure 1. Sample of the performance of the filter. The data length is 100 points, and the FFT is a power of two. The diamond line is the input data; the solid line is the desired output; the dotted line is the output of the filter; the plus line is the output without data flipping; and the cross line is the output with a Hanning window applied to the input data.