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FULL-TURN SYMPLECTIC MAP FROM A GENERATOR IN A FOURIER-SPLINE BASIS*

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Abstract

Given an arbitrary symplectic tracking code, one can construct a full-turn symplectic map that approximates the result of the code to high accuracy. The map is defined implicitly by a mixed-variable generating function. The generator is represented by a Fourier series in angle variables, with coefficients given as B-spline functions of action variables. It is constructed by using results of single-turn tracking from many initial conditions. The method has been applied to a realistic model of the SSC in three degrees of freedom. Orbits can be mapped symplectically for 10^7 turns on an IBM RS6000 model 320 workstation, in a run of about one day.

I. INTRODUCTION

Long term stability of orbits in circular accelerators is usually studied by tracking codes, which integrate the equations of motion through the lattice by some symplectic integration algorithm, proceeding element-by-element. There have been various attempts to summarize the full-turn evolution defined by a tracking code in an analytic formula, a *full-turn map*. If the map represented the code to sufficient accuracy, and could be evaluated in substantially less time than the time for tracking one turn, it could be used for economical studies of long-term evolution.

The method of automatic differentiation [1] allows one to differentiate the tracking algorithm, so as to generate a large number of Taylor coefficients of the corresponding map. The resulting map, given as a truncated Taylor series, cannot be exactly symplectic. In a region of phase space close to the dynamic aperture, the failure of symplecticity may be so large as to raise doubt about the usefulness of the map. This is the case for the highest order Taylor maps generated for the SSC (Superconducting Super Collider).

One possibility is to symplectify the map by producing a mixed- variable generating function that induces an exactly symplectic map that closely approximates the underlying map. This can be done by using formal power developments in Cartesian coordinates to solve the nonlinear equations that define the generator in terms of the map. This method was proposed and carried out long ago [2]. Because of convergence difficulties it proved not to be very useful for some accelerators (for instance the Berkeley Advanced Light Source and the Tevatron), but recently Yan, Channell, and Syphers have reported some success with an application to the SSC [3].

We describe a different way to construct a symplectic full-turn map from a tracking code or other "source map". We again define the map through a mixed-variable generating function, but given as a function of action-angle coordinates rather than Cartesian coordinates. We avoid the use of Taylor series in favor of methods based on Fourier developments and spline interpolation. We believe that these methods are more appropriate at large amplitudes, since they use information on the function to be represented at many points in the region of interest.

This paper is a brief summary of our mapping method. Details and associated references can be found in [4].

II. CONSTRUCTING THE MAP

The map is defined to be a transformation from the "old" variables $(\mathbf{I}, \mathbf{\Phi})$ to the "new" variables $(\mathbf{I}', \mathbf{\Phi}')$. The generating function in this case will be in terms of old action and new angle variables:

$$G(\mathbf{I}, \mathbf{\Phi}') = \sum_{\mathbf{m}} g_{\mathbf{m}}(\mathbf{I})\, e^{i\mathbf{m}\cdot\mathbf{\Phi}'}. \quad (1)$$

The transformation equations are then

$$\mathbf{I}' = \mathbf{I} + G_{\mathbf{\Phi}'}(\mathbf{I}, \mathbf{\Phi}'), \qquad \mathbf{\Phi} = \mathbf{\Phi}' + G_{\mathbf{I}}(\mathbf{I}, \mathbf{\Phi}'). \quad (2), (3)$$

We start with a "source map," which gives the final variables as an explicit function of the initial variables:

$$\mathbf{I}' = \mathbf{I} + \mathbf{R}(\mathbf{I}, \mathbf{\Phi}), \qquad \mathbf{\Phi}' = \mathbf{\Phi} + \mathbf{\Theta}(\mathbf{I}, \mathbf{\Phi}). \quad (4), (5)$$

This map will usually be defined as the result of tracking over one turn, but in the numerical work reported here it was a 12th order Taylor series map.

The Fourier coefficients are obtained from (2) and (4) as

$$\begin{aligned} g_{\mathbf{m}}(\mathbf{I}) &= \frac{1}{(2\pi)^d\, i m_\alpha} \int_0^{2\pi} d\mathbf{\Phi}' G_{\Phi'_\alpha}(\mathbf{I}, \mathbf{\Phi}')\, e^{-i\mathbf{m}\cdot\mathbf{\Phi}'} \\ &= \frac{1}{(2\pi)^d\, i m_\alpha} \int_0^{2\pi} d\mathbf{\Phi}' R_\alpha(\mathbf{I}, \mathbf{\Phi}(\mathbf{I}, \mathbf{\Phi}'))\, e^{-i\mathbf{m}\cdot\mathbf{\Phi}'}. \end{aligned} \quad (6)$$

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Since we do not know $\mathbf{R}$ as a function of $\mathbf{\Phi}'$, we perform a change of variables in the integral to get an integral over $\mathbf{\Phi}$:

$$g_{\mathbf{m}}(\mathbf{I}) = \frac{1}{(2\pi)^d \, i m_\alpha} \int_0^{2\pi} d\mathbf{\Phi} R_\alpha(\mathbf{I}, \mathbf{\Phi}) e^{-i\mathbf{m}\cdot\mathbf{\Phi}} \cdot e^{-i\mathbf{m}\cdot\mathbf{\Theta}(\mathbf{I},\mathbf{\Phi})} \det\left(1 + \mathbf{\Theta}_{\mathbf{\Phi}}(\mathbf{I}, \mathbf{\Phi})\right). \tag{7}$$

The integral is then discretized to obtain

$$g_{\mathbf{m}}(\mathbf{I}) = \frac{1}{i m_\alpha \prod_\beta J_\beta} \sum_{\mathbf{j}} R_\alpha(\mathbf{I}, \mathbf{\Phi}_{\mathbf{j}}) e^{-i\mathbf{m}\cdot\mathbf{\Phi}_{\mathbf{j}}} e^{-i\mathbf{m}\cdot\mathbf{\Theta}(\mathbf{I},\mathbf{\Phi}_{\mathbf{j}})} \det\left(1 + \mathbf{\Theta}_{\mathbf{\Phi}}(\mathbf{I}, \mathbf{\Phi}_{\mathbf{j}})\right), \tag{8}$$

where J_β is the number of Φ_β mesh points in the β dimension, and the summation is over integer vectors $\mathbf{j}$ such that $j_\beta \in \{0, \ldots, J_\beta - 1\}$.

The $\mathbf{m} = \mathbf{0}$ mode must be handled differently. We instead must use $\mathbf{\Theta}$ values. The resulting summation is

$$g_{\mathbf{0}}(\mathbf{I}) = -\frac{1}{\prod_\beta J_\beta} \sum_{\mathbf{j}} \mathbf{\Theta}(\mathbf{I}, \mathbf{\Phi}_{\mathbf{j}}) \det\left(1 + \mathbf{\Theta}_{\mathbf{\Phi}}(\mathbf{I}, \mathbf{\Phi}_{\mathbf{j}})\right). \tag{9}$$

To increase the speed of evaluation of the map, Fourier modes that are smaller than the expected or desired accuracy of the map can be removed from the generating function.

We obtain values of $g_{\mathbf{m}}(\mathbf{I})$ for values on a mesh in $\mathbf{I}$. We then choose a set of basis functions $B_j^{(\alpha)}(I)$ to use in interpolating the coefficients such that

$$g_{\mathbf{m}}(\mathbf{I}) = \sum_{\mathbf{j}} g_{\mathbf{m},\mathbf{j}} \prod_\alpha B_{j_\alpha}^{(\alpha)}\left(I^{(\alpha)}\right). \tag{10}$$

The index α labels the different degrees of freedom. For the $\mathbf{m} \neq 0$ modes, the interpolation is straightforward. For the $\mathbf{m} = 0$ mode, one must be careful to consider the fact that the derivatives of the basis functions are linearly dependent. Details of this can be found in [4]. It is advantageous to choose B-splines for the basis functions. Because they have a small region where they are nonzero, their use greatly increases the speed of evaluation of the map.

III. EVALUATING THE MAP

The map is evaluated by performing a Newton iteration to obtain $\mathbf{\Phi}'$ and then substituting into (2) to get $\mathbf{I}'$. An initial guess for the Newton iteration is provided by an explicit map with a small number of modes retained.

IV. THREE DIMENSIONS

The method can be used in any number of dimensions. In a three dimensional accelerator problem, however, it is not advantageous to do the third dimension in action-angle variables. Instead, note that most of an accelerator ring is time independent. One can construct a map for the time independent part that has the energy deviation as an additional parameter, which is treated on equal footing with the actions. The time-dependent parts (usually r.f. cavities) can then be treated separately as the user chooses. Time-of-flight information is obtained by taking a derivative of the generating function with respect to energy deviation.

V. PRECONDITIONING THE SOURCE MAP

Finally, note that since one wants to perform the action interpolation over a finite domain that does not include the origin in each phase space plane, the plain source map is sometimes not well-suited for direct application of this method. This can be overcome by performing a preliminary canonical transformation on the source map so as to have the new source map take an annulus of initial conditions into a similar (larger) annulus. This can be done easily by a linear transformation or a low-order Taylor series mixed-variable generating function.

VI. RESULTS

As an example, we take the source map to be a 12th order Taylor series map for a realistic model of the SSC. Results for accuracy (agreement with the source map) and iteration time for a three dimensional map are shown in figures 1 through 2. The "mode cutoff" is a measure of the maximum size of the Fourier modes that are being removed from the generating function. The number of actions indicates the number of mesh points in each dimension of action interpolation. The order refers to the order of B-splines used in action interpolation. The curves have approximately slope 1 when the error is dominated by the number of Fourier modes being thrown away. They begin to level off when the error is dominated by the action interpolation (low actions) or failure of symplecticity of the source map (high actions).

We have constructed maps at amplitudes near the dynamic aperature, and have found that we can track stable trajectories for 10^7 terms in about a half a day in two dimensions and about a day in three dimensions. Times are on an IBM RS6000 320H workstation.

Finally, in figure 3 we show "survival plots," and see that our map gives a similar long-term dynamic aperatures to the map it is trying to approximate.

VII. REFERENCES

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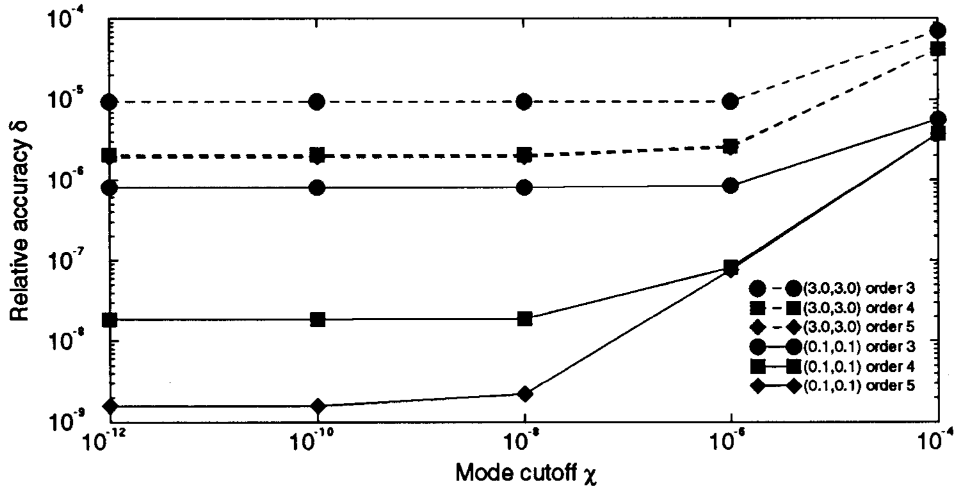

Figure 1: Relative accuracy of 3-D map

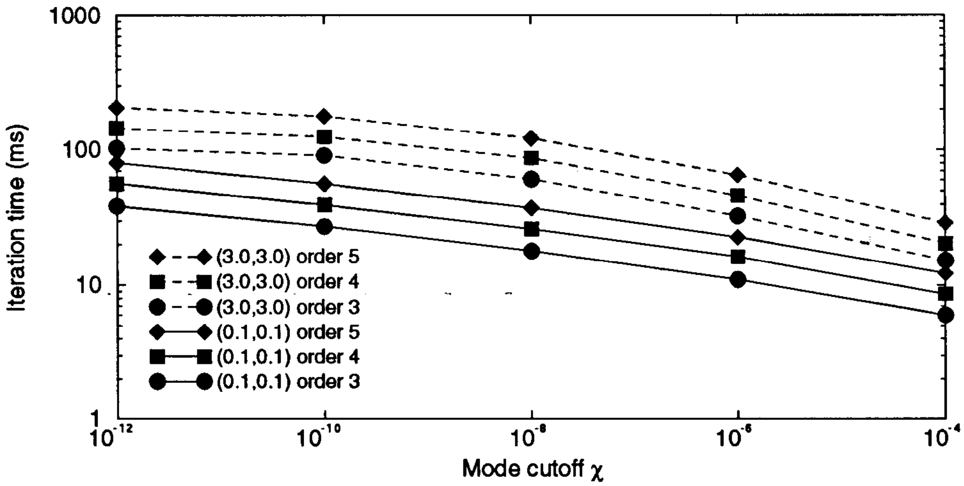

Figure 2: Iteration time of 3-D map.

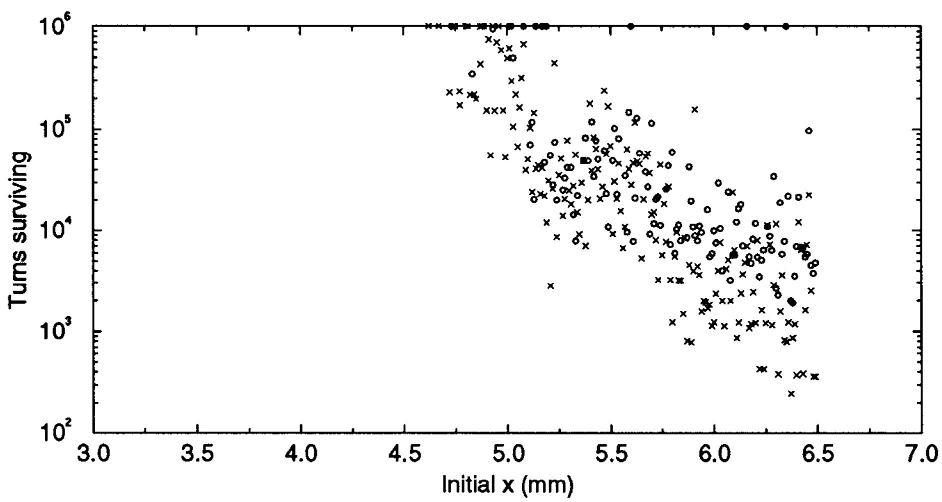

Figure 3: Survival plot. Circles are Taylor series, crosses are the map.