

## EXPLOITING PERIODICITY AND OTHER STRUCTURAL SYMMETRIES IN FIELD SOLVERS

E. M. Nelson\*

Stanford Linear Accelerator Center  
Stanford University, Stanford, CA 94309

### ABSTRACT

Many RF structures have symmetries which can be exploited by field solvers with appropriate boundary conditions. These symmetries allow a reduced problem to be solved, which leads to faster and/or more accurate solutions. Of particular interest to the accelerator community are periodic structures. Quasi-periodic boundary conditions allow modes with any desired phase advance given a single cell of the periodic structure. For symmetric periodic structures there is a variation which requires only a half cell of the periodic structure. These boundary conditions can also be used for rotationally periodic structures, such as cross-field amplifiers and magnetrons. Boundary conditions for some other symmetries, such as reflection symmetry about a plane and about a point, will also be reviewed.

### INTRODUCTION

Symmetries of the microwave structure can reduce the size of a problem. Reduced problems take less time and space to solve numerically. Alternatively, a finer mesh can be used with the reduced problem to obtain a more accurate solution. The symmetries described in the following sections can be exploited using appropriate boundary conditions while leaving the formulation for the interior of the structure unchanged. Other symmetries, such as axisymmetry, lead to different formulations for the interior as well as different boundary conditions.

The connection between Maxwell's equations and a symmetry is that Maxwell's equations, including the boundaries and any material properties, are invariant under the symmetry. Then fields can be found which are simultaneously eigenmodes of both Maxwell's equations and the symmetry operator, but with separate eigenvalues. In the language of quantum mechanics, Maxwell's equations (thought of as an operator) and the symmetry operator commute, so the operators are simultaneously diagonalizable. The size of the problem is reduced by constraining the solutions to be eigenmodes of the symmetry operator with a particular eigenvalue.

Examples of the exploitation of symmetries will be presented. These examples are based on the following formulation of the eigenmode problem: given the region  $\Omega$  and its material properties  $\epsilon$  and  $\mu$ , find the eigenmode fields  $\mathbf{E}$  and the corresponding eigenvalues  $\omega^2/c^2$  such that

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{E}) = \frac{\omega^2}{c^2} \epsilon \mathbf{E} \quad \text{in } \Omega, \quad (1a)$$

$$\nabla \cdot (\epsilon \mathbf{E}) = 0 \quad \text{in } \Omega \quad (1b)$$

$$\text{and } \hat{\mathbf{n}} \times \mathbf{E} = 0 \quad \text{on } \Gamma_{\text{metal}}. \quad (1c)$$

The region  $\Omega$  represents the interior of the structure, and the surface  $\Gamma_{\text{metal}}$  represents the perfectly conducting walls which bound the structure. Specifying either the dirichlet condition  $\hat{\mathbf{n}} \times \mathbf{E} = 0$  or the neumann condition  $\hat{\mathbf{n}} \times (\mu^{-1} (\nabla \times \mathbf{E})) = 0$  on the boundary is sufficient for this type of problem. It should be made clear that despite the specific nature of these examples, the boundary conditions of the following sections are independent of the particular formulation for the interior of the structure.

The following section is a review of reflection symmetry about a plane. While this symmetry is well known and in common use, it will be helpful to introduce the operator notation and describe the steps leading to a reduced problem with this familiar case.

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\* Work supported by Department of Energy, contract DE-AC03-76SF00515

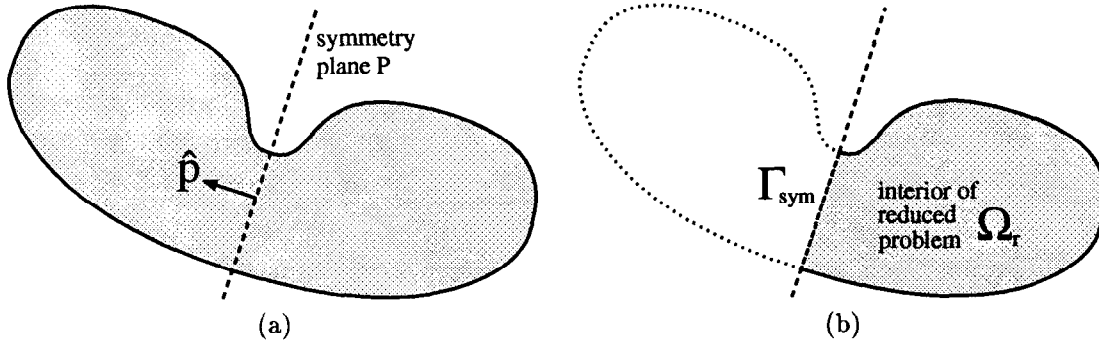


Fig. 1. (a) A structure with symmetry plane P. The shaded region is the interior  $\Omega$ , and the normal to the symmetry plane is  $\hat{\mathbf{p}}$ . (b) The region over which Maxwell's equations must be solved is reduced to  $\Omega_r$  for a symmetric structure. The symmetry boundary is  $\Gamma_{\text{sym}}$ .

### REFLECTION SYMMETRY ABOUT A PLANE

Consider a structure which is symmetric about a plane P, called the symmetry plane, defined by the equation  $\mathbf{x}^T \hat{\mathbf{p}} = p_o$ , where  $\hat{\mathbf{p}}$  is a unit vector normal to the plane P and  $|p_o|$  is its distance from the origin. An example is shown in figure 1a.

Let  $\hat{\mathbf{P}}$  be the reflection operator about the symmetry plane P. The reflection operator  $\hat{\mathbf{P}}$  can act on various types of objects.  $\hat{\mathbf{P}}$  acting on a point  $\mathbf{x}$  yields the point

$$\hat{\mathbf{P}}\mathbf{x} = \mathbf{P}(\mathbf{x} - \hat{\mathbf{p}}p_o) + \hat{\mathbf{p}}p_o = (\mathbf{I} - 2\hat{\mathbf{p}}\hat{\mathbf{p}}^T)(\mathbf{x} - \hat{\mathbf{p}}p_o) + \hat{\mathbf{p}}p_o, \quad (2)$$

where  $\mathbf{P} = (\mathbf{I} - 2\hat{\mathbf{p}}\hat{\mathbf{p}}^T)$  is the reflection matrix and  $\mathbf{I}$  is the identity matrix. The reflection operator acting on a vector field  $\mathbf{E}$  is the vector field  $\hat{\mathbf{P}}\mathbf{E}$ . Evaluating the reflected vector field  $\hat{\mathbf{P}}\mathbf{E}$  at a point  $\mathbf{x}$  gives the vector

$$(\hat{\mathbf{P}}\mathbf{E})(\mathbf{x}) = \mathbf{P}\mathbf{E}(\hat{\mathbf{P}}^{-1}\mathbf{x}). \quad (3)$$

Finally, the reflection operator  $\hat{\mathbf{P}}$  acting on a pseudovector field  $\mathbf{H}$  is  $\hat{\mathbf{P}}\mathbf{H}$ . Evaluating the reflected pseudovector field at the point  $\mathbf{x}$  gives the pseudovector

$$(\hat{\mathbf{P}}\mathbf{H})(\mathbf{x}) = -\mathbf{P}\mathbf{H}(\hat{\mathbf{P}}^{-1}\mathbf{x}). \quad (4)$$

The minus sign is present for pseudovectors because the determinant of the reflection matrix  $\mathbf{P}$  is  $-1$ .

Consider an eigenmode of  $\hat{\mathbf{P}}$  with eigenvalue  $p$ . Denoting one of the fields ( $\mathbf{E}$  or  $\mathbf{H}$ , for example) of the eigenmode by  $\mathbf{A}$ , then  $\hat{\mathbf{P}}\mathbf{A} = p\mathbf{A}$ . Reflecting the mode twice gives the original mode, so

$$\hat{\mathbf{P}}(\hat{\mathbf{P}}\mathbf{A}) = \hat{\mathbf{P}}(p\mathbf{A}) = p^2\mathbf{A} = \mathbf{A}. \quad (5)$$

Thus  $p^2 = 1$ , and the eigenvalues of  $\hat{\mathbf{P}}$  are  $p = \pm 1$ .

Let  $\Gamma_{\text{sym}}$  be the portion of the symmetry plane P in the region  $\Omega$ . This will be a boundary of a reduced problem. Noting that

$$\hat{\mathbf{P}}\mathbf{x} = \mathbf{x} \quad \text{for } \mathbf{x} \in \Gamma_{\text{sym}}, \quad (6)$$

then

$$\begin{aligned} (\hat{\mathbf{P}}\mathbf{E})(\mathbf{x}) &= \mathbf{P}\mathbf{E}(\hat{\mathbf{P}}^{-1}\mathbf{x}) = \mathbf{P}\mathbf{E}(\mathbf{x}) = p\mathbf{E}(\mathbf{x}) \\ (\hat{\mathbf{P}}\mathbf{H})(\mathbf{x}) &= -\mathbf{P}\mathbf{H}(\hat{\mathbf{P}}^{-1}\mathbf{x}) = -\mathbf{P}\mathbf{H}(\mathbf{x}) = p\mathbf{H}(\mathbf{x}) \end{aligned} \quad \text{for } \mathbf{x} \in \Gamma_{\text{sym}} \quad (7)$$

and thus

$$\mathbf{P}\mathbf{E} = p\mathbf{E} \quad \text{and} \quad -\mathbf{P}\mathbf{H} = p\mathbf{H} \quad \text{on } \Gamma_{\text{sym}}. \quad (8)$$

Consider the action of the reflection matrix  $\mathbf{P}$  on a vector  $\mathbf{A}$ . Let  $A_n = \hat{\mathbf{p}} \cdot \mathbf{A}$  be the component of  $\mathbf{A}$  normal to the symmetry plane  $P$ , and let  $\mathbf{A}_t = (\mathbf{I} - \hat{\mathbf{p}}\hat{\mathbf{p}}^T)\mathbf{A}$  be the portion of  $\mathbf{A}$  tangential to the symmetry plane  $P$ . Then

$$(\mathbf{P}\mathbf{A})_t = \mathbf{A}_t \quad \text{and} \quad (\mathbf{P}\mathbf{A})_n = -A_n. \quad (9)$$

That is,  $\mathbf{P}$  reverses the normal component of the vector, but leaves the tangential component of a vector unchanged. Applying this to equation (8) yields

$$\mathbf{E}_t = p\mathbf{E}_t, \quad E_n = -pE_n, \quad \mathbf{H}_t = -p\mathbf{H}_t \quad \text{and} \quad H_n = pH_n \quad \text{on } \Gamma_{\text{sym}}. \quad (10)$$

The case  $p = -1$  corresponds to a perfectly conducting boundary with boundary conditions

$$\text{perfectly conducting:} \quad \mathbf{E}_t = 0 \quad \text{and} \quad H_n = 0 \quad \text{on } \Gamma_{\text{sym}}, \quad (11)$$

and the case  $p = 1$  corresponds to a perfectly insulating boundary with boundary conditions

$$\text{perfectly insulating:} \quad E_n = 0 \quad \text{and} \quad \mathbf{H}_t = 0 \quad \text{on } \Gamma_{\text{sym}}. \quad (12)$$

Similar conditions apply to  $\mathbf{D}$  and  $\mathbf{B}$  on  $\Gamma_{\text{sym}}$ . Like the boundary condition at perfectly conducting walls, the two boundary conditions at the symmetry plane are not independent. For example, given a solution to Maxwell's equations and the boundary condition  $\mathbf{H}_t = 0$ , then the other boundary condition  $E_n = 0$  can be derived if time-varying fields are assumed.

Many field solvers call the perfectly conducting case a metal boundary condition and reserve the words symmetry boundary to mean only the perfectly insulating case. While this is reasonable for calculating the fields in an RF structure, some post-processing calculations, such as power loss due to the finite conductivity of the metal walls, need to distinguish between a real metal wall and the perfectly conducting case of a symmetry plane.

Here is a formulation for the eigenmode problem in a symmetric structure. The problem is reduced to a region  $\Omega_r$  which is half of the original structure, as depicted in figure 1b. For the  $p = -1$  (perfectly conducting) case: given the region  $\Omega_r$  and its material properties  $\epsilon$  and  $\mu$ , find the eigenmode fields  $\mathbf{E}$  and the corresponding eigenvalues  $\omega^2/c^2$  such that

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{E}) = \frac{\omega^2}{c^2} \epsilon \mathbf{E} \quad \text{in } \Omega_r, \quad (13a)$$

$$\nabla \cdot (\epsilon \mathbf{E}) = 0 \quad \text{in } \Omega_r \quad (13b)$$

$$\text{and } \hat{\mathbf{n}} \times \mathbf{E} = 0 \quad \text{on } \Gamma_{\text{metal}} \text{ and } \Gamma_{\text{sym}}. \quad (13c)$$

And for the  $p = 1$  (perfectly insulating) case: given the region  $\Omega_r$  and its material properties  $\epsilon$  and  $\mu$ , find the eigenmode fields  $\mathbf{E}$  and the corresponding eigenvalues  $\omega^2/c^2$  such that

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{E}) = \frac{\omega^2}{c^2} \epsilon \mathbf{E} \quad \text{in } \Omega_r, \quad (14a)$$

$$\nabla \cdot (\epsilon \mathbf{E}) = 0 \quad \text{in } \Omega_r, \quad (14b)$$

$$\hat{\mathbf{n}} \times \mathbf{E} = 0 \quad \text{on } \Gamma_{\text{metal}} \quad (14c)$$

$$\text{and } \hat{\mathbf{n}} \times (\mu^{-1} (\nabla \times \mathbf{E})) = 0 \quad \text{on } \Gamma_{\text{sym}}. \quad (14d)$$

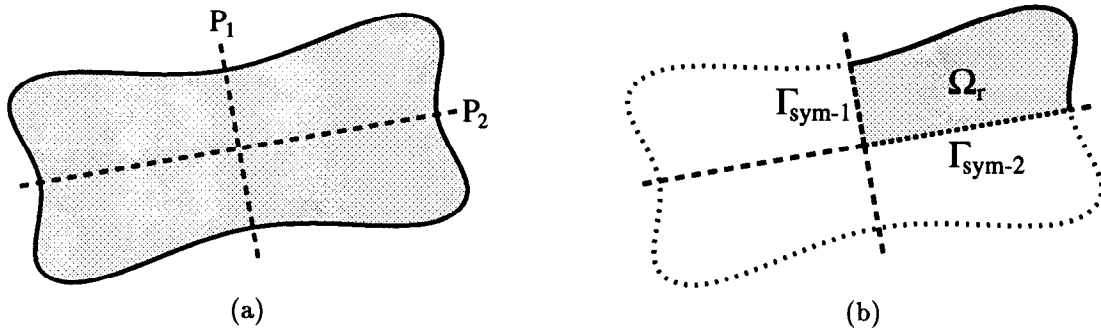


Fig. 2. (a) A structure with two symmetry planes,  $P_1$  and  $P_2$ . (b) The region over which Maxwell's equations must be solved is reduced to  $\Omega_r$ . The symmetry boundaries are  $\Gamma_{\text{sym-1}}$  and  $\Gamma_{\text{sym-2}}$ .

It is possible for a structure to have more than one symmetry plane, in which case there is a symmetry operator for each symmetry plane. If the symmetry planes are perpendicular to each other then the symmetry operators commute and modes can be found which are simultaneously eigenmodes for all of the symmetry operators and of Maxwell's equations. An example with two symmetry planes is shown in figure 2. Let  $\Gamma_{\text{sym}(\text{cond})}$  be the symmetry planes for which the perfectly conducting ( $p = 1$ ) case is chosen, and let  $\Gamma_{\text{sym}(\text{ins})}$  be the symmetry planes for which the perfectly insulating ( $p = -1$ ) case is chosen. Then a formulation for the eigenmode problem is: given the region  $\Omega_r$  and its material properties  $\epsilon$  and  $\mu$ , find the eigenmode fields  $\mathbf{E}$  and the corresponding eigenvalues  $\omega^2/c^2$  such that

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{E}) = \frac{\omega^2}{c^2} \epsilon \mathbf{E} \quad \text{in } \Omega_r, \quad (15a)$$

$$\nabla \cdot (\epsilon \mathbf{E}) = 0 \quad \text{in } \Omega_r, \quad (15b)$$

$$\hat{\mathbf{n}} \times \mathbf{E} = 0 \quad \text{on } \Gamma_{\text{metal}} \text{ and } \Gamma_{\text{sym}(\text{cond})} \quad (15c)$$

$$\text{and } \hat{\mathbf{n}} \times (\mu^{-1} (\nabla \times \mathbf{E})) = 0 \quad \text{on } \Gamma_{\text{sym}(\text{ins})}. \quad (15d)$$

Plane symmetry may be combined with another type of symmetry, for example periodic symmetry. The guiding rule is that both symmetries can be used as long the corresponding symmetry operators commute.

### REFLECTION SYMMETRY ABOUT A POINT

Consider a structure which is symmetric about a point  $\mathbf{x}_o$ , the center of the structure. An example of such a structure is shown in figure 3. Let  $\hat{\mathbf{P}}$  be the reflection operator about the center. The reflection operator acting on a point  $\mathbf{x}$  gives the point

$$\hat{\mathbf{P}}\mathbf{x} = -\mathbf{I}(\mathbf{x} - \mathbf{x}_o) + \mathbf{x}_o = -\mathbf{x} + 2\mathbf{x}_o. \quad (16)$$

The reflection operator acting on a vector field  $\mathbf{E}$  and a pseudovector field  $\mathbf{H}$  gives

$$(\hat{\mathbf{P}}\mathbf{E})(\mathbf{x}) = -\mathbf{E}(\hat{\mathbf{P}}^{-1}\mathbf{x}) \quad \text{and} \quad (\hat{\mathbf{P}}\mathbf{H})(\mathbf{x}) = \mathbf{H}(\hat{\mathbf{P}}^{-1}\mathbf{x}). \quad (17)$$

Let  $\Omega_r$  be half of the interior  $\Omega$  of the structure, and let the symmetry boundaries  $\Gamma_A$ ,  $\Gamma_B$  and possibly the center  $\mathbf{x}_o$  be the portion of the boundary of  $\Omega_r$  which is in  $\Omega$ . The boundaries  $\Gamma_A$  and  $\Gamma_B$  are chosen such that the symmetry operator  $\hat{\mathbf{P}}$  maps  $\Gamma_A$  to  $\Gamma_B$  and vice versa. An example is shown in figure 3c.

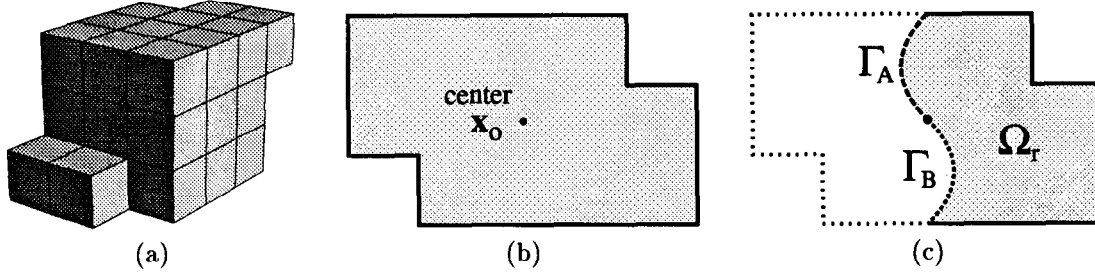


Fig. 3. (a) A structure with reflection symmetry about a point. (b) A 2d structure which is symmetric about its center  $\mathbf{x}_o$ . The shaded region is the interior  $\Omega$ . (c) The region over which Maxwell's equations must be solved is reduced to  $\Omega_r$  for a symmetric structure. The symmetry boundaries are  $\Gamma_A$  and  $\Gamma_B$ .

As in the case of plane symmetry (equation (5)), reflecting an eigenmode of  $\hat{\mathbf{P}}$  twice gives the original mode, so the eigenvalues of  $\hat{\mathbf{P}}$  are  $p = \pm 1$ . The boundary conditions for the boundaries  $\Gamma_A$  and  $\Gamma_B$  are

$$(\hat{\mathbf{P}}\mathbf{E})(\mathbf{x}) = -\mathbf{E}(\hat{\mathbf{P}}^{-1}\mathbf{x}) = p\mathbf{E}(\mathbf{x}) \quad \text{and} \quad (\hat{\mathbf{P}}\mathbf{H})(\mathbf{x}) = \mathbf{H}(\hat{\mathbf{P}}^{-1}\mathbf{x}) = p\mathbf{H}(\mathbf{x}). \quad (18)$$

A special case occurs at the center  $\mathbf{x}_o$  since  $\hat{\mathbf{P}}\mathbf{x}_o = \mathbf{x}_o$ . At the center  $-\mathbf{E}(\mathbf{x}_o) = p\mathbf{E}(\mathbf{x}_o)$  and  $\mathbf{H}(\mathbf{x}_o) = p\mathbf{H}(\mathbf{x}_o)$ , so

$$\mathbf{E}(\mathbf{x}_o) = 0 \quad \text{if } p = 1, \quad (19a)$$

$$\mathbf{H}(\mathbf{x}_o) = 0 \quad \text{if } p = -1. \quad (19b)$$

An example formulation for the  $p = 1$  case is: given the region  $\Omega_r$  and its material properties  $\epsilon$  and  $\mu$ , find the eigenmode fields  $\mathbf{E}$  and the corresponding eigenvalues  $\omega^2/c^2$  such that

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{E}) = \frac{\omega^2}{c^2} \epsilon \mathbf{E} \quad \text{in } \Omega, \quad (20a)$$

$$\nabla \cdot (\epsilon \mathbf{E}) = 0 \quad \text{in } \Omega, \quad (20b)$$

$$\hat{\mathbf{n}} \times \mathbf{E} = 0 \quad \text{on } \Gamma_{\text{metal}}, \quad (20c)$$

$$\mathbf{E}|_{\mathbf{x}} = -\mathbf{E}|_{\hat{\mathbf{P}}\mathbf{x}} \quad \text{for } \mathbf{x} \in \Gamma_A \quad (20d)$$

$$\text{and } \mathbf{E}|_{\mathbf{x}_o} = 0. \quad (20e)$$

Technically there is also a relation between the derivatives of the field at the boundaries  $\Gamma_A$  and  $\Gamma_B$  which is useful for proving various mathematical theorems, but in practice the relation isn't needed to exploit the symmetry in a field solver. Such relations will be neglected in this paper. The corresponding formulation for the  $p = -1$  case is

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{E}) = \frac{\omega^2}{c^2} \epsilon \mathbf{E} \quad \text{in } \Omega, \quad (21a)$$

$$\nabla \cdot (\epsilon \mathbf{E}) = 0 \quad \text{in } \Omega, \quad (21b)$$

$$\hat{\mathbf{n}} \times \mathbf{E} = 0 \quad \text{on } \Gamma_{\text{metal}} \quad (21c)$$

$$\text{and } \mathbf{E}|_{\mathbf{x}} = \mathbf{E}|_{\hat{\mathbf{P}}\mathbf{x}} \quad \text{for } \mathbf{x} \in \Gamma_A. \quad (21d)$$

To exploit this symmetry in a field solver, the field quantities at a point on one boundary, say  $\mathbf{x} \in \Gamma_A$ , are constrained to be the same as the field quantities at the corresponding point on the other boundary,  $\hat{\mathbf{P}}\mathbf{x} \in \Gamma_B$ , times the eigenvalue factor  $p$ . This is easily accomplished if the mesh, or discretization, of the problem has the same symmetry. An example of a finite difference

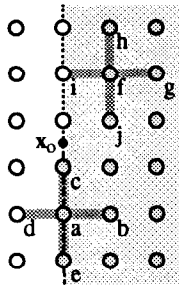


Fig. 4. An example of a finite difference mesh exploiting symmetry about a center  $\mathbf{x}_o$ . The shaded region is part of  $\Omega_r$  and the dotted and dashed lines are the boundaries  $\Gamma_A$  and  $\Gamma_B$ , respectively. The filled circles are nodes where field values are computed. Dark gray lines indicate the field values necessary for a 5-point finite difference operator at nodes  $a$  and  $f$ . The field at node  $d$  is the field at node  $f$  times the eigenvalue  $p$ , while the field at point  $i$  is the field at node  $a$  times  $p$ . Note that the symmetry of the finite difference matrix  $\mathbf{M}$  is preserved since  $\mathbf{M}_{af} = \mathbf{M}_{fa}$ .

mesh is shown in figure 4. The field quantities are computed for points in the region  $\Omega_r$  and on the boundary  $\Gamma_B$ . Whenever a field quantity at a point  $\mathbf{x}$  outside of the region  $\Omega_r$  or the boundary  $\Gamma_B$  is required by the finite difference operator, the field quantity at the point  $\hat{\mathbf{P}}\mathbf{x}$  (which is in the region  $\Omega_r$  or the boundary  $\Gamma_B$ ) multiplied by the eigenvalue  $p$  is used instead. A finite element formulation need only ensure that the global basis functions satisfy the boundary condition between  $\Gamma_A$  and  $\Gamma_B$ . The local basis functions of the finite elements remain the same.

## PERIODIC STRUCTURES

A periodic structure has a symmetry operator  $\hat{\mathbf{R}}$  which rigidly moves the structure by one period. The symmetry operator  $\hat{\mathbf{R}}$  acting on a point  $\mathbf{x}$  can be written generally as

$$\hat{\mathbf{R}}\mathbf{x} = \mathbf{R}\mathbf{x} + \mathbf{x}_o, \quad (22)$$

where the matrix  $\mathbf{R}$  is an orthogonal matrix. That is,  $\mathbf{R}^T\mathbf{R} = \mathbf{I}$ . A common symmetry operation for periodic structures is translation by a cell length  $l$  along an axis, say  $\hat{\mathbf{z}}$ . In this case  $\mathbf{x}_o = l\hat{\mathbf{z}}$  and  $\mathbf{R} = \mathbf{I}$ . Thus  $\hat{\mathbf{R}}\mathbf{x} = \mathbf{x} + l\hat{\mathbf{z}}$ . However, as indicated by the examples in figure 5, the general form of the operator  $\hat{\mathbf{R}}$  allows more than translations. The symmetry operation can include a rotation as shown in figure 5b, and it can include a reflection as shown in figure 5c. A combination of translation and rotation describes helical structures, such as the example shown in figure 5d. These are all periodic structures and, as will be shown below, the modes of a periodic structure can be found by modelling a single period of the structure and using a boundary condition called the quasi-periodic boundary condition.

Periodicity differs from the previous symmetry operators in that  $\hat{\mathbf{R}}^2$  is not the identity operator, so the eigenvalues of the symmetry operator are not simply  $\pm 1$ . Floquet's theorem, described below, gives the eigenvalues allowed for the symmetry operator  $\hat{\mathbf{R}}$  of a periodic structure.

Let  $\mathbf{A}$  represent one of the fields, perhaps  $\mathbf{E}$  or  $\mathbf{H}$ , of a mode which is a solution to the eigenmode problem in the periodic structure. Since Maxwell's equations are second order in space derivatives, there are two independent solutions to the eigenmode problem with frequency  $\omega$ . For example, one solution could be  $\mathbf{A}_1 \propto \cos(kz)$  while the other solution could be  $\mathbf{A}_2 \propto \sin(kz)$ . The fields  $\hat{\mathbf{R}}\mathbf{A}_1$  and  $\hat{\mathbf{R}}\mathbf{A}_2$  are also solutions to the eigenmode problem, and since any solution is a linear combination of  $\mathbf{A}_1$  and  $\mathbf{A}_2$ ,

$$\begin{pmatrix} \hat{\mathbf{R}}\mathbf{A}_1 \\ \hat{\mathbf{R}}\mathbf{A}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{pmatrix}. \quad (23)$$

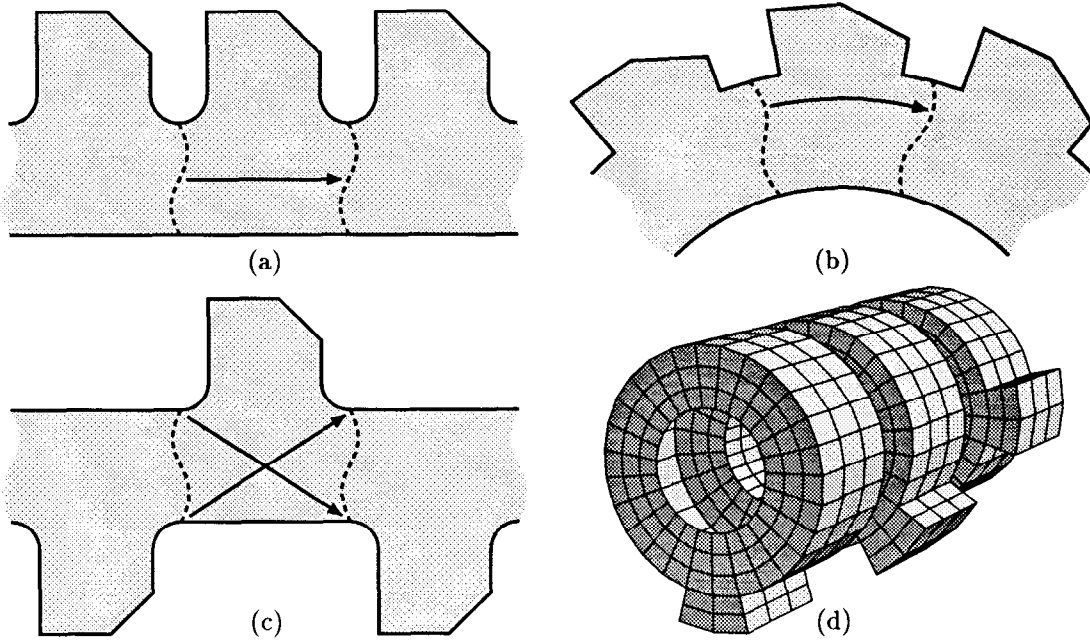


Fig. 5. Some examples of periodic structures. The dashed lines delimit one period, or cell, of the structure. The arrows indicate the action of the symmetry operations: (a) translation, (b) rotation and (c) glide reflection. The symmetry operation for a helical structure (d) is a combination of translation and rotation.

Let  $\mathbf{A}_\alpha = A\mathbf{A}_1 + B\mathbf{A}_2$  be an eigenmode of  $\hat{\mathbf{R}}$  with eigenvalue  $\alpha$ . The eigenvalues  $\alpha$  and the corresponding coefficients  $A$  and  $B$  are obtained from the eigenvalue problem

$$\begin{pmatrix} a_{11} - \alpha & a_{12} \\ a_{21} & a_{22} - \alpha \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0. \quad (24)$$

It can be shown<sup>1</sup> that  $a_{11}a_{22} - a_{12}a_{21} = 1$  for lossless structures. Then the characteristic equation is

$$(a_{11} - \alpha)(a_{22} - \alpha) - a_{12}a_{21} = \alpha^2 - (a_{11} + a_{22})\alpha + 1 = 0. \quad (25)$$

Letting  $\beta = (a_{11} + a_{22})/2$ , the eigenvalues are

$$\alpha = \beta \pm \sqrt{\beta^2 - 1}. \quad (26)$$

If  $|\beta| > 1$  then the eigenvalues are real. One of the eigenvalues has  $|\alpha| > 1$  and corresponds to a mode which grows geometrically along the structure. The other eigenvalue has  $|\alpha| < 1$  and corresponds to a geometrically damped mode. These evanescent modes will not be considered further in this paper. If  $|\beta| \leq 1$  then the eigenvalues are complex,

$$\alpha = \beta \pm i\sqrt{1 - \beta^2}. \quad (27)$$

Since  $|\alpha| = 1$  then the two eigenvalues are  $\alpha = e^{\pm i\psi}$  for some phase advance  $\psi$ . Furthermore, the field  $\mathbf{A}$  is complex and represents a wave propagating along the structure. The real field  $\mathbf{A}(\mathbf{x}, t)$  can be obtained from the complex field  $\mathbf{A}(\mathbf{x})$  using

$$\mathbf{A}(\mathbf{x}, t) = \text{Re} \{ \mathbf{A}(\mathbf{x})e^{-i\omega t} \}. \quad (28)$$

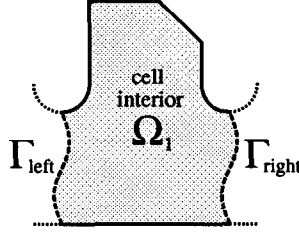


Fig. 6. One cell of the periodic structure of figure 5a.  $\Gamma_{\text{left}}$  and  $\Gamma_{\text{right}}$  are the quasi-periodic boundaries.

To exploit this symmetry the solutions are restricted to be eigenmodes of the symmetry operator  $\hat{\mathbf{R}}$  with a particular eigenvalue  $e^{i\psi}$ . In other words, a phase advance  $\psi$  is selected and the vector field  $\mathbf{E}$  must satisfy

$$\mathbf{E}(\hat{\mathbf{R}}\mathbf{x}) = \mathbf{R}\mathbf{E}(\mathbf{x})e^{i\psi} \quad (29)$$

and the pseudovector field  $\mathbf{H}$  must satisfy

$$\mathbf{H}(\hat{\mathbf{R}}\mathbf{x}) = \pm\mathbf{R}\mathbf{H}(\mathbf{x})e^{i\psi}, \quad (30)$$

where the sign is the determinant of the matrix  $\mathbf{R}$ . The sign is positive except for the examples of figure 5 except for glide reflection, in which case the sign is negative.

Let the region  $\Omega_1$  be the interior of one period, or cell, of the periodic structure. The portion of the cell boundary in the interior  $\Omega$  comprises the quasi-periodic boundaries  $\Gamma_{\text{left}}$  and  $\Gamma_{\text{right}}$ . The symmetry operator  $\hat{\mathbf{R}}$  acting on the boundary  $\Gamma_{\text{left}}$  is the boundary  $\Gamma_{\text{right}}$ . An example is shown in figure 6. There is no unique choice for the cell and its boundaries. The boundaries  $\Gamma_{\text{left}}$  and  $\Gamma_{\text{right}}$  are usually planes, but in general they can be curved surfaces as shown in the example.

Here is a formulation for the eigenmode problem reduced to one cell  $\Omega_1$  of the periodic structure. Given the region  $\Omega_1$ , its material properties  $\epsilon$  and  $\mu$  and a phase advance  $\psi$ , find the complex eigenmode fields  $\mathbf{E}$  and the corresponding eigenvalues  $\omega^2/c^2$  such that

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{E}) = \frac{\omega^2}{c^2} \epsilon \mathbf{E} \quad \text{in } \Omega_1, \quad (31a)$$

$$\nabla \cdot (\epsilon \mathbf{E}) = 0 \quad \text{in } \Omega_1, \quad (31b)$$

$$\hat{\mathbf{n}} \times \mathbf{E} = 0 \quad \text{on } \Gamma_{\text{metal}} \quad (31c)$$

$$\text{and } \mathbf{E}|_{\hat{\mathbf{R}}\mathbf{x}} = \mathbf{R}\mathbf{E}|_{\mathbf{x}}e^{i\psi} \quad \text{for } \mathbf{x} \in \Gamma_{\text{left}}. \quad (31d)$$

## SYMMETRIC PERIODIC STRUCTURES<sup>2</sup>

Now consider a structure which is both periodic and symmetric. In other words, the structure has two non-commuting symmetry planes. An example is shown in figure 7. Let  $\hat{\mathbf{R}}$  be the rigid motion operator which moves the structure or field one period (equation (22)) and let  $\hat{\mathbf{P}}$  be the reflection operator about a symmetry plane  $\Gamma_{\text{sym,left}}$  (equation (2)). There is another symmetry plane  $\Gamma_{\text{sym,right}} = \hat{\mathbf{R}}^{1/2}\Gamma_{\text{sym,left}}$ , where  $\hat{\mathbf{R}}^{1/2}$  is the rigid motion operator which moves the structure or field one half of a period. The reflection operator about the symmetry plane  $\Gamma_{\text{sym,right}}$  is  $\hat{\mathbf{R}}^{1/2}\hat{\mathbf{P}}\hat{\mathbf{R}}^{-1/2} = \hat{\mathbf{R}}\hat{\mathbf{P}}$ .

According to Floquet's theorem, the fields can be decomposed into modes with phase advance  $\psi$ . Consider the electric field  $\mathbf{E}$  of a mode with phase advance  $\psi$  satisfying  $\mathbf{E}(\hat{\mathbf{R}}\mathbf{x}) = \mathbf{R}\mathbf{E}(\mathbf{x})e^{i\psi}$ . The complex conjugate of the mode has the opposite phase advance,

$$\mathbf{E}^*(\hat{\mathbf{R}}\mathbf{x}) = (\mathbf{R}\mathbf{E}(\mathbf{x})e^{i\psi})^* = \mathbf{R}\mathbf{E}^*(\mathbf{x})e^{-i\psi}. \quad (32)$$



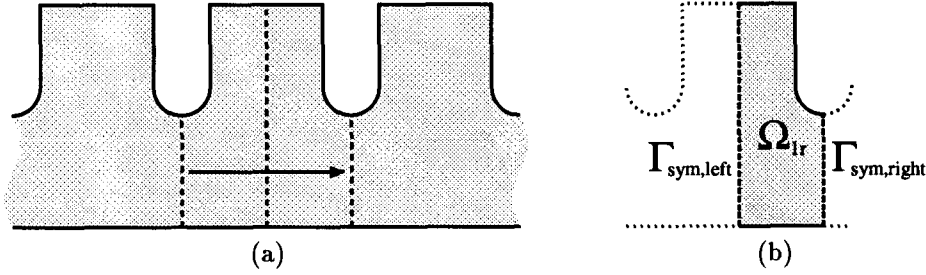


Fig. 7. (a) A symmetric periodic structure. The action of operator  $\hat{\mathbf{R}}$  is indicated by the arrow. (b) The region over which Maxwell's equations must be solved is reduced to  $\Omega_{1r}$ . The symmetry boundaries are  $\Gamma_{\text{sym,left}}$  and  $\Gamma_{\text{sym,right}}$ .

$\hat{\mathbf{P}}\mathbf{E}$  is the mode reflected about the symmetry plane, and it also has the opposite phase advance,

$$(\hat{\mathbf{P}}\mathbf{E})(\hat{\mathbf{R}}\mathbf{x}) = \mathbf{R}(\hat{\mathbf{P}}\mathbf{E})(\mathbf{x})e^{-i\psi}. \quad (33)$$

Assuming  $\mathbf{E}$  is a non-degenerate mode, equations (32) and (33) indicate that  $\mathbf{E}^*$  and  $\hat{\mathbf{P}}\mathbf{E}$  are the same mode. Let

$$\mathbf{E}^* = \alpha \hat{\mathbf{P}}\mathbf{E} \quad (34)$$

for some complex number  $\alpha$ . Conjugating equation (34) and substituting for  $\mathbf{E}^*$  with equation (34) gives

$$\mathbf{E} = \alpha^* \hat{\mathbf{P}}\mathbf{E}^* = \alpha^* \hat{\mathbf{P}}(\alpha \hat{\mathbf{P}}\mathbf{E}) = \alpha^* \alpha \hat{\mathbf{P}}^2 \mathbf{E} = |\alpha|^2 \mathbf{E} \quad (35)$$

which implies  $|\alpha| = 1$ . Without loss of generality choose  $\alpha = 1$ . A different  $\alpha$  would just multiply the mode by an overall phase factor.

At the symmetry plane  $\Gamma_{\text{sym,left}}$ ,  $\hat{\mathbf{P}}\mathbf{x} = \mathbf{x}$  so

$$\mathbf{E}^*(\mathbf{x}) = (\hat{\mathbf{P}}\mathbf{E})(\mathbf{x}) = \mathbf{P}\mathbf{E}(\hat{\mathbf{P}}\mathbf{x}) = \mathbf{P}\mathbf{E}(\mathbf{x}) \quad \forall \mathbf{x} \in \Gamma_{\text{sym,left}}. \quad (36)$$

Let  $E_n$  be the component of  $\mathbf{E}$  normal to  $\Gamma_{\text{sym,left}}$  and let  $\mathbf{E}_t$  be the vector tangential to  $\Gamma_{\text{sym,left}}$ . Then the above conditions are  $E_n^* = -E_n$  and  $\mathbf{E}_t^* = \mathbf{E}_t$ , or

$$\text{Re } E_n = 0 \quad \text{and} \quad \text{Im } \mathbf{E}_t = 0 \quad \forall \mathbf{x} \in \Gamma_{\text{sym,left}}. \quad (37)$$

In other words,  $E_n$  is imaginary and  $\mathbf{E}_t$  is real on  $\Gamma_{\text{sym,left}}$ .

At the other symmetry plane  $\Gamma_{\text{sym,right}}$ ,  $\mathbf{x} = \hat{\mathbf{R}}\hat{\mathbf{P}}\mathbf{x}$ . Replacing  $\mathbf{x}$  with  $\hat{\mathbf{P}}\mathbf{x}$  in equation (32) and using equation (34) gives

$$\begin{aligned} \mathbf{E}^*(\mathbf{x}) &= \mathbf{E}^*(\hat{\mathbf{R}}\hat{\mathbf{P}}\mathbf{x}) = \mathbf{R}\mathbf{E}^*(\hat{\mathbf{P}}\mathbf{x})e^{-i\psi} \\ &= \mathbf{R}(\hat{\mathbf{P}}\mathbf{E})(\hat{\mathbf{P}}\mathbf{x})e^{-i\psi} = \mathbf{R}\mathbf{P}\mathbf{E}(\hat{\mathbf{P}}^2\mathbf{x})e^{-i\psi} = \mathbf{R}\mathbf{P}\mathbf{E}(\mathbf{x})e^{-i\psi} \quad \forall \mathbf{x} \in \Gamma_{\text{sym,right}}. \end{aligned} \quad (38)$$

$\mathbf{R}\mathbf{P}$  is the reflection matrix about the symmetry plane  $\Gamma_{\text{sym,right}}$ , so let  $E_n$  be the component of  $\mathbf{E}$  normal to  $\Gamma_{\text{sym,right}}$  and  $\mathbf{E}_t$  be the vector tangential to  $\Gamma_{\text{sym,right}}$ . Then the above conditions are  $E_n^* = -E_n e^{-i\psi}$  and  $\mathbf{E}_t^* = \mathbf{E}_t e^{-i\psi}$ , or

$$\text{Re } E_n e^{-i\psi/2} = 0 \quad \text{and} \quad \text{Im } \mathbf{E}_t e^{-i\psi/2} = 0 \quad \forall \mathbf{x} \in \Gamma_{\text{sym,right}}. \quad (39)$$

In other words,  $E_n \propto i e^{i\psi/2}$  and  $\mathbf{E}_t \propto e^{i\psi/2}$  on  $\Gamma_{\text{sym,right}}$ .

The eigenmode problem reduced to one half cell  $\Omega_{1r}$  is: given the region  $\Omega_{1r}$ , material properties  $\epsilon$  and  $\mu$  and the phase advance  $\psi$ , find the eigenmode fields  $\mathbf{E}$  and the corresponding eigenvalues  $\omega^2/c^2$  such that

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{E}) = \frac{\omega^2}{c^2} \epsilon \mathbf{E} \quad \text{in } \Omega_{1r}, \quad (40a)$$

$$\nabla \cdot (\epsilon \mathbf{E}) = 0 \quad \text{in } \Omega_{1r}, \quad (40b)$$

$$\hat{\mathbf{n}} \times \mathbf{E} = 0 \quad \text{on } \Gamma_{\text{metal}}, \quad (40c)$$

$$\text{Re } E_n = 0 \quad \text{and} \quad \text{Im } \mathbf{E}_t = 0 \quad \text{on } \Gamma_{\text{sym,left}}, \quad (40d)$$

$$\text{and} \quad \text{Re } E_n e^{-i\psi/2} = 0 \quad \text{and} \quad \text{Im } \mathbf{E}_t e^{-i\psi/2} = 0 \quad \text{on } \Gamma_{\text{sym,right}}. \quad (40e)$$

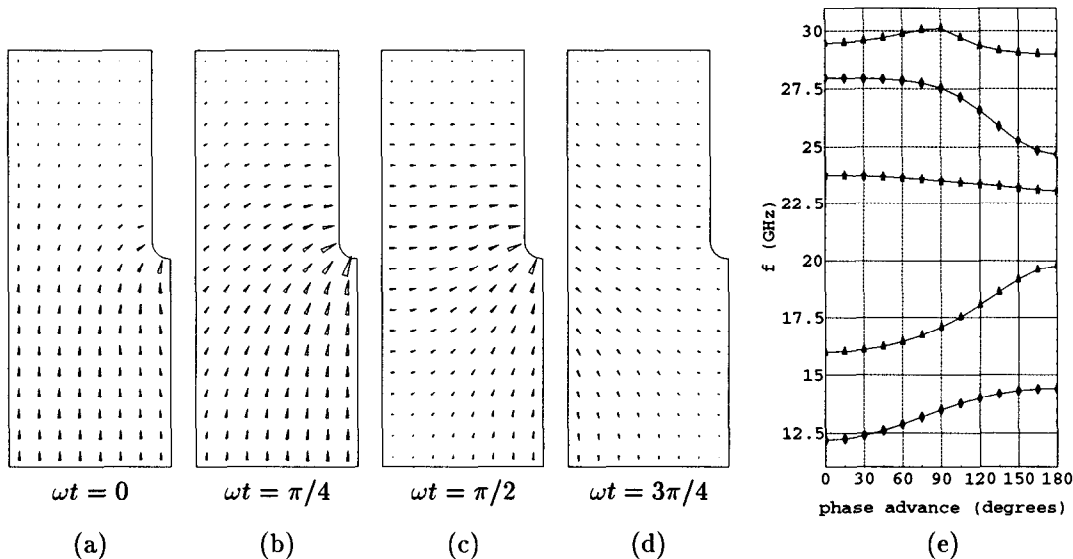


Fig. 8. (a-d) Snapshots of the electric field of the lowest dipole mode with phase advance  $\psi = \pi/2$  for an X-band accelerator structure (disk loaded waveguide). The left and right sides are the symmetry planes. The size and direction of the arrows indicate the magnitude and direction of the fields. (e) Dispersion diagram for the five lowest dipole modes.

An example using symmetric quasi-periodic boundaries is shown in figure 8. Some dipole modes of an X-band accelerator structure were calculated using the finite element field solver *YAP*<sup>3</sup>. The first four figures illustrate the complex nature of a mode with phase advance  $\psi = \pi/2$ . The snapshot at  $\omega t = 0$  is the real part of the calculated field and the snapshot at  $\omega t = \pi/2$  is the imaginary part of the calculated field. Two intermediate snapshots are included to aid visualization of the wave travelling to the right. Since the modes can be calculated for an arbitrary phase advance using these boundary conditions, it is easy to calculate the dispersion diagram shown in figure 8e. If only metal and symmetry boundary conditions were available in the field solver then calculation of the dispersion diagram would be much more work for both the computer and the user.

## CONCLUSION

Boundary conditions for exploiting reflection symmetry about a plane, reflection symmetry about a point, periodicity, and periodicity with a symmetry plane have been described. These conditions do not change the formulation for the interior of the structure and they are easy to implement in a field solver. Taking advantage of the symmetries of an RF structure reduces the size of the problem. This leads to less work for the computer and the user of the field solver.

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