

Reparametrization invariance and the expansion of currents
in the heavy quark effective theory

Matthias Neubert*
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94309

The coefficients appearing at leading and subleading order in the $1/m$ expansion of bilinear heavy quark currents are related to each other by imposing reparametrization invariance on both the effective current operators and the short-distance coefficient functions in the heavy quark effective theory. When combined with present knowledge about the leading order coefficients, the results allow to calculate all coefficients appearing at order $1/m$ to next-to-leading order in renormalization-group improved perturbation theory. They also provide a meaningful definition of the velocity transfer variable $v \cdot v'$ to order $1/m$.

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I. INTRODUCTION

Over the last few years, heavy quark effective field theory has been established as an efficient tool to analyze decay processes involving hadrons containing a heavy quark [1–5]. In such systems the heavy quark is almost on-shell and interacts with the surrounding soup of light quarks, antiquarks and gluons predominantly via the exchange of soft gluons. As $m_Q \gg \Lambda_{\text{QCD}}$ these soft interactions cannot resolve the structure of the heavy quark; in particular, they are blind to its flavor and spin. In this limit the heavy quark acts as a featureless color source. This is the origin of a spin-flavor symmetry, which relates the properties of hadrons containing different heavy quarks [6, 7].

Since, in the $m_Q \rightarrow \infty$ limit, the heavy quark velocity v is conserved with respect to soft QCD interactions, it is appropriate to split the total momentum into a “large” kinetic piece and a “small” residual momentum k , which puts the heavy quark slightly off-shell: $p_Q = m_Q v + k$. Because all dynamics resides in k , it is useful to absorb the mass-dependent piece of the momentum by a field redefinition. To this end, one introduces a velocity-dependent field by [2]

$$Q(x) = e^{im_Q v \cdot x} h_v(x) \quad (1)$$

and imposes the on-shell condition $\not{v} h_v(x) = h_v(x)$, corrections to which are suppressed as Λ_{QCD}/m_Q . The new fields carry the residual momentum k , which by construction does not scale with m_Q . Their strong interactions are described by the so-called heavy quark effective theory (HQET), which essentially provides an expansion in k/m_Q . To lowest order in $1/m_Q$, the effective Lagrangian is [2, 3]

$$\mathcal{L}_v = \bar{h}_v i v \cdot D h_v, \quad (2)$$

where D is the gauge-covariant derivative. Such a Lagrangian has to be written for every heavy quark in the process under consideration. For N_h heavy quarks of the same velocity, the total Lagrangian is then invariant under a $SU(2N_h)$ spin-flavor symmetry group. This symmetry is explicitly broken at order $1/m_Q$ by the presence of higher dimension operators [1, 5].

For the effective theory to provide a converging expansion it is necessary that k be of order Λ_{QCD} . This implies that the heavy quark velocity v must be close to the velocity v_h of the hadron containing the heavy quark:

$$v = v_h + \mathcal{O}(\Lambda_{\text{QCD}}/m_Q). \quad (3)$$

This still allows some freedom in the choice of v , however. Instead of using (v, k) as the heavy quark velocity and residual momentum, one can as well construct HQET using some different set of variables $(v + q/m_Q, k - q)$, as long as q is of order Λ_{QCD} and satisfies $2v \cdot q + q^2/m_Q = 0$, so that the new velocity is still a unit four-vector. The effective theories obtained in these two ways must, of course, be equivalent [8]. This so-called reparametrization invariance of HQET is a very useful concept in that it

relates the coefficients of operators appearing at different order in the $1/m_Q$ expansion [9]. These relations are renormalization-group invariant, i.e, they are true to all orders in perturbation theory and cannot be subject to nonperturbative corrections either.

In this paper, we use reparametrization invariance to derive relations between the coefficients appearing at leading and subleading order in the expansion of heavy quark currents in HQET. Some of these relations were already obtained in Ref. [9], by imposing reparametrization invariance on the operators in the effective theory. However, it was not realized until now that additional relations can be derived by writing also the velocity-dependent short-distance coefficients in a reparametrization invariant form. In fact, we show that *all* coefficients of the effective current operators of dimension four can be determined that way. In Sect. 2 we briefly discuss the heavy quark expansion of currents in HQET. In Sect. 3 we recall the concept of reparametrization invariance and study its implications for the coefficients in the expansion of currents. Sect. 4 deals with an interpretation of the reparametrization invariant extension of the velocity transfer variable $v \cdot v'$. A short summary of the results is given in Sect. 5.

II. EXPANSION OF CURRENTS IN HQET

Let us consider currents of the form $\bar{Q}' \Gamma Q$, which mediate transitions between two heavy quarks Q and Q' of, in general, different flavor. In principle Γ could be an arbitrary combination of Dirac matrices. For the weak currents, however, $\Gamma = \gamma^\mu(1 - \gamma_5)$. We are interested in hadronic matrix elements of these currents between hadron states $H(v_h)$ and $H'(v'_h)$ which contain the heavy quarks. HQET can be used to make the dependence of such matrix elements on the heavy quark masses explicit. In the effective theory each current has a representation as a series of operators built from the new fields h_v and h'_v , replacing Q and Q' . These effective current operators can have dimension higher than three, in which case they are multiplied by inverse powers of the heavy quark masses. In general, one has

$$\bar{Q}' \Gamma Q \hat{=} \sum_i C_i J_i + \sum_j \left[\frac{B_j}{2m_Q} + \frac{B'_j}{2m_{Q'}} \right] O_j + \mathcal{O}(1/m^2), \quad (4)$$

where the symbol $\hat{=}$ is used for equations which are true in matrix elements only, and m stands generically for m_Q or $m_{Q'}$. Both the coefficients and the operators in this expansion depend on Γ . The $\{J_i\}$ are a complete set of dimension three operators with the same quantum numbers as the original current. Similarly, the $\{O_j\}$ form a basis of dimension four operators. Since in the effective theory the fields carry velocity labels, the effective current operators can depend on v and v' . In case of the vector current $\bar{Q}' \gamma^\mu Q$, for instance, the dimension three operators are

$$\begin{aligned} J_1 &= \bar{h}'_v \gamma^\mu h_v, \\ J_2 &= \bar{h}'_v v^\mu h_v, \\ J_3 &= \bar{h}'_v v'^\mu h_v, \end{aligned} \quad (5)$$

while a convenient basis for the dimension four operators is

$$\begin{aligned}
O_1 &= \bar{h}'_v \gamma^\mu i \not{D} h_v, & O_8 &= -\bar{h}'_v i \overleftarrow{\not{D}} \gamma^\mu h_v, \\
O_2 &= \bar{h}'_v v^\mu i \not{D} h_v, & O_9 &= -\bar{h}'_v i \overleftarrow{\not{D}} v^\mu h_v, \\
O_3 &= \bar{h}'_v v'^\mu i \not{D} h_v, & O_{10} &= -\bar{h}'_v i \overleftarrow{\not{D}} v'^\mu h_v, \\
O_4 &= \bar{h}'_v i D^\mu h_v, & O_{11} &= -\bar{h}'_v i \overleftarrow{D}^\mu h_v, \\
O_5 &= \bar{h}'_v \gamma^\mu i v' \cdot D h_v, & O_{12} &= -\bar{h}'_v i v \cdot \overleftarrow{D} \gamma^\mu h_v, \\
O_6 &= \bar{h}'_v v^\mu i v' \cdot D h_v, & O_{13} &= -\bar{h}'_v i v \cdot \overleftarrow{D} v^\mu h_v, \\
O_7 &= \bar{h}'_v v'^\mu i v' \cdot D h_v, & O_{14} &= -\bar{h}'_v i v \cdot \overleftarrow{D} v'^\mu h_v.
\end{aligned} \tag{6}$$

Similar sets of operators can be constructed for the expansion of the axial vector current $\bar{Q}' \gamma^\mu \gamma_5 Q$. In (6) we have not included operators that vanish by the equation of motion $i v \cdot D h_v = 0$ following from the effective Lagrangian (2). They are irrelevant at the level of matrix elements. For simplicity we have evaluated the currents at $x = 0$; otherwise the operators in HQET would acquire a phase according to (1).

Eq. (4) provides a separation of short- and long-distance contributions to current matrix elements. The perturbative corrections arising from hard gluons (with virtualities of order m_Q or $m_{Q'}$) are factorized into the coefficients C_i , B_j and B'_j , which are functions of the heavy quark masses, the velocity transfer $v \cdot v'$, as well as an arbitrary matching scale μ : $C_i = C_i(m_Q, m_{Q'}, v \cdot v', \mu)$ etc. In particular, these functions contain any logarithmic dependence on the heavy quark masses resulting from the running couplings $\alpha_s(m_Q)$ and $\alpha_s(m_{Q'})$. All long-distance effects, on the other hand, are still contained in the hadronic matrix elements of the effective current operators, which are to be evaluated between states of the effective theory. These matrix elements can be parameterized by universal functions of the hadron velocity transfer $v_h \cdot v'_h$, which are independent of the heavy quark masses. They do depend on the matching scale, however, in such a way that the right-hand side of (4) is μ -independent. For the vector and axial vector currents the coefficients C_i are known to next-to-leading order in renormalization-group improved perturbation theory [10]. The coefficients B_j and B'_j , on the other hand, have so far only been calculated in leading logarithmic approximation [11].

III. RELATIONS IMPOSED BY REPARAMETRIZATION INVARIANCE

The effective theory must be invariant under reparametrizations of the heavy quark velocity and residual momentum which leave the total momentum $p_Q = m_Q v + k$ unchanged. Luke and Manohar have investigated the implications following from this simple statement in detail [9]. They found that the velocity and the covariant derivative must always appear in the combination

$$\mathcal{V} = v + \frac{iD}{m_Q}, \tag{7}$$

which can be interpreted as the gauge-covariant extension of the operator \hat{p}_Q/m_Q . A subtlety which has to be taken into account is that the heavy quark spinor fields transform under reparametrizations in a nontrivial way. They become invariant by including a Lorentz boost $\Lambda(\mathcal{V}, v)$, which transforms v into \mathcal{V} . The result is that the effective Lagrangian of HQET, as well as any composite operator in the effective theory, must be built of \mathcal{V} and $\tilde{h}_v = \Lambda(\mathcal{V}, v) h_v$. At order $1/m$, the explicit form of \tilde{h}_v is

$$\tilde{h}_v = \left(1 + \frac{i\not{D}}{2m_Q}\right) h_v = \frac{1 + \mathcal{V}}{2} h_v. \quad (8)$$

Given this result, one can immediately relate some of the coefficients in (4), namely [9]:

$$\begin{aligned} B_1 &= B'_8 = C_1, \\ B_2 &= \frac{1}{2} B_4 = B'_9 = C_2, \\ B_3 &= B'_{10} = \frac{1}{2} B'_{11} = C_3. \end{aligned} \quad (9)$$

Since all dimension four operators in (6) contain a covariant derivative acting on one of the heavy quark fields and are therefore not reparametrization invariant by themselves, it is clear that there must be additional relations. For instance, derivatives acting on h_v can only come in combination with a coefficient $1/m_Q$, while those acting on $\tilde{h}_{v'}$ must come with $1/m_{Q'}$. Hence

$$\begin{aligned} B_j &= 0; \quad j = 8, \dots, 14, \\ B'_j &= 0; \quad j = 1, \dots, 7. \end{aligned} \quad (10)$$

What remains to be determined, then, are the coefficients B_j for $j = 5, 6, 7$ and B'_j for $j = 12, 13, 14$. The important new observation which accomplishes this is that not only the effective current operators, but also the *velocity-dependent coefficient functions* must be written in a reparametrization invariant way. This means that the variable $w = v \cdot v'$ which these functions depend on has to be replaced by the reparametrization invariant *operator*

$$\hat{w} = \mathcal{V}'^\dagger \cdot \mathcal{V} = \left(v' - \frac{i\overleftarrow{D}}{m_{Q'}}\right) \cdot \left(v + \frac{iD}{m_Q}\right), \quad (11)$$

where it is understood that iD acts only on h_v , while $i\overleftarrow{D}$ acts on $\tilde{h}_{v'}$. Inserting the expansion (for simplicity, we suppress the dependence of C_i on the heavy quark masses and on μ)

$$C_i(\hat{w}) = C_i(w) + \frac{\partial C_i(w)}{\partial w} \left[\frac{iv' \cdot D}{m_Q} - \frac{iv \cdot \overleftarrow{D}}{m_{Q'}} \right] + \mathcal{O}(1/m^2) \quad (12)$$

into (4) one does indeed generate the remaining operators O_5 to O_7 and O_{12} to O_{14} . We find

$$\begin{aligned} B_5 &= B'_{12} = 2 \frac{\partial C_1}{\partial w}, \\ B_6 &= B'_{13} = 2 \frac{\partial C_2}{\partial w}, \\ B_7 &= B'_{14} = 2 \frac{\partial C_3}{\partial w}. \end{aligned} \tag{13}$$

Eqs. (9)–(13) summarize our main result: Reparametrization invariance relates *all* coefficients appearing at order $1/m$ in the heavy quark expansion of the vector current to the coefficients appearing at leading order, and to their derivatives with respect to $w = v \cdot v'$. A similar statement applies, of course, for any other current. These relations are valid to all orders in perturbation theory, and they cannot be modified by nonperturbative corrections either.

At this point it is worthwhile to compare our exact results to some approximate expressions for the short-distance coefficients known so far in the literature. In Ref. [11] the coefficients have been calculated in leading logarithmic approximation, working with an average heavy quark mass m . In accordance with the relations (9) and (13), one then obtains

$$\begin{aligned} C_1(w) &= B_1(w) = B'_8(w) = \left(\frac{\alpha_s(m)}{\alpha_s(\mu)} \right)^{a_L(w)}, \\ B_5(w) &= B'_{12}(w) = 2 \frac{\partial C_1(w)}{\partial w} = 2 a'_L(w) \ln \left(\frac{\alpha_s(m)}{\alpha_s(\mu)} \right) C_1(w), \end{aligned} \tag{14}$$

where

$$a_L(w) = \frac{8}{33 - 2n_f} \left[\frac{w}{\sqrt{w^2 - 1}} \ln(w + \sqrt{w^2 - 1}) - 1 \right], \tag{15}$$

and n_f is the number of light quark flavors. All other coefficients vanish in this approximation.¹ Given our exact relations and the fact that the coefficients C_i are known to next-to-leading logarithmic order [10], it is now possible to derive much more accurate expressions for B_j and B'_j .

¹We can also compare to Ref. [12], where the matching contributions of order $\alpha_s(m_{Q'})/m_{Q'}$ arising at $\mu = m_{Q'}$ have been computed. The results given there satisfy the relations (9) and (13). The expression presented for the coefficient C_1 is incorrect, however. The correct result is given in Ref. [10].

IV. REPARAMETRIZATION INVARIANT VELOCITY TRANSFER

At order $1/m$, the effect of the operator \hat{w} in the short-distance coefficients can be readily evaluated at the level of matrix elements. The equation of motion $iv \cdot D h_v = 0$ and the corresponding equation for \bar{h}'_v , allow one to replace the covariant derivatives in (12) by total derivatives acting on the current, e.g.

$$\bar{h}'_v \Gamma i v' \cdot D h_v = i v' \cdot \partial [\bar{h}'_v \Gamma h_v], \quad (16)$$

where Γ is again arbitrary. From translational invariance, and taking into account the phase factors in the definition of the effective heavy quark fields in (1), one finds that the x -dependence of a current matrix element between hadron states $H(v_h)$ and $H'(v'_h)$ is given by $\exp(-i\phi \cdot x)$, where

$$\phi = (m_H v_h - m_Q v) - (m_{H'} v'_h - m_{Q'} v'). \quad (17)$$

Using this, together with the fact that $m_H - m_Q = m_{H'} - m_{Q'}$ to leading order in the $1/m$ expansion, it is straightforward to show that

$$(\hat{w}-1) \langle H'(v'_h) | \bar{h}'_v \Gamma h_v | H(v_h) \rangle = \frac{m_H m_{H'}}{m_Q m_{Q'}} (w_h - 1) \langle H'(v'_h) | \bar{h}'_v \Gamma h_v | H(v_h) \rangle + \mathcal{O}(1/m^2), \quad (18)$$

where $w_h = v_h \cdot v'_h$. Note that the *hadron velocities* appear in this equation. To order $1/m$, it follows that in matrix elements the operator \hat{w} in the short-distance coefficient functions C_i can be replaced by the reparametrization invariant velocity transfer variable

$$\bar{w} = 1 + \frac{m_H m_{H'}}{m_Q m_{Q'}} (v_h \cdot v'_h - 1). \quad (19)$$

If this variable is used in the coefficient functions, the operators O_5 to O_7 and O_{12} to O_{14} no longer appear in the expansion (4) since, for instance,

$$C_1(w) J_1 + 2 \frac{\partial C_1(w)}{\partial w} \left[\frac{O_5}{2m_Q} + \frac{O_{12}}{2m_{Q'}} \right] \cong C_1(\bar{w}) J_1 + \mathcal{O}(1/m^2). \quad (20)$$

Let us explore in more detail the physical meaning of the variable \bar{w} . One might have expected that the reparametrization invariant generalization of the quark velocity transfer would be the velocity transfer of the hadrons, $w_h = v_h \cdot v'_h$. This is not the case, however. Rather, in (19) there appears an additional scaling factor depending on the hadron and quark masses. The kinematic region for \bar{w} extends from $\bar{w} = 1$ at zero recoil ($v_h \cdot v'_h = 1$) up to a maximum value given by

$$\bar{w}_{\max} - 1 = \frac{m_H m_{H'}}{m_Q m_{Q'}} \frac{(m_H - m_{H'})^2}{2m_H m_{H'}} = \frac{(m_Q - m_{Q'})^2}{2m_Q m_{Q'}} + \mathcal{O}(1/m^2). \quad (21)$$

This is just the maximum velocity transfer attainable in a decay of *free* quarks. In fact, it can be readily seen that (19) is precisely (up to terms of order $1/m^2$) the condition

$$(p_H - p_{H'})^2 = (p_Q - p_{Q'})^2 \quad (22)$$

that the momentum transfer to the hadrons equals the momentum transfer to free heavy quarks.

It is not hard to see why, away from zero recoil, the quark velocity transfer \bar{w} seen by hard gluons is different from the hadron velocity transfer w_h . Consider the weak decay $H \rightarrow H' + W$. In the initial state, the heavy quark Q moves on average with the hadron's velocity v_h . When the W boson is emitted, the outgoing heavy quark Q' has in general some different velocity $v_{Q'}$. Over short time scales this velocity remains unchanged, and this is what is seen by hard gluons. After the W emission, however, the light degrees of freedom in the initial hadron still have the initial hadron's velocity. They have to combine with the outgoing heavy quark to form the final state hadron H' . This rearrangement happens over much larger, hadronic time scales by the exchange of soft gluons. In this process the velocity of Q' is changed by an amount of order $1/m$ (its momentum is changed by an amount of order Λ_{QCD}). Hence the hadron velocity transfer differs from the “short-distance” quark velocity transfer by an amount of order $1/m$. The precise relation between w_h and \bar{w} is determined by momentum conservation and is given in (19). At zero recoil, no such rearrangement is needed, and indeed $\bar{w} = w_h = 1$ in this limit.

V. SUMMARY

We have shown that, to order $1/m$ in the heavy quark effective theory, the form of renormalized bilinear heavy quark currents is completely determined by the reparametrization invariant extension of the leading order currents. This is achieved by imposing reparametrization invariance on both the effective current operators and the velocity-dependent short-distance coefficient functions. This way, the velocity transfer variable $w = v \cdot v'$ is promoted into an operator \hat{w} , which in matrix elements can be replaced by a new variable \bar{w} that can be interpreted as being the “short-distance” velocity transfer of free heavy quarks, i.e., the velocity transfer seen by hard gluons. This variable depends only on the hadron velocities and is therefore invariant under reparametrizations. For the vector current, the result reads

$$\begin{aligned} \bar{Q}' \gamma^\mu Q &\hat{=} \bar{h}'_{v'} \left[C_1(\hat{w}) \gamma^\mu + C_2(\hat{w}) \mathcal{V}^\mu + C_3(\hat{w}) \mathcal{V}'^{\mu} \right] \tilde{h}_v \\ &\hat{=} C_1(\bar{w}) \left[J_1 + \frac{O_1}{2m_Q} + \frac{O_8}{2m_{Q'}} \right] + C_2(\bar{w}) \left[J_2 + \frac{O_2 + 2O_4}{2m_Q} + \frac{O_9}{2m_{Q'}} \right] \\ &+ C_3(\bar{w}) \left[J_3 + \frac{O_3}{2m_Q} + \frac{O_{10} + 2O_{11}}{2m_{Q'}} \right] + \mathcal{O}(1/m^2), \end{aligned} \quad (23)$$

with \mathcal{V} and \tilde{h}_v as defined in (7) and (8), respectively. The generalization to other currents is straightforward. The matrix elements of the effective current operators J_i and O_i can be parameterized by universal functions of the hadron velocity transfer in the standard way [4, 13]. Reparametrization invariance relates the anomalous dimensions of the dimension three and dimension four operators in (23), and this leads to relations between the μ -dependence of the associated universal functions.

For the vector and axial vector currents, the coefficients C_i are known to next-to-leading order in renormalization-group improved perturbation theory, for an arbitrary ratio of the heavy quark masses. The ingredients which go into their calculation are the one- and two-loop anomalous dimensions of the operators J_i [4, 14], and the full one-loop matching between QCD and the heavy quark effective theory [10, 15]. In the case of different heavy quark masses one needs in addition the anomalous dimensions and matching in the intermediate effective theory, which governs the region $m_{Q'} < \mu < m_Q$ [16–18]. Detailed lists of the numerical values of C_i as functions of \bar{w} and the heavy quark masses are compiled in Ref. [10]. By virtue of (23) the currents are now known to order $1/m$ with the same accuracy.

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