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THE FIVE GLUON AMPLITUDE AND ONE-LOOP INTEGRALS[a]

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ABSTRACT

We review the conventional field theory description of the string motivated technique. This technique is applied to the one-loop five-gluon amplitude. To evaluate the amplitude a general method for computing dimensionally regulated one-loop integrals is outlined including results for one-loop integrals required for the pentagon diagram and beyond. Finally, two five-gluon helicity amplitudes are given.

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1. Introduction

The search for new physics at current and future hadron colliders demands that we first refine our understanding of events originating in known physics, most importantly QCD-associated background processes. Because the perturbation expansion for jet physics in QCD is not an expansion strictly in the coupling constant, but is rather an expansion in the coupling constant times various infrared logarithms, loop corrections play an important role in matching theoretical expectations to experimental data. Thus far, the one-loop corrections are known only for the most basic processes, matrix elements with four external partons.

In order to minimize the algebra required for one-loop computations involving n external gluons string motivated rules were developed in ref. [1]. Although the method was originally derived from string theory, it has been summarized in terms of simple rules which require no knowledge of string theory[1,2]. Since string theories contain gauge theories in the infinite string tension limit[3] and have a simpler organization of the amplitudes than field theories, a string motivated organization of the amplitude is more compact than a traditional Feynman diagram organization.

Here we discuss the application of the string motivated technique to the computation of the five-gluon amplitude. This requires the evaluation of dimensionally

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regularized pentagon integrals. The computation of pentagon integrals in the case in which all internal lines are massive has been discussed by various authors[4]. In particular van Neerven and Vermaseren have provided an efficient method for calculating such integrals in four dimensions. The techniques of van Neerven and Vermaseren do not apply directly to dimensionally-regularized integrals, however, and the required pentagon integrals have not yet been presented in a closed and useful form, which is to say with all poles in $\epsilon = (4 - D)/2$ manifest, and with all functions of the kinematic invariants expressed in terms of (poly)logarithms. Here we will provide a formula which yields such expressions.[b] We also present a general solution for one-loop integrals beyond the pentagon.

2. Review of String Motivated Methods

The string motivated rules evaluate a one-loop n gluon amplitude in terms of substitution rules acting on a basic kinematic expression. In refs. [1,2] the substitution rules necessary to obtain the values of all diagrams associated with a one-loop n-gluon amplitude have already been given. Here we will not present the rules but will instead briefly review the interpretation of the rules in terms of conventional field theory. The conventional field theory ideas necessary to reproduce the simplicity of the string are[5]:

a) Use of background field gauge. Calculations in QCD have traditionally been performed in ordinary Feynman gauge. Perhaps a reason why the background field method[6] has not been used for gluon amplitudes is that it inherently seems to be a method for computing effective actions and not scattering amplitudes. However, as shown a number of years ago by Abbott, Grisaru and Schaeffer[7], the background field method can in fact be used for S-matrix computations; one simply sews trees onto the loops in some other gauge to obtain the S-matrix elements. In the background field Feynman gauge, vertices can be organized to mimic the simple structure inherent in the string motivated rules leading to large simplifications for QCD amplitudes. A convenient gauge for sewing trees onto the loops is the non-linear Gervais-Neveu gauge[8] (which was also motivated by string theory) since it has simple vertices.

b) Color ordering of vertices. This amounts to rewriting the Yang-Mills structure constant as $f^{abc} = -i\mathrm{Tr}([T^a, T^b]T^c)/\sqrt{2}$ and then evaluating the coefficient of a single color trace ordering[9]. String theory motivates the use of a $U(N_c)$ gauge

[b] We have been informed that R.K. Ellis, W. Giele and E. Yehudai have recently evaluated the pentagon integrals by an independent technique.

group instead of an $SU(N_c)$ gauge group; the extra $U(1)$ decouples but the relevant color algebra is much simpler for $U(N_c)$. A full description of the one-loop color decomposition has been given in ref. [10].

c) Systematic organization of the vertex algebra. In order to minimize the work involved, it is important to organize the vertex algebra in a particular systematic fashion. Because of the way in which the loop momentum enters into the background field vertices it turns out that the integration over loop momentum is trivial. In conventional gauges or other background field gauges the loop momentum enters into the vertices in a much more complicated way, not allowing a simple systematic organization. Once the amplitude has been written in a form where the loop momentum is integrated out, one can use the spinor helicity method[11] to simplify the expressions.

Given this field theory understanding of the string motivated method one might conclude that string theory is no longer required. This, however, misses the point behind the use of string theory; the point is that string theory guides computational organizations of gauge theory amplitudes where efficient organizations of the amplitude are unknown (as the one-loop case was prior to string motivated methods). Further examples where string theory provides useful insight which would be difficult to obtain by conventional means are extensions to multi-loops and calculations of gravity amplitudes[12]. Even with all known field theory tricks such as spinor helicity methods and special gauge choices, in a conventional framework it is difficult to envision the compact organization of the amplitudes implied by string theory.

3. One-Loop Integrals.

The integral we wish to evaluate is the n-gon with general kinematics, whose momentum-space definition is

$$I_n[\tilde{P}(p_\mu)] = (-1)^{n+1}(4\pi)^{2-\epsilon} i \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \frac{\tilde{P}(p_\mu)}{\left(p^2 - M_1^2\right)\left((p-p_1)^2 - M_2^2\right)\cdots\left((p-p_{n-1})^2 - M_n^2\right)}\ . \tag{1}$$

where $\tilde{P}(p_\mu)$ is a polynomial in the loop momentum and where we take $p_i^\mu \equiv \sum_{j=1}^{i} k_j^\mu$ and $k_i^2 = m_i^2$ with the k_i momenta of external particles. For QCD, $m_i = M_i = 0$. In the five point case, after Feynman parametrization the integral becomes

$$I_5[P(\{a_i\})] = \int_0^1 d^5a_i\ \frac{\Gamma(3+\epsilon)P(\{a_i\})\delta(1-\sum_i a_i)}{\left[-\sum\limits_{i=1}^{5}\left((s_{i,i+1} - M_i^2 - M_{i+2}^2)a_i a_{i+2} + (m_i^2 - M_i^2 - M_{i+1}^2)a_i a_{i+1} - M_i^2 a_i^2\right)\right]^{3+\epsilon}}\ . \tag{2}$$

Following 't Hooft and Veltman[4], we make the change of variables $a_i = \alpha_i u_i / \sum_{j=1}^{n} \alpha_j u_j$, (no sum on i) where

$$s_{i,i+1} - M_i^2 - M_{i+2}^2 \;=\; -\frac{1}{\alpha_i \alpha_{i+2}} \;, \qquad m_i^2 - M_i^2 - M_{i+1}^2 \;=\; -\frac{\hat{m}_i^2}{\alpha_i \alpha_{i+1}} \;, \qquad M_i^2 \;=\; -\frac{\hat{M}_i^2}{\alpha_i^2} \;, \tag{3}$$

so that

$$I_5[P(\{a_i\}] = \Gamma(3+\epsilon) \int_0^1 d^5u \, \frac{P(\{\alpha_i u_i\})\, \delta\left(\sum u_i - 1\right) \left(\sum_{j=1}^{5} \alpha_j u_j\right)^{1-m+2\epsilon}}{\left[\sum_{i=1}^{5} \left(u_i u_{i+2} + \hat{m}_i^2 u_i u_{i+1} - \hat{M}_i^2 u_i^2\right)\right]^{3+\epsilon}} \;. \tag{4}$$

The key observation is that this integral can be expressed in terms of derivatives acting on the scalar integral

$$I_5[P_m(\{a_i\}] = \frac{\Gamma(2-m+2\epsilon)}{\Gamma(2+2\epsilon)} \;: P_m\left(\left\{\alpha_i \frac{\partial}{\partial \alpha_i}\right\}\right) :\; I_5[1] \tag{5}$$

where P_m is a homogeneous polynomial of degree m and the normal ordering signifies that all the α_i should be brought to the left of the derivatives. This equation and its extensions forms the basis for obtaining all the tensor integrals as derivatives of the scalar integral and for a differential equation method for evaluating integrals[13].

In order to use this equation we need the solution of the pentagon scalar integral. The general recursive solution for $n \geq 5$ external legs is

$$I_n[1] \;=\; \frac{1}{2N_n} \left[\frac{1}{2} \sum_{i=1}^{n} \alpha_i \frac{\partial \hat{\Delta}_n}{\partial \alpha_i}\Big|_{\mathrm{non-}\alpha_i \text{ variables fixed}} \times I_{n-1}^{(i)}[1] \;+\; (n-5+2\epsilon)\, \hat{\Delta}_n \, I_n^{D=6-2\epsilon}[1] \right] \tag{6}$$

where $I_{n-1}^{(i)}[1]$ is the $(n-1)$-point scalar integral obtained by removing the internal propagator between legs $i-1$ and i (defined mod n), $\hat{\Delta}_n = \det(2k_i \cdot k_j) \times \prod_{i=1}^{n} \alpha_i^2$ is the rescaled Gram determinant (with $i, j = 1 \cdots, n-1$). For $n = 5$ the α_i are defined in Eq. (3). In this case the last term in Eq. (6) is suppressed by a power of ϵ since the $D = 6 - 2\epsilon$ scalar pentagon, $I_5^{D=6-2\epsilon}[1]$, is completely finite. Thus, the explicit value of $I_5^{D=6-2\epsilon}[1]$ is not needed. (It also turns out that it is not needed when applying the differentiation formula (5).) For $n > 5$ the last term vanishes for four-dimensional external kinematics due to the vanishing of the Gram determinant. In this way we obtain a recursive solution for all one-loop scalar integrals in terms of the box integrals (which are relatively easy to evaluate). The overall normalization is $N_n = 2^{n-1} \det \rho$ where $\rho_{ij} = -\frac{1}{2}((p_{j-1} - p_{i-1})^2 - M_i^2 - M_j^2)\alpha_i \alpha_j$ is independent of the α_i when converted to rescaled variables analogous to the pentagon ones in Eq. (3). For the massless pentagon $N_5 = 1$. This solution extends van Neerven and Vermaseren's[4] four-dimensional pentagon solution to dimensional regularization and arbitrary numbers of external legs.

One way to verify the solution (6) for a particular n is by considering the integral (1) with an inverse propagator in the numerator. This integral can be evaluated as either an n-point integral or as an $(n-1)$-point integral. By comparing the two forms and summing over cyclic permutations with coefficients obtained from the solution (6) one can verify its correctness after using $\sum_{i=1}^{n} a_i = 1$. A general proof will be given elsewhere[13].

4. Explicit Results for Amplitudes

For five-point amplitudes a straightforward use of the spinor helicity method is cumbersome. By rewriting ratios of spinor inner products in terms of more conventional kinematic variables and the square-root of the pentagon Gram determinant the spinor helicity method becomes more usable[13]. The five gluon amplitudes can then be obtained by applying the string motivated techniques and using the solution for the pentagon integral. By following this procedure we have obtained the two finite five-gluon one-loop $SU(N_c)$ partial amplitudes:

$$\begin{aligned} A_{5;1}^{1-\mathrm{loop}}(1^+,2^+,3^+,4^+,5^+) = &\left(1+\frac{1}{2}N_s^{\mathrm{adj}}-N_f^{\mathrm{adj}}\right)\\ &\times \frac{i}{48\pi^2}\,\frac{\langle 1\,2\rangle\,[1\,2]\,\langle 2\,3\rangle\,[2\,3]+\langle 4\,5\rangle\,[4\,5]\,\langle 5\,1\rangle\,[5\,1]+\langle 2\,3\rangle\,\langle 4\,5\rangle\,[2\,5]\,[3\,4]}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle}\\ A_{5;1}^{1-\mathrm{loop}}(1^-,2^+,3^+,4^+,5^+) = &\left(1+\frac{1}{2}N_s^{\mathrm{adj}}-N_f^{\mathrm{adj}}\right)\\ &\times \frac{i}{48\pi^2}\,\frac{1}{\langle 3\,4\rangle^2}\left[\frac{[2\,5]^3}{[1\,2]\,[5\,1]}+\frac{\langle 1\,4\rangle^3\,[4\,5]\,\langle 3\,5\rangle}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 4\,5\rangle^2}-\frac{\langle 1\,3\rangle^3\,[3\,2]\,\langle 4\,2\rangle}{\langle 1\,5\rangle\,\langle 5\,4\rangle\,\langle 3\,2\rangle^2}\right] \end{aligned} \tag{7}$$

where N_s^{adj} and N_f^{adj} are the number of adjoint massless real scalars and Weyl fermions. (Fundamental representation fermions or scalars require an additional factor of $1/N_c$.) We follow the same notation and normalizations as in refs. [1,2]. The corresponding double trace $A_{5;3}$ partial amplitudes follow from the formulae in ref. [10]. These amplitudes satisfy the relevant supersymmetry Ward identities[14] (which are satisfied trivially since they hold for the integrand of each string motivated diagram). The remaining helicity amplitudes will be presented elsewhere.

In summary, the string motivated organization of the n-gluon amplitude plays a major role in simplifying the computation of the five-gluon one-loop amplitude. Additional ingredients which allow the computation to be performed are simple formulae for the relevant one-loop integrals and a rewriting of spinor-helicity invariants in terms of more conventional kinematic quantities. These issues will be presented in detail elsewhere[13].

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