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Dimensionally Regulated One-Loop Integrals

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Abstract

We describe methods for evaluating one-loop integrals in $4-2\epsilon$ dimensions. We give a recursion relation that expresses the scalar n-point integral as a cyclicly symmetric combination of $(n-1)$-point integrals. The computation of such integrals thus reduces to the calculation of box diagrams ($n=4$). The tensor integrals required in gauge theory may be obtained by differentiating the scalar integral with respect to certain combinations of the kinematic variables. Such relations also lead to differential equations for scalar integrals. For box integrals with massless internal lines these differential equations are easy to solve.

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Many processes of interest at current and future e^+e^- and hadron colliders involve large numbers of final state particles. Radiative corrections to these processes are needed for precise tests of the standard model. It is therefore useful to have techniques for evaluating one-loop integrals where the number of external legs is large. As the loop integrals appearing in radiative corrections are often infrared and/or ultraviolet divergent, it is desirable to regulate them by performing them in $4 - 2\epsilon$ dimensions.

In this Letter, we derive a relation between the n-point and $(n-1)$-point one-loop integrals, which for $n > 4$ allows the recursive determination of the general n-point scalar integral in $D = 4-2\epsilon$, as a linear combination of box integrals ($n = 4$), provided only that the external momenta are restricted to lie in four dimensions, and neglecting $\mathcal{O}(\epsilon)$ corrections. The required box integrals can generally be evaluated in closed form through $\mathcal{O}(1)$, that is to say, with all poles in ϵ manifest, and with all functions of the kinematic invariants expressed in terms of logarithms and dilogarithms. (A compact expression for the general infrared-finite box integral has recently been given by Denner, Nierste, and Scharf [1]; the infrared-divergent box integrals with all internal lines massless are collected in ref. [2].) Therefore, the higher-point integrals can now be represented in the same closed form. In a separate paper [2], we apply these techniques to determine explicitly the pentagon integral with all external lines massless, or with one external mass.

Various authors [3,4,5,6] have discussed the computation of pentagon and higher-point integrals that can be evaluated in $D = 4$ (i.e. that are infrared finite). In particular, Melrose [3] and independently van Neerven and Vermaseren [5] have expressed the $D = 4$ pentagon integral as a linear combination of five $D = 4$ box integrals, and the relation we find for $n = 5$ may be thought of as the dimensionally-regulated version of their equations. References [3,5] also express the four-dimensional n-point scalar integral for $n \geq 6$ (with external momenta restricted to $D = 4$) as a sum of six $(n-1)$-point integrals; the derivation in ref. [5] extends straightforwardly to $(4 - 2\epsilon)$-dimensional loop-momenta as well. For $n > 6$ these relations are of a somewhat different type than the relations that we find. We have been informed that Ellis, Giele, and Yehudai [7] have recently evaluated the $D = 4 - 2\epsilon$ pentagon integrals by an independent technique.

In gauge theories, tensor integrals appear in which the n-point integral may contain up to n powers of the loop momentum in the numerator of the integrand. It is possible to perform a Brown-Feynman [8] or Passarino-Veltman [9] reduction of the integrand, solving a system of algebraic equations to reduce the tensor integrals to a linear combination of scalar integrals [10]. The framework developed here provides another method for computing tensor integrals. Feynman parametrization converts tensor integrals into integrals where polynomials in the Feynman parameters (of order $\leq n$) are inserted into the numerator of the integrand. (In the string-motivated technique [11] for evaluating QCD amplitudes such a representation is obtained directly.) We will see that such integrals may be obtained by differentiating the scalar integral with respect to particular combinations of the kinematic variables.

The basic integral we wish to evaluate is the dimensionally-regulated one-loop scalar integral with n external momenta k_i, n external masses m_i ($k_i^2 = m_i^2$), and n internal masses M_i, $i = 1, 2, \ldots, n$:

$$\mathcal{I}_n \;=\; \mu^{2\epsilon} \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \frac{1}{\left(p^2 - M_1^2\right)\left((p-p_1)^2 - M_2^2\right)\cdots\left((p-p_{n-1})^2 - M_n^2\right)} \;, \tag{1}$$

where we take

$$p_i^\mu \;\equiv\; \sum_{j=1}^{i} k_j^\mu, \qquad\qquad p_0^\mu \;=\; p_n^\mu \;=\; 0. \tag{2}$$

Introducing Feynman parameters in (1), $\mathcal{I}_n$ becomes

$$\mathcal{I}_n \;=\; \mu^{2\epsilon}\;(n-1)! \int_0^1 d^n a_i \;\int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}}\;\delta(1-\textstyle\sum_i a_i)\;\left[\displaystyle\sum_{i=1}^{n} a_i\left((p-p_{i-1})^2 - M_i^2\right)\right]^{-n}\;, \tag{3}$$

Completing the square in the denominator, Wick rotating, and integrating out the loop momentum yields

$$I_n[1] \;=\; \Gamma(n-2+\epsilon)\int_0^1 d^n a_i\;\delta(1-\textstyle\sum_i a_i)\;\displaystyle\frac{1}{[\mathcal{D}(a_i)]^{n-2+\epsilon}}\;, \tag{4}$$

where the scalar denominator $\mathcal{D}(a_i)$ is

$$\mathcal{D}(a_i) \;=\; \left[\sum_{i=1}^{n} a_i p_{i-1}\right]^2 \;-\; \sum_{i=1}^{n} a_i(p_{i-1}^2 - M_i^2)\;, \tag{5}$$

and we have defined

$$I_n[1] \;\equiv\; i\;(-1)^{n+1}\;(4\pi)^{2-\epsilon}\;\mu^{-2\epsilon}\;\mathcal{I}_n \tag{6}$$

as a convenient normalization for the Feynman parametrized scalar integral. (The expression inside the brackets indicates the Feynman-parameter polynomial in the numerator of the integrand; non-trivial polynomials correspond to tensor integrals.) We rewrite the second set of terms in $\mathcal{D}(a_i)$ using $\sum_i a_i = 1$, in order to make it homogeneous of degree two in the a_i and symmetric in $a_i \leftrightarrow a_j$:

$$\mathcal{D}(a_i) \;=\; \sum_{i,j=1}^{n} S_{ij}\;a_i a_j\;, \tag{7}$$

where the matrix S_{ij} is given by

$$S_{ij} \;=\; \tfrac{1}{2}(M_i^2 + M_j^2 - p_{ij}^2). \tag{8}$$

Here p_{ij} is the sum of $|i-j|$ adjacent momenta,

$$p_{ii} \;\equiv\; 0, \qquad\qquad p_{ij} \;\equiv\; p_{j-1} - p_{i-1} \;=\; k_i + k_{i+1} + \cdots + k_{j-1} \quad \text{for } i<j. \tag{9}$$

Now we are in a position to derive some useful general relations. We start by evaluating the following integral two different ways:

$$J_1 \;\equiv\; i\,(-1)^{n+1}\,(4\pi)^{2-\epsilon} \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \frac{p^2 - M_1^2}{\left(p^2 - M_1^2\right)\left((p-p_1)^2 - M_2^2\right)\cdots\left((p-p_{n-1})^2 - M_n^2\right)}\;. \tag{10}$$

First, we cancel the numerator against the denominator, yielding the $(n-1)$-point integral

$$J_1 \;=\; -I_{n-1}^{(1)}[1]\;. \tag{11}$$

In general, the notation $I_{n-1}^{(i)}[1]$ refers to an $(n-1)$-point integral whose kinematics is 'inherited' from its 'parent' $I_n[1]$ by removing the propagator between legs $i-1$ and i. Second, we evaluate J_1 as an n-point integral. Feynman parametrizing the integral and completing the square in the denominator in the usual way yields

$$\begin{aligned} J_1 \;=\; & i\,(-1)^{n+1}\,(4\pi)^{2-\epsilon}(n-1)! \int_0^1 d^n a_i\;\delta(1-\textstyle\sum_i a_i) \displaystyle\int \frac{d^{4-2\epsilon}q}{(2\pi)^{4-2\epsilon}} \\ & \times \frac{q^2 + \mathcal{D} + \sum_{i=1}^n a_i(p_{i-1}^2 - M_1^2 - M_i^2)}{[q^2 - \mathcal{D}]^n}\;. \end{aligned} \tag{12}$$

Next we integrate out the loop momentum in eq. (12) (after a Wick rotation), and equate the result to the $(n-1)$-point result (11), to get an equation for $I_n[a_i]$:

$$\begin{aligned} -I_{n-1}^{(1)}[1] \;&=\; \left[-\tfrac{1}{2}(4-2\epsilon) + (n-3+\epsilon)\right] I_n^{D=6-2\epsilon}[1] \;+\; \sum_{i=1}^n (p_{i-1}^2 - M_1^2 - M_i^2) I_n[a_i] \\ &=\; (n-5+2\epsilon)\, I_n^{D=6-2\epsilon}[1] \;-\; 2\sum_{i=1}^n S_{1i}\; I_n[a_i]\;. \end{aligned} \tag{13}$$

In this equation we have rewritten the terms coming from $q^2 + \mathcal{D}$ in the numerator of equation (12), using the Feynman-parameter representation of the scalar n-point integral in $D = 6-2\epsilon$ dimensions,

$$I_n^{D=6-2\epsilon}[1] \;=\; \Gamma(n-3+\epsilon) \int_0^1 d^n a_i\;\delta(1-\textstyle\sum_i a_i)\; \displaystyle\frac{1}{[\mathcal{D}(a_i)]^{n-3+\epsilon}}\;. \tag{14}$$

(This representation can be obtained from equation (4) simply by letting $\epsilon \to \epsilon - 1$, which shifts $D = 4-2\epsilon$ to $D = 6-2\epsilon$.) The properties of $I_n^{D=6-2\epsilon}[1]$ as $\epsilon \to 0$ will play a role below.

Similarly, by considering the integral J_i with $(p-p_{i-1})^2 - M_i^2$ in the numerator, we find the set of equations

$$2\sum_{j=1}^n S_{ij}\; I_n[a_j] \;=\; I_{n-1}^{(i)}[1] \;+\; (n-5+2\epsilon)\, I_n^{D=6-2\epsilon}[1]\;. \tag{15}$$

Solving these equations for $I_n[a_i]$ we get

$$I_n[a_i] \;=\; \frac{1}{2}\left[\sum_{j=1}^{n} S^{-1}_{ij}\; I^{(j)}_{n-1}[1] \;+\; (n-5+2\epsilon)\, c_i\; I_n^{D=6-2\epsilon}[1]\right] , \tag{16}$$

where

$$c_i \;=\; \sum_{j=1}^{n} S^{-1}_{ij}. \tag{17}$$

Now sum equation (16) over i and use $\sum_i a_i = 1$ to get

$$I_n[1] \;=\; \frac{1}{2}\left[\sum_{i=1}^{n} c_i\; I^{(i)}_{n-1}[1] \;+\; (n-5+2\epsilon)\, c_0\; I_n^{D=6-2\epsilon}[1]\right] , \tag{18}$$

where

$$c_0 \;=\; \sum_{i=1}^{n} c_i \;=\; \sum_{i,j=1}^{n} S^{-1}_{ij}. \tag{19}$$

Finally, use equation (18) to eliminate $I_5^{D=6-2\epsilon}$ from equation (16), and thereby obtain an equation for the one-parameter integrals $I_n[a_i]$ in terms of $I_{n-1}[1]$ and $I_n[1]$:

$$I_n[a_i] \;=\; \frac{1}{2}\sum_{j=1}^{n} c_{ij}\; I^{(j)}_{n-1}[1] \;+\; \frac{c_i}{c_0}\; I_n[1] , \tag{20}$$

where

$$c_{ij} \;=\; S^{-1}_{ij} - \frac{c_i c_j}{c_0} . \tag{21}$$

The external momenta as well as the loop-momenta in equations (16), (18) and (20) are still $(4-2\epsilon)$-dimensional. In computing one-loop corrections to physical processes, one is generally interested in restricting the external momenta to $D = 4$. On the other hand, for $n > 6$ the coefficients c_0 and c_i given in equations (19) and (17) appear to be singular for $D = 4$ kinematics. (The rank of the $n \times n$ matrix S_{ij} is $n-6$ in $D = 4$ [3], so for $n > 6$, S is not invertible.) In fact c_0 and c_i are nonsingular in the limit of $D = 4$ kinematics (c_0 actually vanishes for $n > 5$). In order to see this, and in order to see how to apply these results to tensor integrals, it is useful to perform two changes of variables: first a change of integration variables in the integral (4), then a change of kinematic variables.

Following 't Hooft and Veltman [4], we make the change of integration variables in equation (4),

$$\begin{aligned} a_i \;&=\; \frac{\alpha_i u_i}{\sum_{j=1}^{n} \alpha_j u_j} , \qquad \text{no sum on } i, \\ a_n \;&=\; \frac{\alpha_n \left(1 - \sum_{j=1}^{n-1} u_j\right)}{\sum_{j=1}^{n} \alpha_j u_j} . \end{aligned} \tag{22}$$

Assuming that all α_i are real and positive (physical regions may be obtained by analytic continuation), the integral becomes

$$I_n\,[1] \;=\; \Gamma(n-2+\epsilon)\left(\prod_{j=1}^{n}\alpha_j\right)\int_0^1 d^n u_i\;\frac{\delta\left(1-\sum u_i\right)\left(\sum_{j=1}^{n}\alpha_j u_j\right)^{n-4+2\epsilon}}{\left[\sum_{i,j=1}^{n}\rho_{ij}u_iu_j\right]^{n-2+\epsilon}}\;, \tag{23}$$

where

$$\rho_{ij} \;=\; S_{ij}\;\alpha_i\alpha_j \;=\; \tfrac{1}{2}\big(M_i^2+M_j^2-p_{ij}^2\big)\alpha_i\alpha_j\;. \tag{24}$$

The form (23) for the integral will be the most useful if we also make a change of kinematic variables such that the scalar denominator (i.e. the matrix ρ) no longer depends on the n variables α_i. First we describe a linearly independent set of the original kinematic variables. The $n(n+1)/2$ Mandelstam variables $s_{ij} \;\equiv\; (k_i+k_j)^2$ are not linearly independent, due to n momentum conservation relations. Instead one may use the square of the sum of ℓ adjacent momenta, $p_{i,i+\ell}^2 \;=\; (k_i+k_{i+1}+\ldots+k_{i+\ell-1})^2$. For n odd, ℓ runs from 1 to $(n-1)/2$. For n even, ℓ runs from 1 to $n/2$, but for $\ell=n/2$ there are only $n/2$, rather than n, such variables. These $n(n-1)/2$ linear combinations of Mandelstam variables form a linearly independent, cyclicly symmetric set. We should add to this set the n internal masses M_i^2. (Alternatively one may use the coefficients $S_{i,i+\ell}$ and S_{ii} appearing in the scalar denominator $\mathcal{D}(a_i)$.) For example, for the hexagon ($n=6$) the independent variables are M_i^2, m_i^2, $s_{i,i+1} \;\equiv\; (k_i+k_{i+1})^2$, for $i=1,2,3,4,5,6$ (all indices are taken mod 6), along with $t_{i,i+1,i+2} \;\equiv\; (k_i+k_{i+1}+k_{i+2})^2$, for $i=1,2,3$.

The following is an example of a change of kinematic variables that eliminates all α_i-dependence from the scalar denominator:

$$\{M_i, m_i, s_{i,i+1}, t_{i,i+1,i+2},\ldots\} \quad\to\quad \{\alpha_i; \hat{M}_i^2, \hat{m}_i^2, \hat{t}_i,\ldots\}\;, \tag{25}$$

where the new variables $\{\alpha_i; \hat{M}_i^2, \hat{m}_i^2, \hat{t}_i,\ldots\}$ are defined by

$$\begin{aligned}
M_i^2 &= -\frac{\hat{M}_i^2}{\alpha_i^2}\;,\\
m_i^2 &= -\frac{\hat{m}_i^2}{\alpha_i\alpha_{i+1}} - \frac{\hat{M}_i^2}{\alpha_i^2} - \frac{\hat{M}_{i+1}^2}{\alpha_{i+1}^2}\;,\\
s_{i,i+1} &= -\frac{1}{\alpha_i\alpha_{i+2}} - \frac{\hat{M}_i^2}{\alpha_i^2} - \frac{\hat{M}_{i+2}^2}{\alpha_{i+2}^2}\;,\\
t_{i,i+1,i+2} &= -\frac{\hat{t}_i}{\alpha_i\alpha_{i+3}} - \frac{\hat{M}_i^2}{\alpha_i^2} - \frac{\hat{M}_{i+3}^2}{\alpha_{i+3}^2}\;,\\
&\ldots \qquad\qquad\qquad\qquad (i=1,2,\ldots,n),
\end{aligned} \tag{26}$$

with all indices taken mod n. This is certainly not the only change of variables possible, and in some cases it is convenient to make other choices. (Indeed, for the special cases $n=8,12,16,\ldots$, equations (26) are not a legitimate change of variables. This problem can be cured by modifying slightly the substitution for just two of the $s_{i,i+1}$, say $s_{12} = -\frac{\lambda_1}{\alpha_1\alpha_3} - \frac{\hat{M}_1^2}{\alpha_1^2} - \frac{\hat{M}_3^2}{\alpha_3^2}$, and $s_{23} = -\frac{\lambda_2}{\alpha_2\alpha_4} - \frac{\hat{M}_2^2}{\alpha_2^2} - \frac{\hat{M}_4^2}{\alpha_4^2}$, so that the set of kinematic variables is now $\{\alpha_i; \lambda_1, \lambda_2, \hat{M}_i^2, \hat{m}_i^2, \hat{t}_i, \ldots\}$.)

With this change of kinematic variables, tensor integrals may be calculated simply by differentiating the scalar integral with respect to the α_i. Consider the Feynman-parametrized integral with an arbitrary monomial of degree m inserted in the numerator,

$$I_n[a_{i_1}a_{i_2}\ldots a_{i_m}] \;\equiv\; \Gamma(n-2+\epsilon)\int_0^1 d^n a_i\;\delta(1-\textstyle\sum_i a_i)\;\frac{a_{i_1}a_{i_2}\ldots a_{i_m}}{[\mathcal{D}(a_i)]^{n-2+\epsilon}}\;. \tag{27}$$

Using equation (23) it can be represented as

$$I_n[a_{i_1}a_{i_2}\ldots a_{i_m}] \;=\; \frac{\Gamma(n-3-m+2\epsilon)}{\Gamma(n-3+2\epsilon)}\left(\prod_{j=1}^n \alpha_j\right)\alpha_{i_1}\ldots\alpha_{i_m}\frac{\partial}{\partial\alpha_{i_1}}\cdots\frac{\partial}{\partial\alpha_{i_m}}\left(\frac{I_n[1]}{\prod_{j=1}^n\alpha_j}\right)\;. \tag{28}$$

If we define the *reduced integrals*

$$\begin{aligned}\hat{I}_n\left[\hat{P}(\{a_i\})\right] \;&\equiv\; \left(\prod_{j=1}^n\alpha_j\right)^{-1} I_n\left[P(\{a_i/\alpha_i\})\right]\;,\\ \hat{I}_n \;\equiv\; \hat{I}_n[1] \;&=\; \left(\prod_{j=1}^n\alpha_j\right)^{-1} I_n[1]\;,\end{aligned} \tag{29}$$

then we may write

$$\hat{I}_n[a_{i_1}a_{i_2}\ldots a_{i_m}] \;=\; \frac{\Gamma(n-3-m+2\epsilon)}{\Gamma(n-3+2\epsilon)}\;\frac{\partial^m\hat{I}_n}{\partial\alpha_{i_1}\cdots\partial\alpha_{i_m}}\;. \tag{30}$$

We now proceed to find simple expressions for the coefficients c_0, c_i and c_{ij} appearing in equations (18), (20), and (16), in terms of the α_i variables. To do this it is first useful to examine how the Gram determinant of the $(n-1)$-vector system associated with the n-point process depends on α_i. The Gram determinant is defined by

$$\Delta_n \;\equiv\; \det{}'(2k_i\cdot k_j), \tag{31}$$

where the prime signifies that one of the n vectors k_i is to be omitted before taking the determinant; due to momentum conservation, $\sum k_i = 0$, any one of the vectors may be omitted.†

Next we introduce the *rescaled* Gram determinant,

$$\hat{\Delta}_n \;\equiv\; \left(\prod_{\ell=1}^n\alpha_\ell^2\right)\Delta_n, \tag{32}$$

† The notation for, and normalization of, the Gram determinant in equation (31) differ from other conventions in the literature, e.g. references [3,12].

which will turn out to have several useful properties. Rewrite $\hat{\Delta}_n$ in terms of the variables α_i and the matrix ρ defined in equation (24), after omitting k_n in the definition of Δ_n:

$$
\begin{aligned}
\hat{\Delta}_n \;&=\; \Big(\prod_{\ell=1}^{n} \alpha_\ell^2\Big) \det_{i,j\neq n} (2k_i \cdot k_j) \;=\; \Big(\prod_{\ell=1}^{n} \alpha_\ell^2\Big) \det_{i,j\neq n} (2p_i \cdot p_j) \\
&=\; 2^{n-1}\Big(\prod_{\ell=1}^{n} \alpha_\ell^2\Big) \det_{i,j\neq 1}\left(\frac{\rho_{ij}}{\alpha_i\alpha_j} - \frac{\rho_{i1}}{\alpha_i\alpha_1} - \frac{\rho_{1j}}{\alpha_1\alpha_j} + \frac{\rho_{11}}{\alpha_1^2}\right) \\
&=\; 2^{n-1}\;\alpha_1^2\; \det_{i,j\neq 1}\left(\rho_{ij} - \rho_{i1}\frac{\alpha_j}{\alpha_1} - \rho_{1j}\frac{\alpha_i}{\alpha_1} + \rho_{11}\frac{\alpha_i\alpha_j}{\alpha_1^2}\right) .
\end{aligned}
\tag{33}
$$

By omitting other vectors k_i in the definition of Δ_n, it is easy to obtain n different expressions for the same quantity $\hat{\Delta}_n$,

$$
\hat{\Delta}_n \;=\; 2^{n-1}\;\alpha_\ell^2\; \det_{i,j\neq \ell}\left(\rho_{ij} - \rho_{i\ell}\frac{\alpha_j}{\alpha_\ell} - \rho_{\ell j}\frac{\alpha_i}{\alpha_\ell} + \rho_{\ell\ell}\frac{\alpha_i\alpha_j}{\alpha_\ell^2}\right) , \qquad \ell = 1,2,\ldots,n. \tag{34}
$$

From equation (34), and the fact that the matrix ρ is independent of the α_i, it is clear that $\hat{\Delta}_n$ is homogeneous of degree 2 in the α_i, and that no α_i appears with a negative power in $\hat{\Delta}_n$. So we may write

$$
\hat{\Delta}_n \;=\; \sum_{i,j=1}^{n} \eta_{ij}\alpha_i\alpha_j, \tag{35}
$$

where η_{ij} is independent of the α_i.

We can relate the two matrices η and ρ. Consider first the diagonal element $\eta_{\ell\ell}$. The α_ℓ^2 term in $\hat{\Delta}_n$ comes from taking only the ρ_{ij} terms in (34), and is given by

$$
\eta_{\ell\ell}\,\alpha_\ell^2 \;=\; 2^{n-1} \det_{i,j\neq\ell}(\rho_{ij})\;\alpha_\ell^2 \;=\; 2^{n-1}\det\rho\;(\rho^{-1})_{\ell\ell}\;\alpha_\ell^2 \tag{36}
$$

(where the second determinant is over all indices). Similarly, we pick off the $\alpha_\ell\alpha_m$ terms ($m \neq \ell$) in (34), by using a $\rho_{i\ell}$ or $\rho_{\ell j}$ term in place of ρ_{im} or ρ_{mj}, to get $\eta_{\ell m} \;=\; 2^{n-1}\det\rho\;(\rho^{-1})_{\ell m}$. Thus η is proportional to the inverse of ρ,

$$
\rho \;=\; N_n\;\eta^{-1}, \qquad\qquad \eta \;=\; N_n\;\rho^{-1}, \tag{37}
$$

where the proportionality constant is $N_n \;\equiv\; 2^{n-1}\det\rho$.

Now we rewrite the coefficients c_0, c_i and c_{ij} in terms of the α_i variables. First of all, using equations (24) and (37) the matrix S^{-1} is given by

$$
S^{-1}_{ij} \;=\; \alpha_i\,\rho^{-1}_{ij}\,\alpha_j \;=\; \frac{\alpha_i\,\eta_{ij}\,\alpha_j}{N_n}\;. \tag{38}
$$

Summing this equation over i and/or j, and using equation (35) for $\hat{\Delta}_n$, plus the definition

$$
\gamma_i \;\equiv\; \sum_{j=1}^{n}\eta_{ij}\alpha_j \;=\; \frac{1}{2}\frac{\partial\hat{\Delta}_n}{\partial\alpha_i}\bigg|_{\mathrm{non-}\alpha_i\ \text{variables fixed}}\;, \tag{39}
$$

we find that

$$c_0 \;=\; \frac{\hat{\Delta}_n}{N_n}\;, \qquad c_i \;=\; \frac{\alpha_i\gamma_i}{N_n}\;, \qquad c_{ij} \;=\; \frac{\alpha_i\alpha_j}{N_n}\left(\eta_{ij} - \frac{\gamma_i\gamma_j}{\hat{\Delta}_n}\right)\;. \tag{40}$$

In terms of the reduced integrals defined in equation (29) — which have simple differentiation properties — equations (18), (20), and (16) also take on a simpler form. It is convenient to make the change of kinematic variables for the integrals $I_{n-1}^{(i)}[1]$, such that the variables α_i are identical to those for the parent integral $I_n[1]$. Then we have

$$\hat{I}_n \;=\; \frac{1}{2N_n}\left[\sum_{i=1}^{n}\gamma_i\,\hat{I}_{n-1}^{(i)} \;+\; (n-5+2\epsilon)\,\hat{\Delta}_n\,\hat{I}_n^{D=6-2\epsilon}\right]\,, \tag{41}$$

$$\frac{1}{n-4+2\epsilon}\,\frac{\partial\hat{I}_n}{\partial\alpha_i} \;=\; \hat{I}_n[a_i] \;=\; \frac{1}{2N_n}\sum_{j=1}^{n}\Bigl(\eta_{ij} - \frac{\gamma_i\gamma_j}{\hat{\Delta}_n}\Bigr)\hat{I}_{n-1}^{(j)} \;+\; \frac{\gamma_i}{\hat{\Delta}_n}\hat{I}_n\;. \tag{42}$$

$$\frac{1}{n-4+2\epsilon}\,\frac{\partial\hat{I}_n}{\partial\alpha_i} \;=\; \hat{I}_n[a_i] \;=\; \frac{1}{2N_n}\left[\sum_{j=1}^{n}\eta_{ij}\;\hat{I}_{n-1}^{(j)} \;+\; (n-5+2\epsilon)\,\gamma_i\;\hat{I}_n^{D=6-2\epsilon}\right]\;. \tag{43}$$

The simple equations (41), (42), and (43) are in some sense the main results of this letter. All the equations derived so far are valid in an arbitrary spacetime dimension $D = 4 - 2\epsilon$; that is, we have not yet assumed that ϵ is small, nor that the external momenta lie in $D = 4$. However, the equations have the most utility in the context of dimensional regularization, i.e. for $D = 4 - 2\epsilon$ with ϵ tending to zero. We now discuss how to use the equations to recursively generate one-loop n-point integrals, up to $\mathcal{O}(\epsilon)$ corrections.

At first sight equation (41) does not appear very useful, due to the presence of the integral $I_n^{D=6-2\epsilon}[1]$ on the right-hand-side. However, for $n = 5$ the coefficient of this term is of order ϵ; and the integral $I_5^{D=6-2\epsilon}[1]$ is finite as $\epsilon \to 0$, because the $D = 6$ scalar pentagon integral possesses neither ultraviolet divergences nor infrared divergences (soft or collinear). So to order ϵ the $D = 4 - 2\epsilon$ scalar pentagon is given by a sum of five scalar boxes,

$$I_5[1] \;=\; \frac{1}{2}\sum_{i=1}^{5}\left(\sum_{j=1}^{5}S_{ij}^{-1}\right)\;I_4^{(i)}[1] \;+\; \mathcal{O}(\epsilon) \;=\; \frac{1}{2N_5}\sum_{i=1}^{5}\alpha_i\gamma_i\,I_4^{(i)}[1] \;+\; \mathcal{O}(\epsilon). \tag{44}$$

It is easy to check that the coefficients of the box integrals in this equation are identical to those in the corresponding $D = 4$ relation in ref. [3]; both sets of coefficients are expressed in terms of matrix elements of $S_{ij} = \frac{1}{2}(M_i^2 + M_j^2 - p_{ij}^2)$. In ref. [5] the box coefficients in the $D = 4$ relation were expressed using Levi-Civita symbols; in ref. [2] we show that the coefficients are nevertheless the same as those in equation (44).

For $n \geq 6$, the coefficient of $I_n^{D=6-2\epsilon}[1]$ in equation (18) is not of order ϵ. On the other hand, if all external momenta are chosen to lie in $D = 4$ dimensions, then we can set $c_0 = 0$. This is because the Gram determinant Δ_n vanishes in $D=4$, due to the linear dependence of the $(n-1)$ vectors $k_1, k_2, \ldots, k_{n-1}$ [3,12,13]. In next-to-leading-order calculations, it is always possible to put this restriction on the external kinematics. We then get

$$I_n[1] \;=\; \frac{1}{2N_n}\sum_{i=1}^{n}\alpha_i\gamma_i\, I_{n-1}^{(i)}[1]\;, \qquad\qquad n\geq 6, \quad D=4 \text{ external kinematics.} \tag{45}$$

For $n=6$ when all integrals appearing in (45) are finite, this equation is identical to the non dimensionally-regulated result of ref. [3], and it can be shown to be equivalent to the corresponding result of ref. [5], expressed in terms of Levi-Civita symbols; the latter derivation extends straightforwardly to $(4-2\epsilon)$-dimensional loop momenta.

For $n>6$, equation (45) differs somewhat from the results of references [3,5] in that it contains n, rather than six, $(n-1)$-point integrals. In the reductions (44) and (45), the cyclic symmetry of $I_n[1]$ is kept manifest. A more important difference arises if one wishes to extract tensor integrals via the differentiation formula (30). Namely, one should not restrict to four-dimensional external kinematics until after carrying out the differentiation, and so the cyclicly symmetric representation (41), with unrestricted kinematics, should be used as the starting point.

As mentioned above, for $n>6$ the representations (19) and (17) of the coefficients c_0 and c_i in terms of the original kinematic variables S_{ij} are problematic when the external momenta are restricted to $D=4$. This is because the rank of the $n\times n$ matrix S_{ij} is $n-6$ in $D=4$ [3], so for $n>6$ the inverse S_{ij}^{-1} does not exist. On the other hand, the α_i-representations of c_0 and c_i in equation (40) are non-singular and well-defined for $D=4$ kinematics. In summary, the combination of equations (44) and (45) recursively determines the general one-loop n-point scalar integral in $D=4-2\epsilon$ dimensions as a linear combination of box integrals.

Equation (18) also has significance for $n\leq 4$, even though the term containing $I_n^{D=6-2\epsilon}[1]$ in the equation may no longer be neglected. For example, for $n=4$ the decomposition (18) of $I_4^{D=4-2\epsilon}[1]$ has the virtue of putting all the $\epsilon\to 0$ divergences into the triangle integrals $I_3^{(i)}[1]$, since the $D=6$ scalar box is infrared and ultraviolet finite. This is important for practical calculations because infrared-divergent triangle integrals are generally simple to evaluate analytically; the infrared-finite triangles and the $D=6$ scalar box can be evaluated numerically if necessary.

Equation (42) gives a set of partial differential equations for n-point scalar integrals. For box integrals ($n=4$), one may also use equation (43), plus the finiteness of the $D=6$ scalar box, to simplify the differential equations through $\mathcal{O}(\epsilon)$:

$$\frac{\partial \hat{I}_4}{\partial\alpha_i} \;=\; \frac{\epsilon}{N_4}\left[\sum_{j=1}^{4}\eta_{ij}\;\hat{I}_3^{(j)} \;+\; (-1+2\epsilon)\,\gamma_i\;\hat{I}_4^{D=6-2\epsilon}\right] \;=\; \frac{\epsilon}{N_4}\sum_{j=1}^{n}\eta_{ij}\;\hat{I}_3^{(j)} \;+\; \mathcal{O}(\epsilon)\;. \tag{46}$$

In ref. [2] we solve these differential equations for box integrals with all internal lines massless, but with nonzero masses for 0, 1, 2, or 3 external lines. (Some of these integrals have been computed previously by other techniques.) Together with the infrared-finite box integral with all four external lines massive [1], these are the complete set of box integrals needed to recursively determine the n-point integrals for next-to-leading-order calculations in QCD with massless quarks.

For example, the box integral with one external mass is needed to obtain the all-massless pentagon via equation (44). The solution to the differential equations for the reduced box integral is [2]

$$\hat{I}_4^{(i)} \;=\; 2r_\Gamma \left[\frac{(\alpha_{i+2}\alpha_{i-2})^\epsilon}{\epsilon^2} + \mathrm{Li}_2\left(1 - \frac{\alpha_{i+1}}{\alpha_{i+2}}\right) + \mathrm{Li}_2\left(1 - \frac{\alpha_{i-1}}{\alpha_{i-2}}\right) - \frac{\pi^2}{6} \right] \;+\; \mathcal{O}(\epsilon), \tag{47}$$

where $r_\Gamma = \Gamma(1+\epsilon)\Gamma^2(1-\epsilon)/\Gamma(1-2\epsilon)$, and we have written the integral in terms of a set of α_i kinematic variables that are appropriate for the all-massless pentagon; the α_i are defined by

$$s_{i,i+1} \;=\; -\frac{1}{\alpha_i \alpha_{i+2}}\,, \qquad i = 1, 2, \ldots, 5 \pmod 5, \tag{48}$$

where $s_{i,i+1}$ are the five independent momentum invariants for the pentagon. The coefficients γ_i are then

$$\gamma_i \;=\; \alpha_{i-2} - \alpha_{i-1} + \alpha_i - \alpha_{i+1} + \alpha_{i+2}, \tag{49}$$

and the normalization constant is $N_5 = 1$. Plugging equation (47) into the formula (44) for the massless scalar pentagon, and using the dilogarithm identity $\mathrm{Li}_2(1-x) + \mathrm{Li}_2(1-x^{-1}) = -\frac{1}{2}\ln^2 x$ to simplify the expression, we find

$$\hat{I}_5^{\mathrm{massless}} \;=\; r_\Gamma \sum_{i=1}^{5} \alpha_i^{1+2\epsilon} \left[\frac{1}{\epsilon^2} + 2\ \mathrm{Li}_2\left(1 - \frac{\alpha_{i+1}}{\alpha_i}\right) + 2\ \mathrm{Li}_2\left(1 - \frac{\alpha_{i-1}}{\alpha_i}\right) - \frac{\pi^2}{6} \right] \;+\; \mathcal{O}(\epsilon). \tag{50}$$

This solution can also be obtained by solving the differential equations (42) or (43). In terms of momentum invariants, the unreduced massless scalar pentagon integral is

$$\begin{aligned} I_5^{\mathrm{massless}}[1] \;=\; & \frac{r_\Gamma\ (-s_{12})^\epsilon (-s_{51})^\epsilon}{(-s_{23})^{1+\epsilon}(-s_{34})^{1+\epsilon}(-s_{45})^{1+\epsilon}} \left[\frac{1}{\epsilon^2} + 2\,\mathrm{Li}_2\Big(1 - \frac{s_{23}}{s_{51}}\Big) + 2\,\mathrm{Li}_2\Big(1 - \frac{s_{45}}{s_{12}}\Big) - \frac{\pi^2}{6} \right] \\ & +\ \text{cyclic permutations}\ +\ \mathcal{O}(\epsilon). \end{aligned} \tag{51}$$

$I_5^{\mathrm{massless}}[1]$ is manifestly real in the region where all $s_{ij} < 0$; its value in physical regions can be obtained by the usual prescription $s_{ij} \to s_{ij} + i\varepsilon$.

There are a few subtleties to obtaining tensor integrals via the derivative formula (30) when $n \geq 4$. Some of the subtleties are associated with the $1/\epsilon$ pole in $\Gamma(n-3-m+2\epsilon)$ for $m \geq n-3$, where m is the degree of the Feynman parameter monomial being integrated. It might appear that the calculation of such integrals to $\mathcal{O}(1)$ would require knowledge of the scalar integral to $\mathcal{O}(\epsilon)$; however, this is not the case. These issues are treated in more detail in ref. [2]; here we simply note that equation (20) for $n = 4$ allows the determination of $I_4[a_i]$ to $\mathcal{O}(1)$, given $I_3[1]$ and $I_4[1]$ to $\mathcal{O}(1)$. The integrals of monomials with degree greater than 1 can then be obtained to $\mathcal{O}(1)$ by differentiating the integrals $I_4[a_i]$. Similarly, for $n = 5$ it is possible to derive an equation that yields $I_5[a_i a_j]$ to $\mathcal{O}(1)$, given $I_3[1]$ and $I_4[1]$ (or equivalently $I_4^{D=6}[1]$) to $\mathcal{O}(1)$. In this case, the $D = 6$ scalar pentagon $I_5^{D=6}[1]$ appears on the right-hand-side of the equation as well, but it is possible to show that for Feynman parameter polynomials that arise from tensor integrals in the loop-momentum, the coefficient of $I_5^{D=6}$ always vanishes. The integrals of monomials with degree greater than 2 can be obtained by differentiating $I_5[a_i a_j]$. One also uses equation (42) for $n = 5$ and $\epsilon \to \epsilon - 1$ to eliminate the derivatives $\partial \hat{I}_5^{D=6-2\epsilon}/\partial \alpha_i$ in favor of box integrals plus the scalar integral $\hat{I}_5^{D=6-2\epsilon}$. After doing this, the coefficient of the $D = 6$ scalar pentagon integral will always vanish, for Feynman parameter polynomials arising from tensor integrals in the loop-momentum.

For tensor integrals with $n \geq 6$ there is another subtlety, associated with the appearance of $\hat{\Delta}_n$ in the denominator of the coefficient c_{ij} in equation (40), since $\hat{\Delta}_n = 0$ for $D = 4$ kinematics and $n \geq 6$. On the other hand, the scalar integrals are manifestly free of singularities as $\hat{\Delta}_n \to 0$ (so long as $s_{i,i+1} \not\to 0$), and by considering the tensor integrals in terms of loop-momenta, one can see that they also can have no singularity as $\hat{\Delta}_n \to 0$. Therefore the $1/\hat{\Delta}_n$ factors that appear in some representations of the tensor integrals (such as equation (42)), should cancel out when all quantities are evaluated explicitly. We have checked that this is true for insertions of up to two Feynman parameters.

In conclusion, we have derived simple equations relating $I_n[1]$, the one-loop n-point scalar integral in $D = 4 - 2\epsilon$, to $I_{n-1}^{(i)}[1]$ and the $(6 - 2\epsilon)$-dimensional integral $I_n^{D=6-2\epsilon}[1]$. In the context of dimensional regulation these equations may be used to recursively determine the scalar integrals for $n > 4$ as a linear combination of box integrals, up to $\mathcal{O}(\epsilon)$ corrections. We also presented an approach to computing Feynman-parametrized tensor integrals via the differentiation of the scalar integral with respect to suitable combinations of the kinematic variables. This approach also leads to simple differential equations for scalar integrals, particularly box integrals. In reference [2] these general results are applied to the specific computation of box integrals with massless internal lines, but an arbitrary number of external masses, and to the computation of pentagon integrals with all lines massless and with one external mass. The latter integrals are of use in the calculation of one-loop contributions to amplitudes such as $gg \to ggg$ and $Z \to q\bar{q}gg$.

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References

[1] A. Denner, U. Nierste, and R. Scharf, Nucl. Phys. B367:637 (1991).

[2] Z. Bern, L. Dixon and D. A. Kosower, preprint SLAC–PUB–5947, SPhT/92–048, UCLA–92–43.

[3] D. B. Melrose, Il Nuovo Cimento 40A:181 (1965).

[4] G. 't Hooft and M. Veltman, Nucl. Phys. B153:365 (1979).

[5] W. van Neerven and J. A. M. Vermaseren, Phys. Lett. 137B:241 (1984).

[6] G. J. van Oldenborgh and J. A. M. Vermaseren, Z. Phys. C46:425 (1990);
G. J. van Oldenborgh, PhD thesis, University of Amsterdam (1990);
A. Aeppli, PhD thesis, University of Zürich (1992).

[7] R. K. Ellis, W. T. Giele, and E. Yehudai, to appear.

[8] L. M. Brown and R. P. Feynman, Phys. Rev. 85:231 (1952).

[9] G. Passarino and M. Veltman, Nucl. Phys. B160:151 (1979).

[10] R. G. Stuart, Comp. Phys. Comm. 48:367 (1988);
R. G. Stuart and A. Gongora, Comp. Phys. Comm. 56:337 (1990).

[11] Z. Bern and D. A. Kosower, Phys. Rev. Lett. 66:1669 (1991);
Z. Bern and D. A. Kosower, Nucl. Phys. B379:451, 1992;
Z. Bern and D. A. Kosower, in *Proceedings of the PASCOS-91 Symposium*, eds. P. Nath and S. Reucroft;
Z. Bern, D. C. Dunbar, Nucl. Phys. B379:562, 1992.

[12] E. Byckling and K. Kajantie, *Particle Kinematics* (Wiley) (1973).

[13] V. E. Asribekov, Sov. Phys. – JETP 15:394 (1962);
N. Byers and C. N. Yang, Rev. Mod. Phys. 36:595 (1964).