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Two-loop corrections to the Isgur-Wise function in QCD sum rules*

Matthias Neubert
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94309

We complete the QCD sum rule analysis of the Isgur Wise form factor $\xi(v\cdot v')$ at next-to-leading order in renormalization-group improved perturbation theory. To this end, the exact result for the two-loop corrections to the perturbative contribution is derived using the heavy quark effective theory. Several techniques for the evaluation of two-loop integrals involving two different types of heavy quark propagators are discussed in detail, among them the methods of integration by parts and differential equations. The order-α_s corrections to the Isgur-Wise function turn out to be small and well under control. At large recoil, they tend to decrease the form factor by $5-10\%$.

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I. INTRODUCTION

In the limit of infinite heavy quark masses, the weak decay form factors describing semileptonic transitions between any two ground-state (pseudoscalar or vector) heavy mesons, $M(v) \to M'(v')\,\ell\,\nu$, are described by a universal form factor $\xi(y)$. This so-called Isgur-Wise function depends on the velocity transfer $y = v\cdot v'$ and is normalized at zero recoil, $\xi(1) = 1$, where the initial and final meson have the same velocity [1]. These remarkable results follow from an implicit spin-flavor symmetry, which QCD reveals for heavy quarks although it is not explicit from its Lagrangian [2]. An expansion about this symmetry limit is afforded by the construction of the heavy quark effective theory (HQET) [3–5]. In the infinite mass limit, its effective Lagrangian is explicitly invariant under spin-flavor symmetry transformations. HQET thus provides a convenient framework in which to analyze the properties of hadrons containing a heavy quark. In particular, it allows a systematic expansion of weak decay form factors in powers of $1/m_Q$ [5–7].

To leading order in this expansion one recovers the Isgur-Wise limit, in which a large set of otherwise unrelated form factors reduces to the Isgur-Wise function. This function describes the dynamical properties of the cloud of light quarks and gluons surrounding the static heavy quarks. Being a hadronic form factor, it can only be investigated using nonperturbative methods. One such method is provided by QCD sum rules [8], which were originally developed for light quark systems and have yielded many nice results which are competitive with lattice computations. Recently, several authors have used QCD sum rules to calculate hadronic matrix elements in HQET [9–18]. The more refined of these analyses included radiative corrections to both the perturbative and nonperturbative contributions. For the Isgur-Wise function, however, the radiative corrections to the perturbative part of the sum rule were only incorporated in leading logarithmic approximation [12]; the complete two-loop corrections to the triangle quark loop were never calculated for the case of heavy quarks. On the other hand, from the well-studied case of pseudoscalar decay constants it is known that next-to-leading logarithmic corrections can be quite substantial and should in principle be taken into account [11–13].

With the advance of HQET, considerable progress has been made in the calculation of radiative corrections. In Ref. [17], for the first time an exact two-loop result was obtained for a heavy meson form factor, in this case for one of the universal functions that appear at order $1/m_Q$ in the heavy quark expansion. The authors of Ref. [18] developed a general method to compute the first two terms in an expansion of a two-loop diagram in HQET as a power series in $(y-1)$, and applied their technique to obtain an expansion of the two-loop corrections to the perturbative part of the Isgur-Wise function close to zero recoil.

In this paper we derive the exact result for the two-loop corrections to $\xi(y)$. To this end, we develop several techniques to evaluate two-loop integrals in HQET involving two heavy quarks with different velocities (v and v') and different residual momenta (k and k'), among them the method of integration by parts [19–21] and the use of differential equations [22]. We also introduce an integral representation for a general two-loop diagram which is particularly convenient for QCD sum rule calculations. These techniques are rather general and can be readily applied to other cases. In Sec. II we briefly review the sum rule analysis of the Isgur Wise function. The calculation of the two-loop perturbative corrections to $\xi(y)$ are described in Sec. III. We discuss in detail the contribution of each individual diagram, so that the interested reader can follow the analysis step by step. After renormalization, we compare our exact result with the expansion around zero recoil given in Ref. [18] and find agreement. Sec. IV deals with the renormalization-group improvement and the numerical analysis of the sum rule. We find that the effects of radiative corrections to the Isgur-Wise functions are moderate and well under control. Sec. V contains the conclusions.

II. SUM RULE FOR THE ISGUR-WISE FUNCTION

The derivation of the QCD sum rule for the Isgur-Wise function has been dealt with at length in Refs. [10, 12, 14], to which we refer the interested reader for details. Here we restrict ourselves to a brief review for the purpose of introducing the necessary notations and recalling the main ideas of the method. One studies the analytic properties of the three-current correlator

$$\begin{aligned}
&\int \mathrm{d}x\,\mathrm{d}z\, e^{i(k'\cdot x-k\cdot z)}\,\langle 0\,|\,T\Big\{[\,\bar q\,\overline{\Gamma}_{M'}h'\,]_x,[\,\bar h'\,\Gamma\,h\,]_0,[\,\bar h\,\Gamma_M q\,]_z\Big\}|\,0\,\rangle \\
&\equiv \Xi(\omega,\omega',v\cdot v')\,\mathrm{Tr}\{\overline{\mathcal{P}}'\Gamma\,\mathcal{P}\}\,, \qquad (1)
\end{aligned}$$

where h and h' describe heavy quarks with velocities v and v' in the effective theory. These quarks have "residual" momenta k and k', which are related to the total external momenta by $P=m_Q v+k$ and $P'=m_{Q'}v'+k'$, where m_Q and $m_{Q'}$ are the heavy quark masses. Depending on the choice $\Gamma_M=-\gamma_5$ or $\Gamma_M=\gamma_\mu-v_\mu$, the heavy-light currents interpolate pseudoscalar or vector mesons, respectively. The Dirac structure Γ of the heavy-heavy current is arbitrary. Usually, however, this is a flavor-changing weak current, in which case $\Gamma=\gamma_\mu(1-\gamma_5)$. The Dirac structure of the correlator is entirely contained in the trace over "spin wave functions"

$$\mathcal{P}=\frac{1+\not v}{2}\,\Gamma_M\,,\quad \overline{\mathcal{P}}'=\overline{\Gamma}_{M'}\,\frac{1+\not v'}{2}\,, \qquad (2)$$

which act as projection operators:

$$\begin{aligned}
\not v\,\mathcal{P}=\mathcal{P}&=-\mathcal{P}\not v\,,\\
-\not v'\,\overline{\mathcal{P}}'=\overline{\mathcal{P}}'&=\overline{\mathcal{P}}'\not v'\,. \qquad (3)
\end{aligned}$$

The coefficient function $\Xi(\omega,\omega',v\cdot v')$ in (1) is analytic in the "off-shell energies" $\omega=2v\cdot k$ and $\omega'=2v'\cdot k'$, with discontinuities for positive values of these variables. In particular, it receives a double-pole contribution from the ground-state mesons M

and M' associated with the heavy-light currents. This pole is located at $\omega = \omega' = 2\bar{\Lambda}$, where $\bar{\Lambda} = m_M - m_Q = m_{M'} - m_{Q'}$. The residue is proportional to the Isgur-Wise function. It follows that [12]

$$\Xi_{\rm pole}(\omega,\omega',y) = \frac{\xi(y,\mu)\,F^2(\mu)}{(\omega - 2\bar{\Lambda} + i\epsilon)(\omega' - 2\bar{\Lambda} + i\epsilon)}\,, \tag{4}$$

where $y = v\cdot v'$, and F corresponds to the scaled meson decay constant in the effective theory ($F \sim f_M\sqrt{m_M}$). Note that both F and the Isgur-Wise function are defined in terms of matrix elements in the effective theory and are therefore scale-dependent quantities.

In the deep Euclidean region, the correlator can be calculated perturbatively by using the Feynman rules of the heavy quark effective theory [5]. The idea of QCD sum rules is that, at the transition from the perturbative to the nonperturbative regime, confinement effects can be accounted for by including the leading power corrections in the operator product expansion of the three-point function. They are proportional to vacuum expectation values of local quark-gluon operators, the so-called condensates [8]. One then writes the theoretical expression for the correlator in terms of a double dispersion integral,

$$\Xi_{\rm th}(\omega,\omega',y) = \int\!{\rm d}\nu\,{\rm d}\nu'\,\frac{\rho_{\rm th}(\nu,\nu',y)}{(\nu-\omega-i\epsilon)(\nu'-\omega'-i\epsilon)} + \mbox{subtractions}, \tag{5}$$

and performs a Borel transformation in ω and ω' (see Appendix A for the definition of the Borel operator). This yields an exponential damping factor in the dispersion integral and eliminates possible subtraction terms. Because of the flavor symmetry it is natural to set the associated Borel parameters equal: $\tau = \tau' \equiv 2T$. Following Refs. [12, 15], one then introduces new variables $\omega_\pm = \frac{1}{2}(\nu \pm \nu')$, performs the integral over ω_-, and employs quark-hadron duality to equate the integral over ω_+ up to a threshold ω_0 to the Borel transform of the pole contribution in (4). This yields the Borel sum rule

$$\xi(y,\mu)\,F^2(\mu)\,e^{-2\bar{\Lambda}/T} = \int\limits_0^{\omega_0}\!{\rm d}\omega_+\,e^{-\omega_+/T}\,\widetilde{\rho}_{\rm th}(\omega_+,y) \equiv K(T,\omega_0,y)\,. \tag{6}$$

The effective spectral density $\widetilde{\rho}_{\rm th}$ arises after integration of the double spectral density over ω_-.

To lowest order in perturbation theory, the theoretical expression for the right-hand side of the sum rule is given by [9, 10, 12]

$$K(T,\omega_0,y) = \frac{3}{8\pi^2}\left(\frac{2}{y+1}\right)^2 \int\limits_0^{\omega_0}\!{\rm d}\omega_+\,\omega_+^2\,e^{-\omega_+/T} - \langle\bar{q}q\rangle + \left(\frac{y-1}{y+1}\right)\frac{\langle\alpha_s GG\rangle}{48\pi T} + \frac{(2y+1)}{3}\,\frac{m_0^2\,\langle\bar{q}q\rangle}{4T^2}\,. \tag{7}$$

We have included the leading nonperturbative contributions in the operator product expansion, which are proportional to the quark condensate (dimension $d = 3$), the gluon condensate ($d = 4$), and the mixed quark-gluon condensate ($d = 5$). In the numerical analysis in Sec. IV we will use the standard values (at $\mu = 1$ GeV)

$$\begin{aligned} \langle\bar q q\rangle &= -(0.23\,\mathrm{GeV})^3\,, \\ \langle\alpha_s GG\rangle &= 0.04\,\mathrm{GeV}^4\,, \\ \langle g_s\bar q\sigma_{\mu\nu}G^{\mu\nu}q\rangle &= m_0^2\,\langle\bar q q\rangle\,, \quad m_0^2 = 0.8\,\mathrm{GeV}^2\,. \end{aligned} \tag{8}$$

At zero recoil, a Ward identity relates the three-point function (1) to the correlator of two heavy-light currents, from which one derives the sum rule for the parameter F. It allows one to replace the product $F^2 e^{-2\bar\Lambda/T}$ in (6) by $K(T,\omega_0,1)$. Then the final form of the sum rule for the Isgur-Wise function explicitly reveals its normalization at zero recoil:

$$\xi(y,\mu) = \frac{K(T,\omega_0,y)}{K(T,\omega_0,1)}\,. \tag{9}$$

In the following section we will derive the complete expression for the perturbative corrections to the function $K(T,\omega_0,y)$ arising at order α_s. The one-loop corrections to the quark condensate were calculated in Ref. [12]. The mixed and gluon condensate are already of order g_s or g_s^2, and consequently one does not have to include radiative corrections to these terms at order α_s. What is missing are thus the order-α_s corrections to the perturbative contribution. There are restrictive constraints on the result of this two-loop calculation. The normalization of the Isgur-Wise function requires that, at zero recoil, one must recover the expression for the two-loop corrections to the perturbative part of the sum rule for F. This implies [11–13]

$$K_{\mathrm{pert}}(T,\omega_0,1) = \frac{3}{8\pi^2}\int\limits_0^{\omega_0}\mathrm{d}\omega_+\,\omega_+^2\,e^{-\omega_+/T}\left\{1 + \frac{\alpha_s}{\pi}\left[2\ln\frac{\mu}{\omega_+} + \frac{4\pi^2}{9} + \frac{17}{3}\right]\right\}. \tag{10}$$

Furthermore, the two-loop calculation must reproduce the known anomalous dimension of both $F(\mu)$ and $\xi(y,\mu)$ [2, 23, 5]. We will see at the end of Sec. III how these constraints are fulfilled.

III. TWO-LOOP CALCULATION

The two-loop corrections to the perturbative contribution to the three-current correlator (1) are shown in Fig. 1. We shall analyze these diagrams separately below. Throughout the calculation we use Feynman gauge. For practical purposes, it is useful to realize that the dependence of the perturbative spectral density $\widetilde\rho_{\mathrm{pert}}(\omega_+,y)$ on ω_+ is known on dimensional grounds:

$$\tilde{\rho}_{\rm pert}(\omega_+,y) = \omega_+^2\left[\rho_1(y) + \rho_2(y)\,\ln\frac{\mu}{\omega_+}\right],$$

$$\Rightarrow \quad \int_0^\infty \mathrm{d}\omega_+\, e^{-\omega_+/T}\,\tilde{\rho}_{\rm pert}(\omega_+,y) = 2T^3\left[\rho_1(y) + \rho_2(y)\left(\ln\frac{\mu}{T} + \gamma_E - \frac{3}{2}\right)\right]. \tag{11}$$

The coefficient functions $\rho_i(y)$ are independent of ω_+. It thus suffices to calculate directly the Borel transform of the correlator, corresponding to the second line in this equation. The spectral density can then be read off immediately. Below, we will always denote Borel-transformed quantities by a "hat".

A. Gluons attached to heavy quark lines

Consider the three diagrams D_1 to D_3 in Fig. 1. The evaluation of the first graph gives

$$\begin{aligned} D_1 = {} & 16N_c C_F\, g_s^2\, y\,\mathrm{Tr}\{\,\overline{\mathcal{P}}'\Gamma\,\mathcal{P}\,\gamma^\alpha\} \\ & \times \int \mathrm{d}\tilde{s}\,\mathrm{d}\tilde{t}\,\frac{s_\alpha}{(\omega+2v\cdot s)(\omega+2v\cdot t)(\omega'+2v'\cdot s)(\omega'+2v'\cdot t)\,s^2\,(s-t)^2}, \end{aligned} \tag{12}$$

where $C_F = (N_c^2-1)/2N_c$, and $\mathrm{d}\tilde{s} \equiv (2\pi)^{-D}\mathrm{d}^D s$. The two-loop integral is most conveniently performed by using a Fourier representation for the light quark and gluon propagators, and an exponential integral representation for the heavy quark propagators. A detailed description of this method, as well as its application to the above integral, can be found in Appendix A. After Borel transformation we find

$$\hat{D}_1 = -\frac{4yA}{(D-4)}\,[2(y+1)]^{D/2-2}\,G(0,0,\tfrac{D}{2}-1;y) \tag{13}$$

where we have abbreviated

$$A = -\frac{16N_c C_F\, g_s^2}{(4\pi)^D}\,\frac{(2T)^{2D-5}}{[2(y+1)]^{D-2}}\,\Gamma(\tfrac{D}{2})\,\Gamma(\tfrac{D}{2}-1)\,\mathrm{Tr}\{\,\overline{\mathcal{P}}'\Gamma\,\mathcal{P}\,\}\,. \tag{14}$$

Here and in the following we will often encounter integrals of the form

$$G(a,b,c;y) = \int_0^1 \mathrm{d}u\,\frac{(1-u)^a\,u^b}{(1+2yu+u^2)^c}, \tag{15}$$

which are related to generalized hypergeometric functions of the velocity transfer y.

The contribution of the second diagram is

$$\begin{aligned} D_2 = {} & 16N_c C_F\, g_s^2\,\mathrm{Tr}\{\,\overline{\mathcal{P}}'\Gamma\,\mathcal{P}\,\gamma^\alpha\} \\ & \times \int \mathrm{d}\tilde{s}\,\mathrm{d}\tilde{t}\,\frac{s_\alpha}{(\omega+2v\cdot s)^2(\omega+2v\cdot t)(\omega'+2v'\cdot s)\,s^2\,(s-t)^2}\,. \end{aligned} \tag{16}$$

Introducing a shifted loop momentum by $t' = t + s$, one can first perform the integral over t' and then carry out the integral over s by using the master equations (A1) and (A2) for one-loop integrals in HQET (see Appendix A). After Borel transformation, the result is

$$\hat{D}_2 = \frac{2A}{(D-4)(D-3)}\,[2(y+1)]^{D/2-2}\,. \tag{17}$$

Obviously the third diagram, D_3, gives the same contribution.

Next we set $D = 4 + 2\epsilon$ and expand in ϵ, using some of the integrals collected in Appendix B. We find

$$\sum_{i=1}^{3} \hat{D}_i = A\left\{\frac{1}{\epsilon}\Big[2 - y\,r(y)\Big] + 2\Big[1 - y\,r(y)\Big]\ln\left[2(y+1)\right] + 2y h(y) - 4 + \mathcal{O}(\epsilon)\right\}. \tag{18}$$

The functions $r(y)$ and $h(y)$ are given by

$$\begin{aligned} r(y) &= \frac{\ln(y_+)}{\sqrt{y^2-1}}\,, \\ h(y) &= \frac{1}{\sqrt{y^2-1}}\Big[L_2(1-y_-^2) - L_2(1-y_-)\Big] + \frac{3}{4}\sqrt{y^2-1}\,r^2(y)\,, \end{aligned} \tag{19}$$

where $y_\pm = y \pm \sqrt{y^2-1}$, and $L_2(x)$ is the dilogarithm (see Appendix B). They satisfy $r(1) = h(1) = 1$.

B. Gluons attached to the light quark line

Next consider the self-energy contribution for the light quark shown in diagram D_4 in Fig. 1. It gives

$$\begin{aligned} D_4 = &-4(D-2)N_c C_F\, g_s^2\, \mathrm{Tr}\{\,\overline{\mathcal{P}}'\Gamma\,\mathcal{P}\,\gamma^\alpha\gamma^\beta\gamma^\gamma\} \\ &\times \int \mathrm{d}\tilde{s}\,\mathrm{d}\tilde{t}\,\frac{s_\alpha\, t_\beta\, s_\gamma}{(\omega + 2v\cdot s)(\omega' + 2v'\cdot s)(s^2)^2\, t^2\,(s-t)^2}\,. \end{aligned} \tag{20}$$

After performing the integral over t, the remaining one-loop integral can be readily evaluated using the master equation (A1). After Borel transformation, one finds the simple result

$$\hat{D}_4 = -\frac{A}{D-4} = -\frac{A}{2\epsilon}\,. \tag{21}$$

C. Gluons attached to both heavy and light quark lines

Let us now turn to the most cumbersome part of the calculation, namely the loop correction of the heavy-light vertices. The contribution of the diagram D_5 in Fig. 1 is

$$D_5 = 8N_c C_F\, g_s^2\, \mathrm{Tr}\{\,\overline{\mathcal{P}}'\Gamma\,\mathcal{P}\,\gamma^\alpha \not{v}\gamma^\beta\}$$
$$\times \int \mathrm{d}\tilde{s}\,\mathrm{d}\tilde{t}\,\frac{t_\alpha\, s_\beta}{(\omega+2v\cdot s)(\omega+2v\cdot t)(\omega'+2v'\cdot s)\, s^2\, t^2\,(s-t)^2}\,. \tag{22}$$

It is convenient to split the calculation into three parts by use of the trace identity

$$\mathrm{Tr}\{\,\overline{\mathcal{P}}'\Gamma\,\mathcal{P}\,\gamma^\alpha \not{v}\gamma^\beta\} = \mathrm{Tr}\Big\{\,\overline{\mathcal{P}}'\Gamma\,\mathcal{P}\Big(g^{\alpha\beta}+\tfrac{1}{2}\,[\gamma^\alpha,\gamma^\beta]+2v^\alpha\gamma^\beta\Big)\Big\}\,. \tag{23}$$

Let us denote the corresponding contributions by $D_5^{(i)}$ and discuss them in turn.

1. Calculation of $D_5^{(1)}$

By rewriting its numerator, the integral appearing in $D_5^{(1)}$ can be further decomposed into three parts:

$$\int \mathrm{d}\tilde{s}\,\mathrm{d}\tilde{t}\,\frac{s^2+t^2-(s-t)^2}{(\omega+2v\cdot s)(\omega+2v\cdot t)(\omega'+2v'\cdot s)\, s^2\, t^2\,(s-t)^2} \equiv I_1+I_2+I_3\,. \tag{24}$$

The integral I_1 can be evaluated by introducing a new variable $s'=s+t$ and using the master equation (A1). We find

$$I_1 = \frac{i(-1)^{D-3}}{(4\pi)^{D/2}}\,\Gamma(4-D)\,\Gamma(\tfrac{D}{2}-1)\int\limits_0^\infty \mathrm{d}u\,\frac{1}{V^2}$$
$$\times \int \mathrm{d}\tilde{t}\,\frac{1}{(\omega+2v\cdot t)(\Omega/V+2\hat{V}\cdot t)^{4-D}\, t^2}\,, \tag{25}$$

where $V=(1+2yu+u^2)^{1/2}$, $\Omega=\omega+u\omega'$, and $\hat{V}_\alpha=(v+uv')_\alpha/V$ is a unit vector. The integral over t can again be performed using the master equation, resulting in a double parameter integral. The result simplifies upon Borel transformation. We find

$$\hat{I}_1 = \frac{2C}{(D-2)}\,\frac{G(0,0,\frac{D}{2}-1;y)}{[2(y+1)]^{D/2-1}}\;;\quad C\equiv\frac{(2T)^{2D-5}}{(4\pi)^D}\,\Gamma(\tfrac{D}{2})\,\Gamma(\tfrac{D}{2}-1)\,. \tag{26}$$

The calculation of the remaining integrals is straightforward. For I_2 we introduce $t'=t+s$ and make repeated use of the master equations. I_3 factorizes into the product of one-loop integrals. To evaluate its Borel transform it is convenient to combine denominators by a Feynman parameter. This gives

$$\hat{I}_2 = -\frac{2C}{(D-3)(D-2)}\,\frac{1}{[2(y+1)]^{D/2-1}}\,,$$
$$\hat{I}_3 = -\frac{2C}{(D-2)}\,G(2-D,0,\tfrac{D}{2}-1;y)\,. \qquad (27)$$

The calculation of the parameter integral in $\hat{I}_3$ is discussed in Appendix B.

2. Calculation of $D_5^{(2)}$

For the two-loop integral appearing in $D_5^{(2)}$ we use the integral representations discussed in Appendix A. We find that

$$\tfrac{1}{2}\,\mathrm{Tr}\{\,\overline{\mathcal{P}}'\Gamma\,\mathcal{P}\,[\gamma^\alpha,\gamma^\beta]\,\}\int \mathrm{d}\tilde{s}\,\mathrm{d}\tilde{t}\,\frac{t_\alpha\,s_\beta}{(\omega+2v\cdot s)(\omega+2v\cdot t)(\omega'+2v'\cdot s)\,s^2\,t^2\,(s-t)^2}$$
$$= (y-1)\,\mathrm{Tr}\{\,\overline{\mathcal{P}}'\Gamma\,\mathcal{P}\,\}\,I_4\,, \qquad (28)$$

where after Borel transformation

$$\hat{I}_4 = \frac{C}{D-3}\left\{ G(3-D,0,\tfrac{D}{2};y) - \frac{G(1,0,\frac{D}{2};y)}{[2(y+1)]^{D/2-1}} \right.$$
$$\left. - \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})\,\Gamma(\frac{D}{2}-1)}\int_1^\infty \mathrm{d}u_1 \int_1^\infty \mathrm{d}u_2\,\frac{(u_1u_2-1)^{D/2-2}}{[u_1+2(y+1)(u_2-1)]^{D-1}}\right\}. \qquad (29)$$

The double integral becomes trivial in the limit $D\to 4$. The evaluation of the first parameter integral is outlined in Appendix B.

Remarkable cancellations take place when one adds up the contributions from I_1 to I_4. We find the simple result

$$\hat{D}_5^{(1)} + \hat{D}_5^{(2)} = A\left\{ -\frac{1}{2\epsilon} + \mathcal{O}(\epsilon)\right\}. \qquad (30)$$

3. Calculation of $D_5^{(3)}$

This part of the amplitude involves the hardest integral:

$$I_\beta = \int \mathrm{d}\tilde{s}\,\mathrm{d}\tilde{t}\,\frac{2v\cdot t\,s_\beta}{(\omega+2v\cdot s)(\omega+2v\cdot t)(\omega'+2v'\cdot s)\,s^2\,t^2\,(s-t)^2}\,. \qquad (31)$$

It can be simplified by using the method of integration by parts, which allows one to reduce a given loop integral to a series of simpler integrals [19–21]. In this case, we can relate I_β to a sum of four integrals involving five (instead of six) types of propagators. To this end, we evaluate the identity

$$\int \mathrm{d}\tilde{s}\,\mathrm{d}\tilde{t}\,\frac{\partial}{\partial t_\alpha}\,\frac{2v\cdot t\,s_\beta\,(t-s)_\alpha}{(\omega+2v\cdot s)(\omega+2v\cdot t)(\omega'+2v'\cdot s)\,s^2\,t^2\,(s-t)^2}=0 \tag{32}$$

to obtain

$$\begin{aligned}
-(D-4)\,I_\beta = & \int \mathrm{d}\tilde{s}\,\mathrm{d}\tilde{t}\,\frac{s_\beta}{(\omega+2v\cdot s)(\omega'+2v'\cdot s)\,s^2\,t^2\,(s-t)^2} \\
&+\int \mathrm{d}\tilde{s}\,\mathrm{d}\tilde{t}\,\frac{2v\cdot t\,s_\beta}{(\omega+2v\cdot s)(\omega+2v\cdot t)(\omega'+2v'\cdot s)(t^2)^2\,(s-t)^2} \\
&-\int \mathrm{d}\tilde{s}\,\mathrm{d}\tilde{t}\,\frac{2v\cdot t\,s_\beta}{(\omega+2v\cdot s)(\omega+2v\cdot t)(\omega'+2v'\cdot s)(t^2)^2\,s^2} \\
&-\int \mathrm{d}\tilde{s}\,\mathrm{d}\tilde{t}\,\frac{\omega\,s_\beta}{(\omega+2v\cdot t)^2(\omega'+2v'\cdot s)\,s^2\,t^2\,(s-t)^2} \\
&= J_\beta^{(1)}+J_\beta^{(2)}+J_\beta^{(3)}+J_\beta^{(4)}\,.
\end{aligned} \tag{33}$$

The first three integrals can be calculated along the lines discussed above. After Borel transformation, we find

$$\begin{aligned}
\hat{J}_\beta^{(1)} &= \frac{4C}{(D-4)(D-2)}\,\frac{(v+v')_\beta}{[2(y+1)]^{D-2}}\,, \\
\hat{J}_\beta^{(2)} &= -\frac{C}{[2(y+1)]^{D/2-2}}\Bigg\{\bigg[(v+v')_\beta-\frac{2v_\beta}{D-2}\bigg]\,\frac{G(0,0,\frac{D}{2}-1;y)}{2(y+1)} \\
&\qquad + G(0,1,\tfrac{D}{2};y)\,v_\beta + G(0,0,\tfrac{D}{2};y)\,v'_\beta\Bigg\}\,, \\
\hat{J}_\beta^{(3)} &= 2C\,\Big[G(3-D,1,\tfrac{D}{2})\,v_\beta + G(3-D,0,\tfrac{D}{2})\,v'_\beta\Big]\,,
\end{aligned} \tag{34}$$

with C as defined in (26). Notice that because of the factor $(D-4)$ on the left-hand side of (33) these expressions have to be evaluated up to first order in ϵ.

The evaluation of $J_\beta^{(4)}$ is more involved. We have used the method of differential equations to calculate this integral [22]. This interesting technique will be discussed below. For the moment we just present the result:

$$\begin{aligned}
\hat{J}_\beta^{(4)} = & -\frac{2Cv'_\beta}{D-2}\left\{\frac{1}{[2(y+1)]^{D/2-1}}-(D-4)\,G(0,D-4,\tfrac{D}{2}-1;y)\right\} \\
&-2C\int_0^1 \mathrm{d}u\,\Big(1-u^{D-4}\Big)\,\frac{(uv+v')_\beta}{(1+2yu+u^2)^{D/2}}\,.
\end{aligned} \tag{35}$$

We have not written the last integral in terms of G-functions in order to show explicitly that it is of order $(D-4)$.

Next we set $D=4+2\epsilon$ and expand the above expressions, keeping terms of order ϵ. The integrals encountered are collected in Appendix B. They yield rather nontrivial functions of y. However, again remarkable cancellations appear if one sums the various contributions to the right-hand side of (33). Not only do the poles in $1/\epsilon$ contained in $J_\beta^{(1)}$ and $J_\beta^{(3)}$ cancel, but also most of the y-dependent terms. Our final result is, in fact, very simple. It reads

$$\begin{aligned} -(D-4)\,\hat{I}_\beta\,\mathrm{Tr}\{\,\overline{\mathcal{P}}'\Gamma\,\mathcal{P}\,\gamma^\beta\} = {} & \frac{C}{[2(y+1)]^{D-2}}\,\mathrm{Tr}\{\,\overline{\mathcal{P}}'\Gamma\,\mathcal{P}\,\} \\ & \times\Big\{4-2\epsilon\Big[2+\frac{2\pi^2}{3}+(y^2-1)\,r^2(y)\Big]+\mathcal{O}(\epsilon^2)\Big\}. \end{aligned} \tag{36}$$

After Borel transformation, the diagram D_6 gives the same contribution as D_5. Hence

$$\hat{D}_5+\hat{D}_6 = A\left\{\frac{1}{\epsilon}-2-\frac{2\pi^2}{3}-(y^2-1)\,r^2(y)+\mathcal{O}(\epsilon)\right\}. \tag{37}$$

D. Calculation of $J_\beta^{(4)}$ using a differential equation

In this paragraph we illustrate the application of differential equations to the analysis of multi-loop diagrams. Such techniques were introduced in Ref. [22] to evaluate integrals with massive propagators. They are readily adaptable to HQET. The idea is to derive a differential equation for a particular loop integral whose inhomogeneous term can be calculated in terms of simpler integrals. The original loop integral is then obtained from the solution of the differential equation.

We start by rewriting $J_\beta^{(4)} = x\frac{\partial}{\partial x}J_\beta(x)|_{x=1}$, where

$$J_\beta(x) = \int\!\mathrm{d}\tilde{s}\,\mathrm{d}\tilde{t}\,\frac{s_\beta}{(x\omega+2v\cdot t)(\omega'+2v'\cdot s)\,s^2\,t^2\,(s-t)^2}\,. \tag{38}$$

To derive a differential equation for $J_\beta(x)$, we use again the method of integration by parts. Starting from the identity

$$\int\!\mathrm{d}\tilde{s}\,\mathrm{d}\tilde{t}\,\frac{\partial}{\partial t_\alpha}\,\frac{s_\beta\,t_\alpha}{(x\omega+2v\cdot t)(\omega'+2v'\cdot s)\,s^2\,t^2\,(s-t)^2} = 0 \tag{39}$$

we find

$$\Big[(D-4)-x\frac{\partial}{\partial x}\Big]J_\beta(x) = f_\beta(x)\,, \tag{40}$$

where

$$\begin{aligned} f_\beta(x) = {} & \int\!\mathrm{d}\tilde{s}\,\mathrm{d}\tilde{t}\,\frac{s_\beta}{(x\omega+2v\cdot t)(\omega'+2v'\cdot s)\,s^2\,[(s-t)^2]^2} \\ & -\int\!\mathrm{d}\tilde{s}\,\mathrm{d}\tilde{t}\,\frac{s_\beta}{(x\omega+2v\cdot t)(\omega'+2v'\cdot s)\,t^2\,[(s-t)^2]^2} \end{aligned} \tag{41}$$

does indeed only contain simpler integrals. The general solution of the differential equation is

$$J_\beta(x) = x^{D-4}\Big\{ \int\limits_x^\infty \mathrm{d}z\, z^{3-D} f_\beta(z) + k_\beta \Big\}\,, \tag{42}$$

with k_β being independent of x. We are interested in the Borel transform of this equation. A straightforward calculation gives

$$z^{3-D}\hat{f}_\beta(z) = \frac{(2T)^{2D-5}}{(4\pi)^D}\,\Gamma^2(\tfrac{D}{2}-1)\bigg\{ \frac{v'_\beta}{(1+2yz+z^2)^{D/2-1}} \\ -\frac{D-2}{D-4}\left(1-z^{4-D}\right)\frac{z^{D-3}\,(v+zv')_\beta}{(1+2yz+z^2)^{D/2}}\bigg\}. \tag{43}$$

In order to determine the constant of integration k_β we consider the limit $x\to\infty$ in (38), in which

$$\begin{aligned}\lim_{x\to\infty} x J_\beta(x) &= \frac{1}{\omega}\int \mathrm{d}\tilde{s}\,\mathrm{d}\tilde{t}\,\frac{s_\beta}{(\omega'+2v'\cdot s)\,s^2\,t^2\,(s-t)^2} \\ &\overset{\text{B.T.}}{\to} \frac{(2T)^{2D-5}}{(4\pi)^D}\,\Gamma(\tfrac{D}{2}-1)\,\Gamma(\tfrac{D}{2}-2)\,v'_\beta\,.\end{aligned} \tag{44}$$

By evaluating (42) for $x\gg 1$, on the other hand, we find (for $D>3$)

$$\lim_{x\to\infty} x\hat{J}_\beta(x) = \frac{(2T)^{2D-5}}{(4\pi)^D}\,\Gamma(\tfrac{D}{2}-1)\,\Gamma(\tfrac{D}{2}-2)\,v'_\beta + \hat{k}_\beta \lim_{x\to\infty} x^{D-3}\,. \tag{45}$$

Hence $\hat{k}_\beta = 0$ follows. From the solution of the differential equation we now obtain

$$\hat{J}_\beta^{(4)} = x\frac{\partial}{\partial x}\hat{J}_\beta(x)\bigg|_{x=1} = -\hat{f}_\beta(1) + (D-4)\int\limits_1^\infty \mathrm{d}z\, z^{3-D}\hat{f}_\beta(z)\,. \tag{46}$$

Substituting here $z=1/u$ leads to (35).

E. Summary and renormalization

We are now in a position to sum up the various two-loop corrections that contribute at order α_s to the Borel-transformed correlator. We find

$$\hat{\Xi}_1 \equiv \sum_{i=1}^6 \hat{D}_i = A\bigg\{ \frac{3}{2}\Big(\frac{1}{\epsilon} - \frac{4\pi^2}{9} - \frac{8}{3}\Big) - \Big[y\,r(y)-1\Big]\Big(\frac{1}{\epsilon} + 2\ln\left[2(y+1)\right]\Big) \\ +2\Big[y\,h(y)-1\Big] - (y^2-1)\,r^2(y) + \mathcal{O}(\epsilon)\bigg\}. \tag{47}$$

Next we expand A from (14) around $D=4$, keeping terms of order ϵ, and relate $\hat{\Xi}_1$ to the lowest-order correlator

$$\hat{\Xi}_0 = \frac{3T^3}{4\pi^2}\left(\frac{2}{y+1}\right)^2 \mathrm{Tr}\{\,\overline{\mathcal{P}}'\Gamma\,\mathcal{P}\,\}\,. \tag{48}$$

This yields

$$\begin{aligned}\hat{\Xi}_1 = \frac{\alpha_s}{\pi}\,\hat{\Xi}_0\bigg\{&-\frac{1}{\hat\epsilon} + \left(2\ln\frac{\mu}{T} + 2\gamma_E - 3\right) + \frac{4\pi^2}{9} + \frac{17}{3}\\ &-\frac{\gamma(y)}{2}\bigg[-\frac{1}{\hat\epsilon} + \left(2\ln\frac{\mu}{T} + 2\gamma_E - 3\right)\bigg] + c_{\mathrm{pert}}(y) + \mathcal{O}(\epsilon)\bigg\}\,,\end{aligned} \tag{49}$$

where

$$\frac{1}{\hat\epsilon} = \frac{1}{\epsilon} + \gamma_E - \ln\frac{4\pi}{\mu^2}\,. \tag{50}$$

We have introduced the functions

$$\gamma(y) = \frac{4}{3}\Big[y\,r(y) - 1\Big]\,, \tag{51}$$

$$\begin{aligned}c_{\mathrm{pert}}(y) &= \frac{\gamma(y)}{2}\left[4\ln 2 - 3 + \ln\frac{y+1}{2}\right] - \frac{4}{3}\Big[y\,h(y) - 1\Big] + \ln\frac{y+1}{2} + \frac{2}{3}\,(y^2-1)\,r^2(y)\\ &= \left(\frac{16}{9}\ln 2 - \frac{49}{54}\right)(y-1) - \left(\frac{8}{15}\ln 2 - \frac{197}{600}\right)(y-1)^2 + \ldots\,,\end{aligned}$$

both of which vanish at $y=1$. The first two terms in the expansion of $c_{\mathrm{pert}}(y)$ around zero recoil were previously calculated in Ref. [18], and we confirm the result obtained there.

The $1/\hat\epsilon$ poles in (49) cancel upon renormalization of the heavy-light and heavy-heavy currents in (1). In the $\overline{\mathrm{MS}}$ subtraction scheme, the corresponding renormalization factors are [2, 23, 5]

$$\mathcal{Z}_{\mathrm{hl}} = 1 - \frac{\alpha_s}{2\pi\hat\epsilon}\,, \quad \mathcal{Z}_{\mathrm{hh}} = 1 + \frac{\alpha_s}{2\pi\hat\epsilon}\,\gamma(y)\,. \tag{52}$$

That means that our two-loop calculation reproduces correctly the known running of the hadronic form factors $F(\mu)$ and $\xi(y,\mu)$. By comparing (49) with (11) we can now write our final result for the renormalized correlator in form of a dispersion integral and introduce the continuum threshold ω_0 to obtain

$$\begin{aligned}K_{\mathrm{pert}}(T,\omega_0,y) &= \frac{3}{8\pi^2}\left(\frac{2}{y+1}\right)^2 \int\limits_0^{\omega_0} \mathrm{d}\omega_+\,\omega_+^2\,e^{-\omega_+/T}\\ &\times\left\{1 + \frac{\alpha_s}{\pi}\left[2\ln\frac{\mu}{\omega_+} + \frac{4\pi^2}{9} + \frac{17}{3} - \gamma(y)\,\ln\frac{\mu}{\omega_+} + c_{\mathrm{pert}}(y)\right]\right\}.\end{aligned} \tag{53}$$

This is the exact expression for the perturbative part of the correlator at order α_s. It is now seen that at zero recoil one indeed recovers (10).

IV. RENORMALIZATION-GROUP IMPROVEMENT AND NUMERICAL ANALYSIS

The theoretical expression for the correlator depends on the subtraction scale μ, indicating a scheme-dependence associated with the subtraction of the $1/\hat{\varepsilon}$ poles. This just reflects that the hadronic parameters $F(\mu)$ and $\xi(y,\mu)$, which are defined in terms of matrix elements of currents in the effective theory, are scheme-dependent quantities. At next-to-leading order in renormalization-group improved perturbation theory one can define renormalized, scheme-independent form factors by [24]

$$\begin{aligned} F_{\rm ren} &= \Big[\alpha_s(\mu)\Big]^{2/9}\bigg\{1 - \frac{\alpha_s(\mu)}{\pi}\,\Big[Z_{\rm hl} + \delta_{\rm hl}\Big]\bigg\}\,F(\mu)\,, \\ \xi_{\rm ren}(y) &= \Big[\alpha_s(\mu)\Big]^{-a_L(y)}\bigg\{1 - \frac{\alpha_s(\mu)}{\pi}\,\Big[Z_{\rm hh}(y) + \delta_{\rm hh}(y)\Big]\bigg\}\,\xi(y,\mu)\,, \end{aligned} \tag{54}$$

where $a_L(y) = \frac{2}{9}\,\gamma(y)$ [5], and we have used that the number of light quark flavors in the effective theory is $n_f = 3$. The next-to-leading logarithmic corrections consist of two parts. The coefficients Z are renormalization-group invariant quantities. For $n_f = 3$, they are given by [24–27]

$$\begin{aligned} Z_{\rm hl} &= -\frac{185}{324} - \frac{7\pi^2}{243}\,, \\ Z_{\rm hh}(y) &= \frac{\gamma(y)}{3}\bigg\{\frac{53}{54} - \frac{\pi^2}{12} - y\,r(y) + \frac{y}{\sqrt{y^2-1}}\,\Big[L_2(1-y_-^2) + \ln^2(y_-)\Big]\bigg\} \\ &\quad - \frac{8}{9}\int\limits_0^\theta \mathrm{d}\psi\,\Big[\psi\coth\psi - 1\Big]\bigg\{\psi\coth^2\theta + \frac{\sinh\theta\,\cosh\theta}{\sinh^2\theta - \sinh^2\psi}\,\ln\frac{\sinh\theta}{\sinh\psi}\bigg\} \\ &= \Big(\frac{752}{729} - \frac{8\pi^2}{81}\Big)(y-1) - \Big(\frac{368}{1215} - \frac{4\pi^2}{135}\Big)(y-1)^2 + \ldots\,, \end{aligned} \tag{55}$$

where $y_- = y - \sqrt{y^2-1}$, and the hyperbolic angle θ is defined by $y = \cosh\theta$. The coefficients δ, on the other hand, are scheme-dependent. They arise from matching of QCD onto the effective theory and combine with the scheme-dependent terms in (53) to give a renormalization-group invariant result. In the $\overline{\rm MS}$ subtraction scheme one has [24, 26]

$$\delta_{\rm hl} = \frac{2}{3}\,, \quad \delta_{\rm hh}(y) = 0\,. \tag{56}$$

After renormalization-group improvement, the sum rule for the renormalized Isgur-Wise function takes the form

$$\xi_{\rm ren}(y) = \Big[\alpha_s(T)\Big]^{-a_L(y)}\,\frac{\widehat{K}(T,\omega_0,y)}{\widehat{K}(T,\omega_0,1)}\,, \tag{57}$$

where we have summed the leading logarithms down to a characteristic scale given by the Borel parameter T. The renormalization-group invariant function $\widehat{K}$ is given by

$$
\begin{aligned}
\widehat{K}(T,\omega_0,y) = {} & \frac{3T^3}{8\pi^2}\left(\frac{2}{y+1}\right)^2 \int\limits_0^{\omega_0/T} \mathrm{d}x\, x^2\, e^{-x} \\
& \times\left\{1+\frac{\alpha_s(T)}{\pi}\left[\frac{4\pi^2}{9}+\frac{13}{3}-2Z_{\rm hl}-[2-\gamma(y)]\ln x+c_{\rm pert}(y)-Z_{\rm hh}(y)\right]\right\} \\
& -\langle\bar q q\rangle(T)\left\{1+\frac{\alpha_s(T)}{\pi}\left[\frac{2}{3}-2Z_{\rm hl}+\gamma(y)\left[\mathrm{Ei}\left(-\frac{\omega_0}{T}\right)-\gamma_E\right]+c_{\langle\bar q q\rangle}(y)-Z_{\rm hh}(y)\right]\right\} \\
& +\left(\frac{y-1}{y+1}\right)\frac{\langle\alpha_s GG\rangle}{48\pi T}+\frac{(2y+1)}{3}\,\frac{m_0^2\,\langle\bar q q\rangle}{4T^2}\,. \qquad (58)
\end{aligned}
$$

We have included the order-α_s corrections to the quark condensate as calculated in Ref. [12].[1] The function $c_{\langle\bar q q\rangle}(y)$ has a similar form as $c_{\rm pert}(y)$. It reads

$$
\begin{aligned}
c_{\langle\bar q q\rangle}(y) &= \frac{\gamma(y)}{2}\left[4\ln 2+\ln\frac{y+1}{2}\right]-\frac{4}{3}\Big[y\,h(y)-1\Big]-\frac{2}{3}\,(y-1)\,r(y) \\
&= \left(\frac{16}{9}\ln 2-\frac{56}{27}\right)(y-1)-\left(\frac{8}{15}\ln 2-\frac{112}{225}\right)(y-1)^2+\dots \qquad (59)
\end{aligned}
$$

A remark is in order concerning the appearance of γ_E and the exponential integral in (58). The effective spectral density for the quark condensate receives contributions proportional to $\delta(\omega_+)$ and $1/\omega_+ + \delta(\omega_+)\ln(\omega_+)$. The latter ones have to be regularized in the dispersion integral, leading to

$$
\lim_{\eta\to+0}\int\limits_0^\infty \mathrm{d}\omega_+\, e^{-\omega_+/T}\left[\frac{1}{\omega_++\eta}+\delta(\omega_+-\eta)\ln\frac{\omega_+}{\mu}\right]=\ln\frac{T}{\mu}-\gamma_E\,. \qquad (60)
$$

If one introduces a continuum threshold, one obtains an extra contribution $\mathrm{Ei}(-\omega_0/T)$ from the second term, where $\mathrm{Ei}(-x) = -\int_x^\infty \frac{\mathrm{d}t}{t}e^{-t}$ is the exponential integral. This contribution is very small and has been neglected in Ref. [12].

For the numerical analysis of the sum rule (57) we use the vacuum condensates as given in (8), as well as $\Lambda_{\overline{\rm MS}} = 0.25$ GeV (for $n_f = 3$) in the running coupling $\alpha_s(T)$. In Fig. 2(a) we show the range of predictions for the renormalized Isgur-Wise function obtained by varying the continuum threshold over the range $2.0 < \omega_0 < 2.6$ GeV, and the Borel parameter inside the "sum rule window" $0.8 < T < 1.2$ GeV, where the theoretical calculation is reliable. This window is determined by requiring that the

[1]The function $c_{\langle\bar q q\rangle}(y)$ was called $\frac{2}{3}c_{\overline{\rm MS}}(y)$ in Ref. [12]. Note that we have simplified the dilogarithms appearing in $h(y)$ as compared to this reference.

nonperturbative contributions to the sum rule be less than 30% of the perturbative ones ($T > 0.8$ GeV), and that the pole contribution account for at least 30% of the perturbative part of the correlator ($T < 1.2$ GeV). The lower limit on T also ensures that $\alpha_s(T)$ is small enough to allow for a perturbative expansion. The above range of values for ω_0 was obtained from the study of the correlator of two heavy-light currents. As in previous analyses, we observe excellent stability of the sum rule. The sensitivity of the Isgur-Wise function to different choices of the continuum model is investigated in detail in Refs. [12, 15]. We do not discuss this subject here, since we are mainly interested in the effects of radiative corrections. We just note that the theoretical uncertainty in the sum rule prediction is probably larger than indicated by the width of the band in Fig. 2(a).

To study the importance of radiative corrections we first have to relate $\xi_{\rm ren}(y)$ to a more "physical" form factor, which includes the logarithmic dependence on the heavy quark masses. Otherwise it is not possible to consider the limit $\alpha_s \to 0$. For simplicity, we work with a single scale $\bar m$ and define

$$\xi_{\rm phys}(\bar m, y) = \Big[\alpha_s(\bar m)\Big]^{a_L(y)} \bigg\{ 1 + \frac{\alpha_s(\bar m)}{\pi}\, Z_{\rm hh}(y) \bigg\}\, \xi_{\rm ren}(y) \,. \tag{61}$$

We use $\bar m \simeq 2.3$ GeV as a characteristic scale for $b \to c$ transitions [24]. Fig. 2(b) shows the next-to-leading order result for this form factor in comparison with the "bare" Isgur-Wise function computed by neglecting radiative corrections. We also show the leading logarithmic approximation to $\xi_{\rm phys}(\bar m, y)$, which is obtained by ignoring terms of order α_s in (58) and keeping only the first factor in (61). It is apparent from this figure that the radiative corrections to the Isgur-Wise function are well under control. The large y-independent corrections, which enhance the sum rule prediction for the decay constant F by 50% [11–13], cancel out in the ratio (57). The remaining recoil-dependent radiative corrections are small. At large recoil, they tend to decrease the form factor by $5 - 10\%$. Part of this effect comes from leading logarithms and is associated with the velocity-dependent anomalous dimension $a_L(y)$ of the heavy-heavy current in the effective theory.

V. CONCLUSIONS

We have presented the complete QCD sum rule analysis of the Isgur-Wise form factor $\xi(v \cdot v')$ at next-to-leading order in renormalization-group improved perturbation theory. To this end, we have derived the exact result for the two-loop corrections to the triangle quark loop. Such a calculation, which was never done before for a form factor of heavy mesons, becomes feasible by using the heavy quark effective theory. We have developed some general techniques for dealing with two-loop integrals with two different types of heavy quark propagators. Using the method of integration by parts, complicated integrals can be reduced to simpler ones in a recursive way. Integrals which cannot be reduced any further can be evaluated by using differential equations. We have applied this technique to the loop corrections to the heavy-light vertices. We have also presented an integral representation for two-loop integrals which is particularly convenient for QCD sum rule calculations. These methods can be applied to other sum rule calculations and will eventually lead to more accurate predictions for heavy meson form factors than were available before.

Our numerical analysis shows that, unlike in the case of meson decay constants, radiative corrections to the Isgur-Wise function are small and well under control. This is an important result which puts the sum rule analysis of $\xi(v \cdot v')$ on a firm footing. The smallness of the two-loop corrections in this particular case was not unexpected, however, since the normalization of the Isgur-Wise function at zero recoil prohibits any recoil-independent radiative effects. This does not imply that such corrections are always negligible. In fact, some of the universal functions appearing at order $1/m_Q$ in the heavy quark expansion receive their leading contributions at order α_s. Then the two-loop perturbative contribution is important and cannot be neglected for a reliable analysis [17].

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APPENDIX A: LOOP INTEGRALS IN HQET

1. One-loop integrals

We summarize some important equations for one-loop tensor integrals in HQET. Integrals involving massless propagators only can be found, *i.e.*, in Ref. [28]. Integrals involving two types of heavy quark propagators were considered in the second reference in [14]. There the master equation

$$
\begin{aligned}
I_{\mu_1\ldots\mu_n}(\alpha,\beta,\gamma) &= \int \mathrm{d}\tilde{t}\, \frac{t_{\mu_1}\ldots t_{\mu_n}}{(-t^2)^\alpha(\omega+2v\cdot t)^\beta(\omega'+2v'\cdot t)^\gamma} \\
&= \frac{i}{(4\pi)^{D/2}}\, I_n(\alpha,\beta,\gamma) \int\limits_0^\infty \mathrm{d}u\, \frac{u^{\gamma-1}}{[\Omega(u)]^{\beta+\gamma}} \left[-\frac{\Omega(u)}{V(u)}\right]^{D-2\alpha+n} K_{\mu_1\ldots\mu_n}(u)\,,
\end{aligned}
\tag{A1}
$$

was derived, where $\mathrm{d}\tilde{t}=(2\pi)^{-D}\mathrm{d}^D t$, and

$$
\begin{aligned}
I_n(\alpha,\beta,\gamma) &= \frac{\Gamma(2\alpha+\beta+\gamma-D-n)\,\Gamma(D/2-\alpha+n)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma)}\,, \\
\Omega(u) &= \omega+u\,\omega'\,, \\
V(u) &= (1+u^2+2u\,v\cdot v')^{1/2}\,.
\end{aligned}
$$

For $n=0,1,2$ the tensors $K_{\mu_1\ldots\mu_n}(u)$ are given by

$$
\begin{aligned}
K(u) &= 1\,, \\
K_\mu(u) &= -\hat{V}_\mu(u)\,, \\
K_{\mu\nu}(u) &= \hat{V}_\mu(u)\,\hat{V}_\nu(u) - \frac{g^{\mu\nu}}{D-2\alpha+2}\,,
\end{aligned}
$$

with $\hat{V}_\mu(u)=(v+uv')_\mu/V(u)$ being a unit vector. We note that the master equation is valid for arbitrary values of α, β, and γ.

In the case of one heavy quark the master equation reduces to [14, 25]

$$
\begin{aligned}
I_{\mu_1\ldots\mu_n}(\alpha,\beta) &= \int \mathrm{d}\tilde{t}\, t_{\mu_1}\ldots t_{\mu_n} \left(\frac{1}{-t^2}\right)^\alpha \left(\frac{\omega}{\omega+2v\cdot t}\right)^\beta \\
&= \frac{i}{(4\pi)^{D/2}}\, I_n(\alpha,\beta)\,(-\omega)^{D-2\alpha+n}\, K_{\mu_1\ldots\mu_n}\,,
\end{aligned}
\tag{A2}
$$

where

$$
I_n(\alpha,\beta) = \frac{\Gamma(2\alpha+\beta-D-n)\,\Gamma(D/2-\alpha+n)}{\Gamma(\alpha)\,\Gamma(\beta)}\,.
$$

$K_{\mu_1\ldots\mu_n}$ is obtained from above by replacing $\hat{V}_\mu(u)$ by v_μ.

The integral representation (A1) is particularly convenient for a Borel transformation in ω and ω'. Defining the Borel operator by

$$\frac{1}{\tau}\,\hat{B}^{(\omega)}_\tau = \lim_{\substack{n\to\infty\\ -\omega\to\infty}} \frac{\omega^n}{\Gamma(n)}\left(-\frac{\mathrm{d}}{\mathrm{d}\omega}\right)^n \; ; \quad \tau = \frac{-\omega}{n}\ \text{fixed}\,,$$

we note that

$$\hat{B}^{(\omega')}_{\tau'}\hat{B}^{(\omega)}_\tau\,[-\Omega(u)]^{-a} = \frac{\tau^{2-a}}{\Gamma(a)}\,\delta\Big(u-\frac{\tau}{\tau'}\Big)\,.$$

2. Two-loop integrals

Let us now turn to two-loop integrals in HQET. The case with one heavy quark has been discussed in detail in Ref. [25]. Here we consider integrals with two species of heavy quark propagators. They have the general form

$$I^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_m}(\alpha,\beta,\gamma,\delta;a,b,c) = \tag{A3}$$
$$\int\mathrm{d}\tilde{s}\,\mathrm{d}\tilde{t}\,\frac{s_{\mu_1}\ldots s_{\mu_m}\,t^{\nu_1}\ldots t^{\nu_n}}{(\omega+2v\cdot s)^\alpha(\omega+2v\cdot t)^\beta(\omega'+2v'\cdot s)^\gamma(\omega'+2v'\cdot t)^\delta(-s^2)^a(-t^2)^b[-(s-t)^2]^c}$$

We will derive a representation for this integral which is particularly convenient for further analysis. The first step consists in performing a Wick rotation of the loop momenta,

$$s\;\to\;(is^0,\vec{s}\,)\;,\quad t\;\to\;(it^0,\vec{t}\,)\,,$$

so that

$$\begin{aligned} -s^2\;&\to\;(s^0)^2+\vec{s}^{\,2}\equiv s_E^2\,,\\ v\cdot s\;&\to\;iv^0s^0-\vec{v}\cdot\vec{s}\equiv iv_E\cdot s_E\,, \end{aligned}$$

where $s_E=(s^0,\vec{s}\,)$ and $v_E=(v^0,i\vec{v}\,)$ are vectors in a Euclidean space. Note that this definition of a Euclidean velocity ensures that $v_E^2=1$ and $v_E\cdot v_E'=y$, where $y=v\cdot v'$ in Minkowski space. After the Wick rotation, we represent the massless propagators as Fourier integrals in a D-dimensional Euclidean space and use an exponential integral representation for the heavy quark propagators:

$$\begin{aligned} \frac{1}{(s_E^2)^a} &= \frac{\Gamma(\frac{D}{2}-a)}{\pi^{D/2}\,\Gamma(a)}\int\mathrm{d}^Dx\,\frac{e^{2is_E\cdot x}}{(x^2)^{D/2-a}}\,,\\ \frac{1}{(\omega+2iv_E\cdot s_E)^\alpha} &= \frac{(-1)^\alpha}{\Gamma(\alpha)}\int_0^\infty\mathrm{d}\lambda\,\lambda^\alpha\,e^{\lambda(\omega+2iv_E\cdot s_E)}\,. \qquad (\omega<0) \end{aligned}$$

The most general two-loop integral involves three D-dimensional integrations over x_i and four one-dimensional integrations over λ_i. The advantage of these representations is that the integrals over the loop momenta can immediately be performed and give rise to two D-dimensional δ-functions, which eliminate two of the integrations over x_i. Furthermore, note that two of the integrals over λ_i become trivial upon Borel transformation, since

$$\hat{B}_\tau^{(\omega)}\, e^{\lambda\omega} = \delta(\lambda - \tau^{-1})\,.$$

The tensor structure in the numerator in (A3) can be generated by taking derivatives with respect to x_i. Recall that for every timelike index there is a factor i encountered during the Wick rotation. For every spacelike index, on the other hand, one encounters a factor i when rotating back to Minkowski space. Together with a factor i^2 from the loop integrations there is thus a factor of i^{2+m+n} to be taken into account.

Let us illustrate this technique for some of the integrals encountered in Sec. III. We start with the integral in (12). It contains only two types of massless propagators, and consequently there are no integrations over x_i left after evaluation the δ-functions arising from the loop integrations. In the notation of (A3) we find

$$I_\alpha(1,1,1,1;1,0,1) = \frac{\Gamma(\frac{D}{2})\,\Gamma(\frac{D}{2}-1)}{(4\pi)^D} \int\limits_0^\infty \mathrm{d}\lambda_1\,\mathrm{d}\lambda_2\,\mathrm{d}\lambda_3\,\mathrm{d}\lambda_4\, e^{(\lambda_1+\lambda_2)\omega+(\lambda_3+\lambda_4)\omega'}$$
$$\times \frac{x_{1\alpha}}{(x_1^2)^{D/2}}\,\frac{1}{(x_2^2)^{D/2-1}}\,,$$

where $x_1 = (\lambda_1+\lambda_2)v + (\lambda_3+\lambda_4)v'$, and $x_2 = \lambda_2 v + \lambda_4 v'$. After Borel transformation we obtain

$$\hat{I}_\alpha(1,1,1,1;1,0,1) = \frac{C(v+v')_\alpha}{[2(y+1)]^{D/2}} \int\limits_0^1 \mathrm{d}z_1\,\mathrm{d}z_2\, \frac{1}{(z_1^2+z_2^2+2yz_1z_2)^{D/2-1}}$$

with C as defined in (26). Substituting $z_2 = uz_1$, an integration by parts in z_1 yields

$$\int\limits_0^1 \mathrm{d}z_1\,\mathrm{d}z_2\, \frac{1}{(z_1^2+z_2^2+2yz_1z_2)^{D/2-1}} = -\frac{2}{(D-4)}\, G(0,0,\tfrac{D}{2}-1;y)\,,$$

which leads to (13). The integral $J_\beta^{(2)}$ in (33) can be evaluated along the same lines. In this case there are only three integrations over λ_i, and one is thus left with a single parameter integral after Borel transformation.

A more complicated integral is that appearing in (28). Following the general procedure outlined above we derive

$$I_\beta^\alpha(1,1,1,0;1,1,1) = -\frac{\Gamma^2(\frac{D}{2})\,\Gamma(\frac{D}{2}-1)}{(4\pi)^{D/2}} \int\limits_0^\infty \mathrm{d}\lambda_1\,\mathrm{d}\lambda_2\,\mathrm{d}\lambda_3\, e^{(\lambda_1+\lambda_2)\omega+\lambda_3\omega'}$$
$$\times \int \mathrm{d}\tilde{x}_1\, \frac{x_{1\beta}}{(x_1^2)^{D/2}}\,\frac{x_2^\alpha}{(x_2^2)^{D/2}}\,\frac{1}{(x_3^2)^{D/2-1}}\,,$$

where $x_2 = x_1+(\lambda_1+\lambda_2)v+\lambda_3 v'$, and $x_3 = x_1+\lambda_1 v+\lambda_3 v'$. The integral over x_1 has the form of a Euclidean one-loop integral and can be performed in the standard manner by introduction of two Feynman parameters z_i. One then contracts the Lorentz indices with those in the trace in (28) to compute the integral I_4. The result is

$$I_4 = \frac{\Gamma(D-1)}{(4\pi)^D} \int_0^\infty \mathrm{d}\lambda_1\, \mathrm{d}\lambda_2\, \mathrm{d}\lambda_3\, \lambda_2\lambda_3\, e^{(\lambda_1+\lambda_2)\omega+\lambda_3\omega'} \int \mathrm{d}z_1\, \mathrm{d}z_2\, \frac{z_1(z_1^2\bar{z}_1 z_2\bar{z}_2)^{D/2-1}}{[M^2(\lambda_i,z_i)]^{D-1}}\,,$$

where $\bar{z}_i = 1 - z_i$, and

$$M^2(\lambda_i,z_i) = z_1(z_2 p^2 + \bar{z}_2 q^2) - z_1^2(z_2 p + \bar{z}_2 q)^2$$

with $p = \lambda_2 v$ and $q = (\lambda_1 + \lambda_2)v + \lambda_3 v'$. After Borel transformation the integral can be cast into the following form:

$$\hat{I}_4 = \frac{(2T)^{2D-5}}{(4\pi)^D}\,\Gamma(D-1) \int_0^1 \mathrm{d}\lambda\, \lambda^{2-D} \int_\lambda^\infty \mathrm{d}u_1 \int_{1/\lambda}^\infty \mathrm{d}u_2\, \frac{(u_1 u_2 - 1)^{D/2-2}}{[u_1 + 2(y+1)(u_2-1)]^{D-1}}\,.$$

For $D < 3$ one can use an integration by parts in λ to obtain (29). By analytic continuation, this result can then be evaluated around $D = 4$.

APPENDIX B: PARAMETER INTEGRALS

We collect some useful formulae for the evaluation of parameter integrals. We start with a remark on divergent integrals such as those appearing in (27) and (29). Assuming first that D is sufficiently small, one can use an integration by parts to rewrite these in terms of integrals which have a well defined expansion around $D = 4$. For instance, for $D < 4$ one can show that

$$(D-4)\,G(3-D,0,\tfrac{D}{2};y) = D\left[y\,G(4-D,0,\tfrac{D}{2}+1;y) + G(4-D,1,\tfrac{D}{2}+1;y)\right] - 1\,.$$

Similarly, $G(2-D,0,\frac{D}{2}-1;y)$ can be related to $G(3-D,0,\frac{D}{2};y)$ plus nonsingular terms for $D < 3$. By analytic continuation, one can then evaluate the resulting expressions in the vicinity of $D = 4$.

We now present a list of parameter integrals which are encountered when one expands the results presented in Sec. III around $D = 4$. When evaluating these integrals it is useful to introduce a hyperbolic angle θ by $y = \cosh\theta$. Then

$$V^2(u) = 1 + 2yu + u^2 = (u + e^\theta)(u + e^{-\theta})$$

factorizes. Setting $R_0 = V^2(1) = 2(y+1)$, we find:

$$F_1 = \int_0^1 \mathrm{d}u \, \frac{1}{V^2(u)} = \frac{r(y)}{2} \,,$$

$$F_2 = R_0 \int_0^1 \mathrm{d}u \, \frac{1}{[V^2(u)]^2} = \frac{1 - r(y)}{2(y-1)} + 1 \,,$$

$$F_3 = R_0 \int_0^1 \mathrm{d}u \, \frac{1+u}{[V^2(u)]^2} = -\frac{y+1}{2} r(y) - \frac{\ln R_0 + l(y)}{2} \,,$$

$$F_4 = R_0 \int_0^1 \mathrm{d}u \, \frac{\ln(1-u)}{[V^2(u)]^2} = -\frac{r(y)}{4} - \frac{1}{4(y-1)} \Big\{ [r(y)+1] \ln R_0 - 2\, l(y) \Big\} \,,$$

$$F_5 = \int_0^1 \mathrm{d}u \, \frac{\ln V^2(u)}{V^2(u)} = -\frac{r(y)}{2} \ln R_0 + h(y) \,,$$

$$F_6 = R_0 \int_0^1 \mathrm{d}u \, \frac{\ln V^2(u)}{[V^2(u)]^2} = \frac{1}{2(y-1)} \Big\{ 2y - 2h(y) + [r(y) - 1]\,(1 + \ln R_0) \Big\} \,,$$

$$F_7 = \int_0^1 \mathrm{d}u \, \frac{\ln(1-u)}{(1-u)} \left[\frac{R_0^2}{[V^2(u)]^2} - 1 \right] = \frac{\ln^2 R_0}{8} - \frac{\pi^2}{12} - \frac{y^2-1}{8} r^2(y) - \frac{y+1}{2} r(y)$$
$$- \frac{y \ln R_0}{2(y-1)} - \frac{(2-y)(y+1)}{4(y-1)} \Big[r(y) \ln R_0 - 2\, l(y) \Big] \,,$$

$$F_8 = R_0^2 \int_0^1 \mathrm{d}u \, \frac{\ln V^2(u) - \ln R_0}{(1-u)\,[V^2(u)]^2} = -\frac{\pi^2}{6} - \frac{y^2-1}{4} r^2(y) - \frac{y+1}{2} r(y) \ln R_0$$
$$+ (y+1)\,(1 - \ln R_0) - \frac{1}{2} + (y+1) \Big[F_5 + F_6 - F_2 \ln R_0 \Big] \,.$$

The functions $r(y)$ and $h(y)$ have been defined in (19). In addition, we have introduced

$$l(y) = \frac{1}{\sqrt{y^2-1}} \Big[L_2(-y + \sqrt{y^2-1}) - L_2(-y - \sqrt{y^2-1}) \Big] \,,$$

which satisfies $l(1) = 2 \ln 2$. Here $L_2(x) = -\int_0^x \frac{\mathrm{d}t}{t} \ln(1-t)$ is the dilogarithm. The first derivatives of these functions at $y = 1$ are $r'(1) = -\frac{1}{3}$, $h'(1) = \frac{1}{18}$, and $l'(1) = \frac{1}{6} - \frac{2}{3} \ln 2$. Finally, we note the following useful identity ($n \geq 1$):

$$\int_0^1 \mathrm{d}u \, \ln^n(1-u) \frac{D(y+u)}{[V^2(u)]^{D/2+1}} = n \int_0^1 \mathrm{d}u \, \frac{\ln^{n-1}(1-u)}{(1-u)} \Big[R_0^{-D/2} - [V^2(u)]^{-D/2} \Big] \,.$$

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FIGURES

FIG. 1. Two-loop diagrams contributing at order α_s to the perturbative part of the sum rule for the Isgur-Wise form factor. Heavy quark propagators are drawn as double lines, while the wavy line represents the weak current.

FIG. 2. (a) Sum rule prediction for the renormalized Isgur-Wise function $\xi_{\mathrm{ren}}(y)$. The width of the band arises from variation of ω_0 and T as specified in the text. (b) The "physical" form factor $\xi_{\mathrm{phys}}(\bar{m}, y)$ computed in next-to-leading order in renormalization-group improved perturbation theory (solid), in leading logarithmic approximation (dashed), and without including any QCD corrections (dotted). We use the central values $\omega_0 = 2.3$ GeV and $T = 1.0$ GeV.

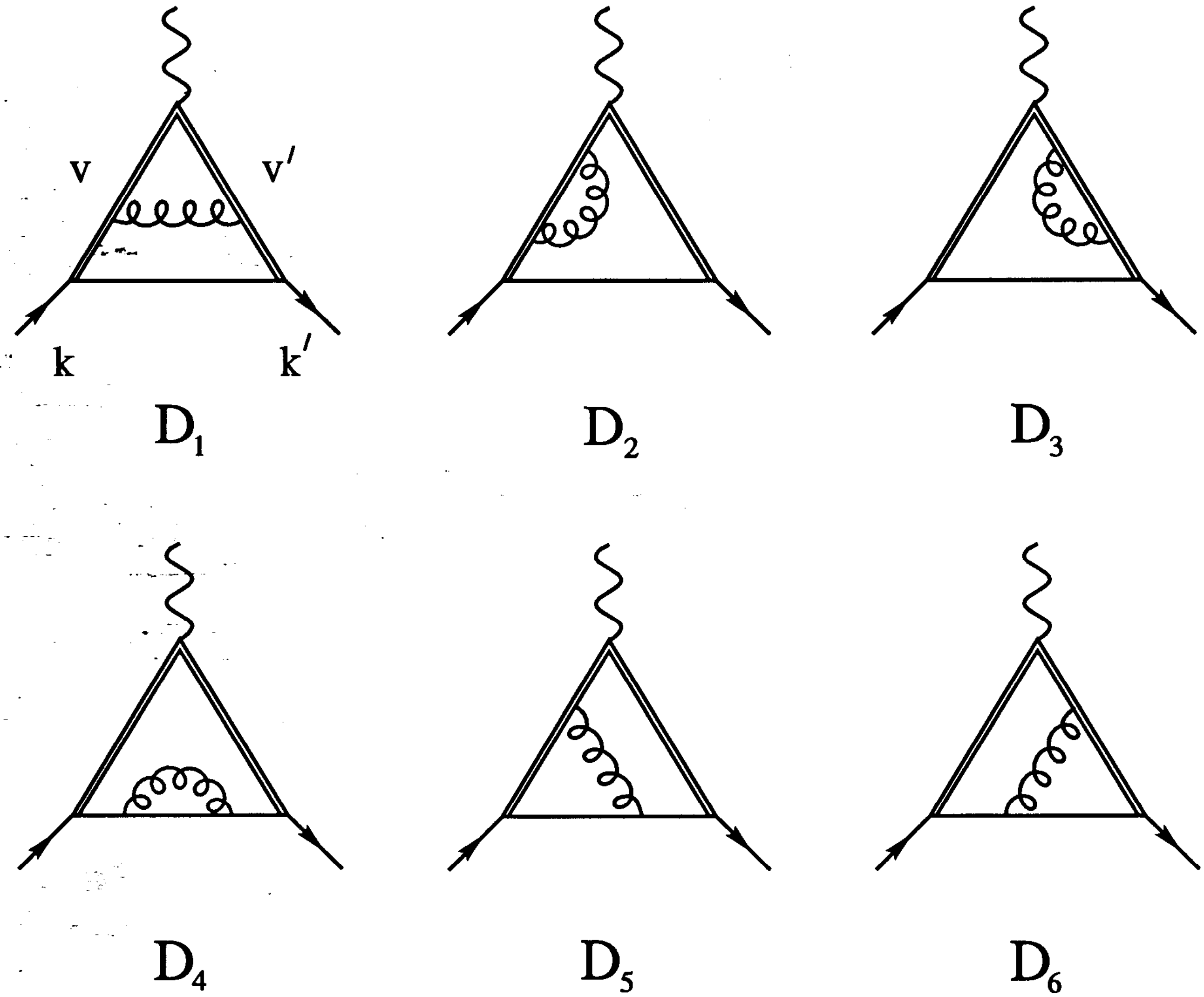
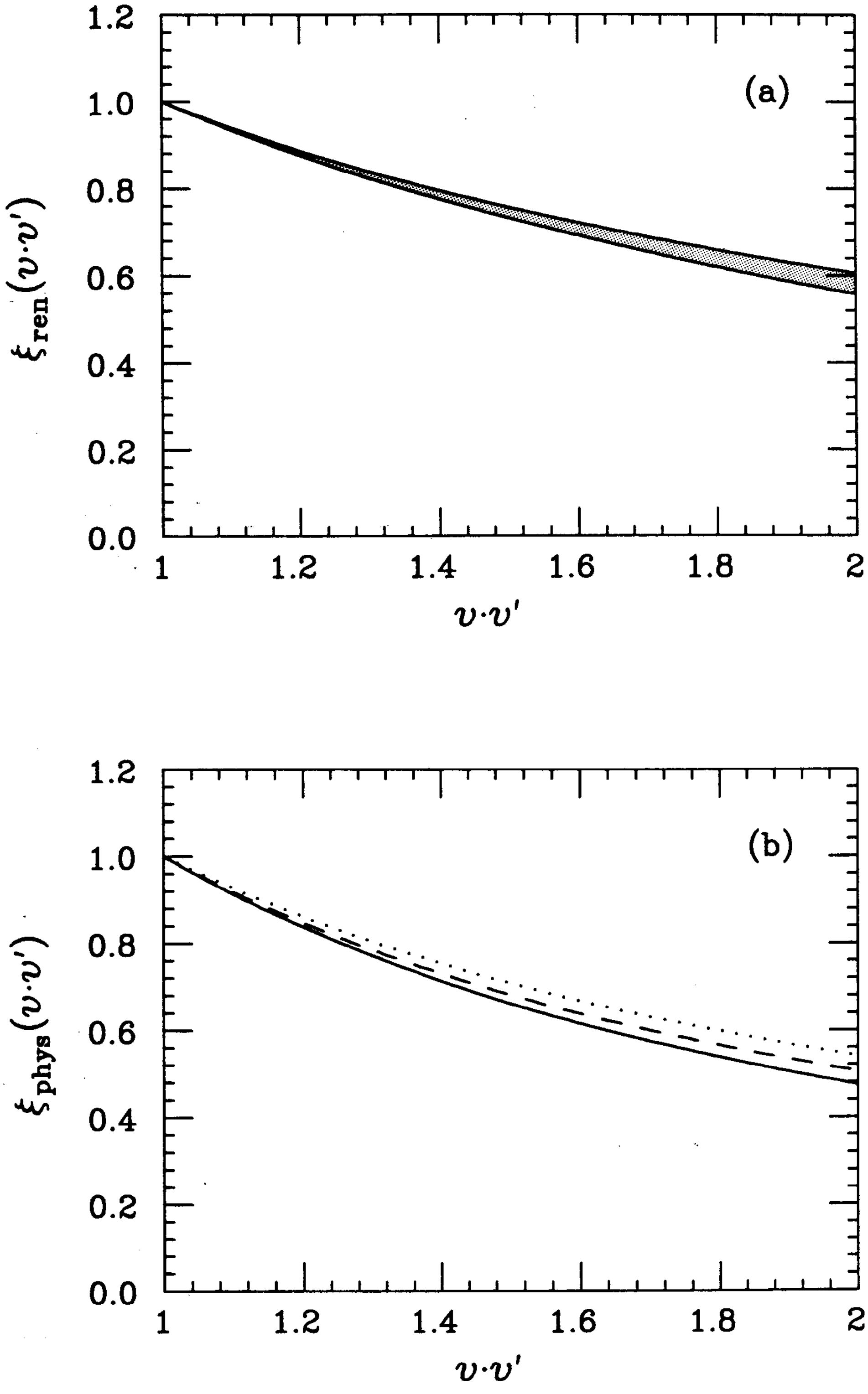


Fig.1



1.2
1.0
0.8
0.6
0.4
0.2
0.0
1
1.2
1.4
1.6
1.8
2
(a)
$\xi_{\mathrm{ren}}(v\cdot v')$
$v\cdot v'$
(b)
$\xi_{\mathrm{phys}}(v\cdot v')$
$v\cdot v'$

Fig. 2