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Field Theory Without Feynman Diagrams: A Demonstration Using Actions Induced by Heavy Particles

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ABSTRACT

In this paper the Bern-Kosower-type rules for effective actions, derived in an earlier paper, are used to find the actions for low-momentum gauge fields induced by loops of heavy scalars and Dirac fermions. Because of the special organization inherent in the first-quantized methods, certain gauge invariant structures in the effective action appear automatically, each multiplying a separate Feynman parameter integral; others can be extracted with minimal effort. In the abelian case the low-momentum effective action can be completely analyzed for an arbitrary number of external photons; the calculations are significantly simpler than those used in Feynman diagrams. On the basis of these results it is argued that the advantage of the Bern-Kosower rules lies not simply in their use of the Feynman parameter representation but also in the special way the Feynman parameter integrands are organized. It is suggested that the benefits demonstrated in this sample calculation are also useful when computing full S-matrix elements.

1. Introduction

In the past year significant advances have been made in techniques for calculating one-loop scattering amplitudes in gauge theories. There has been considerable interest and debate concerning the new Bern-Kosower technique for perturbative calculations in gauge theory [1–6,9], which converts amplitudes with external gauge bosons into Feynman parameter integrals and uses a novel diagrammatic expansion. In a recent paper [9] (which I will henceforth refer to as Paper A), I showed, using first-quantized path integrals, that Bern-Kosower-type rules could be derived from field theory for computation of one-loop effective actions. Recently Lam has reminded us [6] that any Feynman diagram may immediately be written as a parameter integral [7,8], raising the question as to whether the Bern-Kosower rules represent a real advance in calculational techniques. In this work I will argue that, at least at one loop, the new rules do indeed have advantages; their power lies not merely in their Feynman parameterized form but in their special organization.

In this paper, I will study the semi-classical effective action for low-momentum abelian and non-abelian gauge fields due to massive scalar or spinor particles. My purpose is to illustrate several special features of the Bern-Kosower techniques in a setting where the concepts are clear and the calculations are trivial. The most elegant way to do the specific calculations of this paper is to use the gauge invariant methods of Schwinger [12] and Shore [13]; however these results are only derived for covariantly constant fields and do not easily generalize to arbitrary momentum. On the other hand, traditional Feynman diagrams, even when parameterized as in [7], are somewhat clumsy, despite the conceptual simplicity of their building blocks, because their individual pieces are not gauge invariant. The Bern-Kosower rules have the conceptual simplicity and broader applicability of the Feynman approach while benefiting from the more explicitly gauge invariant structure of the Schwinger proper-time representation.

A particular feature I will emphasize is one that I discussed briefly in section 5 of Paper A: the manifestly gauge invariant organization of the formalism. Feynman diagrams are simply not gauge invariant in their various pieces, whether one uses Lam's technique or more standard approaches. By contrast, the starting point of the Bern-Kosower formalism [1,2], eq. (2.1), is explicitly gauge invariant. After applying the integration-by-parts (IBP) procedure, which is described by Bern and Kosower [1,2] and in Paper A, one finds that the full abelian amplitude can be easily organized into gauge invariant combinations of Lorentz invariants, each multiplying a single integral. In Feynman diagrams, one must compute many more parameter integrals, cancel many terms against each other, and reshuffle the algebra to make the answer look explicitly gauge invariant. In the Bern-Kosower approach the cancellations are removed at the start by the IBP, and the appropriate organization comes out automatically. Non-abelian theories are more complicated: the effective action contains more gauge invariant structures, not all of which are easy to identify from the organization of the amplitude, and there are contact terms from the IBP which must be accounted for, which are nevertheless well-organized, as they are computed from the pinch rules of Paper A. All of these points will be illustrated below.

In Sec. 2 I discuss the Bern-Kosower Master Formula and its properties, and state a theorem which I use in later sections. Sec. 3 contains a discussion of the effective action of abelian gauge fields due to a massive scalar or spinor; a simple formula for the leading term in $1/m^2$ at order-g^N is written down. In Sec. 4, a similar computation for non-abelian gauge fields is carried out to order-g^4. Discussion and conclusions are presented in Sec. 5.

2. The Bern-Kosower Master Formula and its Properties

The computation of the effective action begins from the following expression, called the Bern-Kosower Master Formula. It was first arrived at from string theory by Bern and Kosower [1,2], and later rederived in Paper A from field theory. (It may also be viewed as a gauge invariant form of eq. (2.1) of ref. [7].) The abelian Master Formula is

$$\Gamma_N(k_1,\ldots,k_N) = \frac{(ig)^N}{(4\pi)^{D/2}} \int_0^\infty \frac{dT\; e^{-m^2T}}{T^{1-N+D/2}} \Big(\prod_{i=2}^N \int_0^1 du_i\Big) \exp\Big(\sum_{i<j=1}^N k_i\cdot k_j G_B^{ji}\Big)$$
$$\times \exp\Big[\sum_{i<j=1}^N \big(-i\,(k_i\cdot\epsilon_j - k_j\cdot\epsilon_i)\dot{G}_B^{ji} + \epsilon_i\cdot\epsilon_j \ddot{G}_B^{ji}\big)\Big]\Big|_{\text{linear in }\epsilon_i} \tag{2.1}$$

which is valid in D-dimensional Euclidean spacetime. (Here $u_1 \equiv 0$.) This expression gives the abelian one-loop effective action for N external photons and a scalar loop; on shell it gives the one-loop photon S-matrix. The ϵ_i and k_i are the polarization and momentum vectors of the i^{th} photon; the scalar has mass m. The last line in (2.1) is called the "generating kinematic factor" (GKF); only terms containing each of the N polarization vectors ϵ_i exactly once are to be kept. Normally one carries out an integration-by-parts procedure (IBP), described in refs. [1,2] and discussed in Paper A, which removes all factors of $\ddot{G}_B^{ij}$, leading to an "improved generating kinematic factor" (IGKF) which is a sum of terms each containing N $\dot{G}_B$ functions. For large N, there are many equivalent ways to carry out the IBP, and consequently many possible forms for the IGKF.

For a Dirac spinor loop one multiplies (2.1) by -2 and makes the following replacement wherever possible in the IGKF:

$$\prod_{k=1}^{d} \dot{G}_B^{i_k i_{k-1}} \rightarrow \left(\prod_{k=1}^{d} \dot{G}_B^{i_k i_{k-1}} - \prod_{k=1}^{d} G_F^{i_k i_{k-1}} \right) \tag{2.2}$$

where $i_0 \equiv i_d$. (This manifestation of *world-line* supersymmetry is easily derived from an explicitly supersymmetric formalism. See also ref. [10].) The functions G_F^{ij}, G_B^{ij}, $\dot{G}_B^{ij}$ and $\ddot{G}_B^{ij}$ are [1,9]

$$\begin{aligned} G_B^{ij} &= G_B(t_i, t_j) \equiv T\Big(|u_i - u_j| - (u_i - u_j)^2\Big) \ ; \\ \dot{G}_B^{ij} &= \partial_{t_i} G_B(t_i, t_j) \equiv \big(\mathrm{sign}(u_i - u_j) - 2(u_i - u_j)\big) \ ; \\ \ddot{G}_B^{ij} &= \partial_{t_i}^2 G_B(t_i, t_j) \equiv \frac{2}{T}\big(\delta(u_i - u_j) - 1\big) \\ G_F^{ij} &= G_F(t_i, t_j) \equiv \mathrm{sign}(t_i - t_j) = \mathrm{sign}(u_i - u_j) \end{aligned} \tag{2.3}$$

where the t_i are proper times on a loop of total proper time T, and $u_i = t_i/T$. Note that $G_B^{ij} = G_B^{ji}$ and $\ddot{G}_B^{ij} = \ddot{G}_B^{ji}$ are symmetric, while $\dot{G}_B^{ij} = -\dot{G}_B^{ji}$ and $G_F^{ij} = -G_F^{ji}$ are antisymmetric.

There is a basic though limited theorem about the structure of the IGKF which I will use throughout the remainder of this article. Before stating it I must define some terminology.

A *chain* is any series of $\dot{G}_B$ functions arranged so that the second index of one $\dot{G}_B$ is the first index of the next, as in the expression

$$\dot{G}_B^{12} \dot{G}_B^{23} \dot{G}_B^{34} \cdots \dot{G}_B^{n-2,n-1} \dot{G}_B^{n-1,n} \ . \tag{2.4}$$

Every factor of $\dot{G}_B^{ij}$ in the GKF or IGKF is accompanied by a kinematic invariant of the form $\epsilon_i \cdot \epsilon_j$, $\epsilon_i \cdot k_j$, or $k_i \cdot k_j$. I will sometimes use the word *chain* to refer not only to the chain of $\dot{G}_B$ functions but also to the kinematic invariants multiplying them.

A chain which completely closes on itself will be called a *closed chain*, and will be denoted by its indices placed (in order) between braces; for example,

$$\epsilon_1 \cdot k_3 \epsilon_3 \cdot \epsilon_2 k_2 \cdot k_6 \epsilon_6 \cdot k_1 \dot{G}_B^{13} \dot{G}_B^{32} \dot{G}_B^{26} \dot{G}_B^{61} \tag{2.5}$$

is a closed chain denoted $\{1326\}$ or $\{2316\}$ or $\{1623\}$, *etc.* The kinematic factors which may appear as part of a closed chain will be specified in the theorem below.

A chain whose last index j appears in a closed chain but which itself is not part of a closed chain is called a *tail.* For example, in

$$\epsilon_5 \cdot k_1 \epsilon_1 \cdot k_3 \epsilon_3 \cdot \epsilon_2 k_2 \cdot k_1 \epsilon_6 \cdot k_3 \dot{G}_B^{51} \dot{G}_B^{13} \dot{G}_B^{32} \dot{G}_B^{26} \dot{G}_B^{63} \tag{2.6}$$

the structure $\epsilon_5 \cdot k_1 \epsilon_1 \cdot k_3 \dot{G}_B^{51} \dot{G}_B^{13}$ is a tail attached to the closed chain $\{326\}$; the complete notation for (2.6) will be $|513\rangle\,\{326\}$. (It is proved in Appendix D that in any tail $|ab \cdots ij\rangle$ appearing in the GKF or IGKF, the last $\dot{G}_B^{ij}$ function is associated with a kinematic invariant $\epsilon_i \cdot k_j$ or $k_i \cdot k_j$.) Tails may have branches, as in $|41\rangle\,|513\rangle\,\{326\}$, and closed chains may have an arbitrary number of tails, as in $|41\rangle\,|513\rangle\,|76\rangle\,|86\rangle\,\{326\}$:

$$\dot{G}_B^{41} \dot{G}_B^{51} \dot{G}_B^{13} \dot{G}_B^{76} \dot{G}_B^{86} \dot{G}_B^{32} \dot{G}_B^{26} \dot{G}_B^{63} \ . \tag{2.7}$$

It is useful to define the invariants

$$
\begin{aligned}
D_p(i_p\cdots,i_2,i_1) =&(k_{i_p\mu_1}\epsilon_{i_p}^{\mu_p}-\epsilon_{i_p\mu_1}k_{i_p}^{\mu_p})\cdots(k_{i_2\mu_3}\epsilon_{i_2}^{\mu_2}-\epsilon_{i_2\mu_3}k_{i_2}^{\mu_2})(k_{i_1\mu_2}\epsilon_{i_1}^{\mu_1}-\epsilon_{i_1\mu_2}k_{i_1}^{\mu_1})\\
=&\epsilon_{i_p}\cdot k_{i_{p-1}}\cdots\epsilon_{i_2}\cdot k_{i_1}\epsilon_{i_1}\cdot k_{i_p}+(-1)^p\epsilon_{i_p}\cdot k_{i_1}\cdots\epsilon_{i_2}\cdot k_{i_3}\epsilon_{i_1}\cdot k_{i_2}\\
&+\text{ all possible exchanges of } (\epsilon_i\leftrightarrow -k_i)
\end{aligned}
\tag{2.8}
$$

for $p\geq 3$, and

$$
D_2(i,j)=(\epsilon_i\cdot k_j\epsilon_j\cdot k_i-\epsilon_i\cdot\epsilon_j k_i\cdot k_j)\ . \tag{2.9}
$$

In general, every term of the factor $D_p(p,\ldots,2,1)$ which appears in the IGKF multiplies the closed chain of $\dot{G}_B$ functions

$$
\delta_p(p,\ldots,2,1)=\dot{G}_B^{1p}\cdots\dot{G}_B^{32}\dot{G}_B^{21}\ . \tag{2.10}
$$

To state the theorem, I need to define a set Q_N. An element $q\in Q_N$ is defined as follows: Let S be any subset of the integers $1,\ldots,N$. Divide S into ordered subsets S_k, where each S_k has at least two elements, and where S_k is defined only up to cyclic permutation and inversion of its ordering. The set $\{S_k\}$ is an element $q\in Q_N$. Let $n(q)\in\{2,3,\ldots,N\}$ be the number of indices appearing in S. Considering all possible S and S_k gives all possible q for fixed N; the set of all possible q's will be called Q_N. For example, the elements of Q_4 are

$$
\begin{gathered}
\{ij\},1\leq i<j\leq 4;\ \ \{ijk\},1\leq i<j<k\leq 4;\\
\{12\}\{34\};\ \ \{13\}\{24\};\ \ \{14\}\{23\};\ \ \{1234\};\ \ \{1324\};\ \ \{1243\}\ .
\end{gathered}
\tag{2.11}
$$

For $q\in Q_N$ let $D(q)$ be the product of D_p functions whose arguments are the sets S_k contained in q, and let $\delta(q)$ be the associated product of closed chains δ_p of $\dot{G}_B$

functions. For example, for $q = \{421\}$,

$$D(q)\delta(q) = D_3(4,2,1)\dot{G}_B^{14}\dot{G}_B^{42}\dot{G}_B^{21} \ , \tag{2.12}$$

while for $q = \{42\}\{31\}$,

$$D(q)\delta(q) = D_2(4,2)D_2(3,1)\dot{G}_B^{24}\dot{G}_B^{42}\dot{G}_B^{13}\dot{G}_B^{31} \ . \tag{2.13}$$

The statement of the theorem is the following:

Theorem: The IGKF at order g^N has the form

$$\mathrm{IGKF}(N) = i^N \sum_{q \in Q_N} \left[D(q)\delta(q)\eta(q) \sum_{\{T\}} T(q) \right] + \text{ other terms}, \tag{2.14}$$

where $\eta(q) = \pm 1$, and where the second sum is over all tails or products of tails of the form

$$T(q) = \prod_{r \notin q} \epsilon_r \cdot k_{j_r} \dot{G}_B^{r j_r} \ ; \tag{2.15}$$

the product is over only those indices r which do not appear in q, and the $j_r = 1, 2, \ldots, N$ are chosen so that T itself contains no closed chains. The "other terms" in the IGKF all possess at least one tail containing a factor $\epsilon_i \cdot \epsilon_j$ (with i, j not in q.) This result is independent of the choice of algorithm for the integration-by-parts procedure.

The proof of the theorem is given in appendix D. It is somewhat tedious, however, and readers may instead wish to convince themselves by working a few examples, or by following along with Sec. 4 where the four-gluon case is discussed in detail.

One may conjecture an extension to the theorem, which to this point I have been unable to prove, though I have verified it up to $N = 4$.

Conjecture: If the integration-by-parts procedure is carried out using a particular algorithm (as yet unknown), the IGKF at order g^N has the form

$$\mathrm{IGKF}(N) = i^N \sum_{q \in Q_N} \left[D(q)\delta(q) \sum_{\{T\}} T(q)\eta(T(q);q) \right] \tag{2.16}$$

where the second sum is over all tails $T(q)$ of the chains represented by $D(q)$, such that each ϵ_i appears once either in $D(q)$ or T, and where $\eta(T(q);q) = 0, \pm 1$ depends on q and on $T(q)$. If $T(q) = 1$ or a product of $\epsilon_i \cdot k_j \dot{G}_B^{ij}$ factors, then the factor $D(q)T(q)$ appears in the IGKF *no matter what algorithm is used for the integration by parts*, so $\eta(T(q);q) = \pm 1$.

Proof or disproof of this conjecture might be useful, in that it might help us to understand the IBP procedure, and might identify a particularly efficient IBP algorithm.

3. The Abelian Case

In this section I will compute the one-loop effective action of QED due to a massive scalar or Dirac spinor particle in the limit of low momentum. For N external photons, I will calculate the leading term in k^2/m^2 of the N-point one-loop amplitude, where k is the characteristic momentum of the gauge bosons and m is the mass of the particle in the loop. After studying the vacuum polarization in some detail, I will use theorem (2.14) to write the low-momentum effective action for all N as an expression equivalent to Schwinger's famous result [12]. All calculations will be performed in Euclidean space.

First, I review the result of Paper A for the infinite term in the vacuum polarization of scalar QED. I will explicitly present many steps in this calculation so as to illustrate how unnecessarily difficult the standard Feynman diagram techniques actually are.

Let us work for the moment in an arbitrary number of dimensions D. For $N = 2$ the Master Formula yields [9]

$$\Gamma_2(k_1, k_2) = \frac{(ig)^2}{(4\pi)^{D/2}} \int_0^\infty \frac{dT}{T^{D/2-1}} \int_0^1 du\ e^{k_1\cdot k_2 G_B(u) - m^2 T} \Big[\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1 [\dot{G}_B(u)]^2 +\ \epsilon_1 \cdot \epsilon_2 \ddot{G}_B(u)\Big]\ . \tag{3.1}$$

which contains both diagrams of figure 1. This expression is similar to that which one would arrive at using the formalism of Lam and Lebrun [7], but it is organized in a very special way. In particular, if one straightforwardly computes the leading term in (3.1), and expands in powers of k^2/m^2, one finds

$$
\begin{aligned}
\Gamma_2(k_1,k_2) =& \frac{-g^2}{(4\pi)^{D/2}}\int\limits_0^1 du \\
& \times \left[\epsilon_1\cdot k_2\epsilon_2\cdot k_1(1-2u)^2\Gamma(\frac{4-D}{2})\left[m^2-k_1\cdot k_2(u-u^2)\right]^{\frac{D-4}{2}}\right. \\
& \quad \left. +\ 2\epsilon_1\cdot\epsilon_2\big(\delta(u)-1\big)\Gamma(\frac{2-D}{2})\left[m^2-k_1\cdot k_2(u-u^2)\right]^{\frac{D-2}{2}}\right] \\
=& \frac{-g^2}{(4\pi)^{D/2}}\Gamma(\frac{4-D}{2})m^{D-4}\int\limits_0^1 du\ \Bigg[\epsilon_1\cdot k_2\epsilon_2\cdot k_1(1-2u)^2[1+\mathcal{O}(\frac{k^2}{m^2})] \\
& \quad -\frac{2}{D-2}2\epsilon_1\cdot\epsilon_2 m^2\big(\delta(u)-1\big) \\
& \qquad \times\left[1-\frac{D-2}{2}\frac{k_1\cdot k_2}{m^2}(u-u^2)+\mathcal{O}(\frac{k^4}{m^4})\right]\Bigg] \\
=& \frac{-g^2}{(4\pi)^{D/2}}\Gamma(\frac{4-D}{2})m^{D-4}\int\limits_0^1 du\ \Bigg[-2\epsilon_1\cdot\epsilon_2 m^2\frac{2}{D-2}\big(\delta(u)-1\big) \\
& \quad +\ \epsilon_1\cdot k_2\epsilon_2\cdot k_1(1-2u)^2+\ 2\epsilon_1\cdot\epsilon_2 k_1\cdot k_2\big(\delta(u)-1\big)(u-u^2)\Bigg] \\
=& \frac{-g^2}{(4\pi)^{D/2}}\Gamma(\frac{4-D}{2})m^{D-4}\Bigg[2\epsilon_1\cdot\epsilon_2 m^2\frac{2}{D-2}(1-1) \\
& \quad +\ \epsilon_1\cdot k_2\epsilon_2\cdot k_1(\frac{1}{3})-\ 2\epsilon_1\cdot\epsilon_2 k_1\cdot k_2(\frac{1}{6})\Bigg] \\
=& \frac{g^2}{3(4\pi)^{D/2}}\Gamma(\frac{4-D}{2})m^{D-4}\Big[\epsilon_1\cdot\epsilon_2 k_1\cdot k_2-\ \epsilon_1\cdot k_2\epsilon_2\cdot k_1\Big]
\end{aligned}
\tag{3.2}
$$

The leading terms are highly divergent in any integer dimension $D \geq 2$ but are not gauge invariant; after these cancel one must pull out the sub-leading divergences and only then take the limit $k^2/m^2 \to 0$. Notice the four separate (though, in this case, simple) Feynman parameter integrals. If we had not taken the $k^2/m^2 \to 0$ limit then it would have been taken somewhat more effort to show that the leading terms cancelled and the result was gauge invariant.

The more clever method, introduced by Bern and Kosower via string theory [1,2,9], is to integrate the $\ddot{G}_B$ term in (3.1) by parts to arrive at

$$\begin{aligned}
\Gamma_2(k_1,k_2) &= \frac{(ig)^2}{(4\pi)^{D/2}}\Big[\epsilon_1\cdot\epsilon_2\; k_1\cdot k_2 - \epsilon_1\cdot k_2\epsilon_2\cdot k_1\Big] \\
&\quad\times\int_0^1 du\;[\dot{G}_B(u)]^2\int_0^\infty \frac{dT}{T^{D/2-1}}e^{k_1\cdot k_2 G_B(u)-m^2T} \\
&= \frac{g^2}{(4\pi)^{D/2}}\Gamma(\frac{4-D}{2})\Big[\epsilon_1\cdot\epsilon_2\; k_1\cdot k_2 - \epsilon_1\cdot k_2\epsilon_2\cdot k_1\Big] \\
&\quad\times\int_0^1 du\;[\dot{G}_B(u)]^2[m^2 - k_1\cdot k_2(u-u^2)]^{\frac{D-4}{2}} \\
&\to \frac{g^2}{(4\pi)^{D/2}}\Gamma(\frac{4-D}{2})m^{D-4}\Big[\epsilon_1\cdot\epsilon_2\; k_1\cdot k_2 - \epsilon_1\cdot k_2\epsilon_2\cdot k_1\Big] \\
&\quad\times\int_0^1 du\;[\dot{G}_B(u)]^2
\end{aligned} \tag{3.3}$$

and the single Feynman parameter integral yields a factor of $\frac{1}{3}$. Notice that the IBP removed the higher-order, gauge noninvariant divergences, left the expression in a manifestly gauge invariant form, and gave us a single Feynman parameter integral to perform which, in the limit $k^2/m^2 \to 0$, easily gave us the correct coefficient.

If we specialize to four dimensions, the vacuum polarization is logarithmically divergent. It is interesting to see this emerge through the use of a single Pauli-Villars regulator of mass M, which appears in the Master Formula through the replacement $e^{-m^2T} \to (e^{-m^2T} - e^{-M^2T})$.

$$
\begin{aligned}
\Gamma_2(k_1,k_2) =& \frac{(ig)^2}{(4\pi)^2}\int\limits_0^\infty \frac{dT}{T}\int\limits_0^1 du\ e^{k_1\cdot k_2 G_B(u)}(e^{-m^2T}-e^{-M^2T}) \\
&\times\Big[\epsilon_1\cdot k_2\epsilon_2\cdot k_1[\dot{G}_B(u)]^2+\ \epsilon_1\cdot\epsilon_2\ddot{G}_B(u))\Big] \\
=& \frac{g^2}{(4\pi)^2}\Big[\epsilon_1\cdot\epsilon_2\ k_1\cdot k_2-\epsilon_1\cdot k_2\epsilon_2\cdot k_1\Big] \\
&\times\int\limits_0^1 du\ [\dot{G}_B(u)]^2\int\limits_0^\infty \frac{dT}{T}e^{k_1\cdot k_2 G_B(u)}(e^{-m^2T}-e^{-M^2T}) \\
\to& \frac{g^2}{(4\pi)^2}\log\big(\frac{M^2}{m^2}\big)\Big[\epsilon_1\cdot\epsilon_2\ k_1\cdot k_2-\epsilon_1\cdot k_2\ \epsilon_2\cdot k_1\Big]\int\limits_0^1 du\ [\dot{G}_B(u)]^2
\end{aligned}
\tag{3.4}
$$

In the standard techniques, the use of Pauli-Villars regulators for vacuum polarizations requires three regulator fields and a careful balancing of their masses to ensure both quadratic and logarithmic divergences are controlled [11].

The same calculation for a Dirac spinor gives

$$
\begin{aligned}
\Gamma_2(k_1,k_2) =& -2\frac{g^2}{(4\pi)^2}\Big[\epsilon_1\cdot\epsilon_2\ k_1\cdot k_2-\epsilon_1\cdot k_2\epsilon_2\cdot k_1\Big] \\
&\times\int\limits_0^\infty \frac{dT}{T}\int\limits_0^1 du\ e^{k_1\cdot k_2 G_B(u)-m^2T}\Big([\dot{G}_B(u)]^2-[G_F(u)]^2\Big)\ . \\
=& -\frac{2g^2}{(4\pi)^2}\log\big(\frac{M^2}{m^2}\big)\Big[\epsilon_1\cdot\epsilon_2 k_1\cdot k_2-\epsilon_1\cdot k_2\epsilon_2\cdot k_1\Big]\int\limits_0^1 du\ [(1-2u)^2-1] \\
=& \frac{4g^2}{3(4\pi)^2}\log\big(\frac{M^2}{m^2}\big)\Big[\epsilon_1\cdot\epsilon_2\ k_1\cdot k_2-\epsilon_1\cdot k_2\epsilon_2\cdot k_1\Big]\ .
\end{aligned}
\tag{3.5}
$$

Granted, these calculations are not hard to do from Feynman diagrams. Let us therefore move on to something more complex.

First, let us see why all amplitudes with an odd number of abelian gauge bosons vanish. To calculate the amplitude with N photons, one expands (2.1) to order

g^N. As we saw, every term in the IGKF contains exactly N $\dot{G}_B$'s (or, for spinor loops, n G_F's and $(N-n)$ $\dot{G}_B$'s.) When every gauge boson is integrated over the entire loop, as it is in an abelian theory, we can make the change of variables $u_i \to v_i = 1 - u_i$. The integration region and the exponential $\exp[\sum k_i \cdot k_j G_B^{ji}]$ are unchanged; the IGKF changes by a factor of $(-1)^N$, showing that if N is odd the result of the integration must give zero.

I now turn to the calculation of the four-photon amplitude in figure 2. (For the remainder of this paper I will work in four dimensions. In the Bern-Kosower formalism all calculations beyond $N = 2$ are finite, so no regulator is needed.) The starting expression is now rather long; however, we may discard most terms from the start by using the theorem (2.14).

The key is that any term C in the IGKF possessing a tail, as defined above eq. (2.6), may be discarded in the low momentum limit. The reason for this is that when $k^2/m^2 \to 0$ the only dependence on the Feynman parameters is in the IGKF itself. The first index of any tail $|ijk \cdots mn\rangle \in C$ can occur only once in C, as part of the function $\dot{G}_B^{ij}$, so C can be written as $\partial_{t_i} C'$. For example

$$\dot{G}_B^{31}\dot{G}_B^{12}\dot{G}_B^{23}\dot{G}_B^{41} = \partial_{t_4}[\dot{G}_B^{31}\dot{G}_B^{12}\dot{G}_B^{23}G_B^{41}] \tag{3.6}$$

is a total derivative, while

$$\dot{G}_B^{31}\dot{G}_B^{24}\dot{G}_B^{13}\dot{G}_B^{42} \text{ and } \dot{G}_B^{31}\dot{G}_B^{12}\dot{G}_B^{24}\dot{G}_B^{43} \tag{3.7}$$

are not. Since $\dot{G}_B^{ij}$ and G_B^{ij} are periodic functions, and the integration region of each u_i is the entire loop, a total derivative like (3.6) vanishes when integrated.

We may therefore keep only those terms which are made purely of closed chains without tails; in other words, rewriting the theorem (2.14),

$$\mathrm{IGKF}(N) = i^N \sum_{q\in Q_N^0} \left[D(q)\delta(q)\eta(q)\right] + \text{ total derivatives;} \tag{3.8}$$

where Q_N^0 is the subset of Q_N whose elements contain all N integers.

Thus, in the limit $k^2/m^2 \to 0$, the four-photon scalar-loop amplitude is

$$\begin{aligned}\Gamma_4 = \frac{2g^4}{(4\pi)^2 m^4} \int_0^1 du_4 \int_0^1 du_3 \int_0^1 du_2 \\ \Big\{ & D_2(2,1)D_2(4,3)[\dot{G}_B^{21}]^2[\dot{G}_B^{43}]^2 \\ & + D_2(3,1)D_2(4,2)[\dot{G}_B^{31}]^2[\dot{G}_B^{42}]^2 \\ & + D_2(4,1)D_2(3,2)[\dot{G}_B^{41}]^2[\dot{G}_B^{32}]^2 \\ & + D_4(4,3,2,1)\dot{G}_B^{43}\dot{G}_B^{32}\dot{G}_B^{21}\dot{G}_B^{14} \\ & + D_4(4,2,3,1)\dot{G}_B^{42}\dot{G}_B^{23}\dot{G}_B^{31}\dot{G}_B^{14} \\ & + D_4(3,4,2,1)\dot{G}_B^{34}\dot{G}_B^{42}\dot{G}_B^{21}\dot{G}_B^{13} \Big\}\end{aligned} \tag{3.9}$$

The Feynman parameter integrals are simple and yield

$$\begin{aligned}\Gamma_4 = \frac{2g^4}{(4\pi)^2 m^4} \Big\{ & (\frac{1}{3})^2 \Big[D_2(2,1)D_2(4,3) + D_2(3,1)D_2(4,2) + D_2(4,1)D_2(3,2)\Big] \\ & + \frac{1}{45}\Big[D_4(4,3,2,1) + D_4(4,2,3,1) + D_4(3,4,2,1)\Big]\Big\}\end{aligned} \tag{3.10}$$

In the Feynman diagram calculation each of the three types of diagrams has terms proportional to $\log[M^2/m^2]$ and to $1/m^2$; in the Bern-Kosower formalism these terms are automatically removed by the IBP procedure and never need to be computed. Even if one were to identify all terms in the Feynman diagram calculation that are lower order than $1/m^4$ and discard them, the remaining terms would multiply a substantial number of integrands, in contrast to (3.9) in which only two appear.

For a Dirac spinor world-line supersymmetry gives us

$$
\begin{aligned}
\Gamma_4 = -2\frac{2g^4}{(4\pi)^2 m^4} \int_0^1 du_4 \int_0^1 du_3 \int_0^1 du_2 \\
&\Big\{ D_2(2,1)D_2(4,3)([\dot{G}_B^{21}]^2 - [G_F^{21}]^2)([\dot{G}_B^{43}]^2 - [G_F^{43}]^2) \\
&+ D_2(3,1)D_2(4,2)([\dot{G}_B^{31}]^2 - [G_F^{31}]^2)([\dot{G}_B^{42}]^2 - [G_F^{42}]^2) \\
&+ D_2(4,1)D_2(3,2)([\dot{G}_B^{41}]^2 - [G_F^{41}]^2)([\dot{G}_B^{32}]^2 - [G_F^{32}]^2) \\
&+ D_4(4,3,2,1)(\dot{G}_B^{43}\dot{G}_B^{32}\dot{G}_B^{21}\dot{G}_B^{14} - G_F^{43}G_F^{32}G_F^{21}G_F^{14}) \\
&+ D_4(4,2,3,1)(\dot{G}_B^{42}\dot{G}_B^{23}\dot{G}_B^{31}\dot{G}_B^{14} - G_F^{42}G_F^{23}G_F^{31}G_F^{14}) \\
&+ D_4(3,4,2,1)(\dot{G}_B^{34}\dot{G}_B^{42}\dot{G}_B^{21}\dot{G}_B^{13} - G_F^{34}G_F^{42}G_F^{21}G_F^{13})\Big\}
\end{aligned}
\tag{3.11}
$$

leading to

$$
\begin{aligned}
\Gamma_4 = -4\frac{g^4}{(4\pi)^2 m^4} \Big\{ (\frac{2}{3})^2 \Big[D_2(2,1)D_2(4,3) + D_2(3,1)D_2(4,2) + D_2(4,1)D_2(3,2) \Big] \\
- \frac{14}{45} \Big[D_4(4,3,2,1) + D_4(4,2,3,1) + D_4(3,4,2,1) \Big] \Big\}
\end{aligned}
\tag{3.12}
$$

Clearly D_2 bears some relation to $F^{\mu\nu}F_{\mu\nu}$, and D_4 to $F^{\mu\nu}F_{\nu\rho}F^{\rho\sigma}F_{\sigma\mu}$. Let us make this relationship more precise. In Paper A the derivation of (2.1) required a gauge field $A^\mu = \sum \epsilon_i^\mu e^{ik_i \cdot x}$. The appropriate field strength is therefore

$$
F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = i \sum_{i=1}^{N} (k_i^\mu \epsilon_i^\nu - k_i^\nu \epsilon_i^\mu) \tag{3.13}
$$

while $(F_\mu{}^\nu)^p \equiv F^{\mu_p}_{\mu_1} \cdots F^{\mu_2}_{\mu_3} F^{\mu_1}_{\mu_2}$ is given by

$$
\begin{aligned}
i^p & \sum_{i_1, i_2, \cdots, i_p=1}^{N} (k_{i_p \mu_1} \epsilon^{\mu_p}_{i_p} - \epsilon_{i_p \mu_1} k^{\mu_p}_{i_p}) \cdots (k_{i_2 \mu_3} \epsilon^{\mu_2}_{i_2} - \epsilon_{i_2 \mu_3} k^{\mu_2}_{i_2})(k_{i_1 \mu_2} \epsilon^{\mu_1}_{i_1} - \epsilon_{i_1 \mu_2} k^{\mu_1}_{i_1}) \\
& = i^p \sum_{i_1, i_2, \cdots, i_p=1}^{N} D_p(i_p, \cdots, i_2, i_1) \\
& = 2p i^p \Big(\sum_{i_1, i_2, \cdots, i_p=1}^{N} \Big)^* D_p(i_p, \cdots, i_2, i_1)
\end{aligned}
\tag{3.14}
$$

where the starred sum indicates that only terms in which i_p is the largest index of the set and in which $i_{p-1} > i_1$ are to be included (this eliminates all equivalent factors of D_p.)

With this result we may identify the expression (3.10) as

$$
\begin{aligned}
\Gamma_4 = & \frac{2g^4}{(4\pi)^2 m^4} \Big\{ \frac{1}{2 \cdot 4^2 \cdot 3^2} (F^{\mu\nu} F_{\nu\mu})^2 + \frac{1}{8 \cdot 45} F^{\mu\nu} F_{\nu\rho} F^{\rho\sigma} F_{\sigma\mu} \Big\} \\
= & \frac{g^4}{(4\pi)^2 m^4} \Big\{ \frac{7}{180} (E^2 - B^2)^2 + \frac{1}{45} (E \cdot B)^2 \Big\}
\end{aligned}
\tag{3.15}
$$

where I have used

$$
F^{\mu\nu} F_{\nu\mu} = 2(E^2 - B^2) \,; F^{\mu\nu} F_{\nu\rho} F^{\rho\sigma} F_{\sigma\mu} = 2(E^2 - B^2)^2 + 4(E \cdot B)^2 \,. \tag{3.16}
$$

(The general relation between electromagnetic fields and $(F_\mu{}^\nu)^p$ is given in appendix B.) For the Dirac spinor the four-photon amplitude is

$$
\begin{aligned}
\Gamma_4 = & -2 \frac{2g^4}{(4\pi)^2 m^4} \Big\{ \frac{2^2}{2 \cdot 4^2 \cdot 3^2} (F^{\mu\nu} F_{\nu\mu})^2 - \frac{14}{8 \cdot 45} F^{\mu\nu} F_{\nu\rho} F^{\rho\sigma} F_{\sigma\mu} \Big\} \\
= & \frac{g^4}{(4\pi)^2 m^4} \Big\{ -\frac{4}{45} (E^2 - B^2)^2 + \frac{28}{45} (E \cdot B)^2 \Big\}
\end{aligned}
\tag{3.17}
$$

This is of course the famous Euler-Heisenberg Lagrangian.

We may also use (3.14) to write down the effective action for general N. Referring again to the theorem (3.8), and computing the signs $\eta(q)$ by comparison between particular terms in (2.1) and particular terms in $(F_\mu{}^\nu)^p$, *the complete effective action in this limit may be written down.* The coefficients of the various terms are simply given by the integrals of the $\dot{G}_B$ chains and G_F chains. Specifically, note that any structure $D(q)\delta(q)$ in (3.8) contains at least one term of the form

$$\prod_{m=1}^{N} \epsilon_m \cdot k_{j_m} \dot{G}_B^{m j_m} \; . \tag{3.18}$$

On the other hand, the GKF and IGKF always contain the terms

$$(-i)^N \prod_{m=1}^{N} \left[\sum_{j_m=1}^{N} \epsilon_m \cdot k_{j_m} \dot{G}_B^{m j_m} \right] . \tag{3.19}$$

We may use this to determine the overall sign $\eta(q)$ for each q. Now, with the help of (3.14), we may conclude that at order-g^N the low-momentum photon effective action due to a massive scalar loop is given by summing over all partitions $\Pi(N)$ of N into even integers — I will write $\Pi(N) = \{n_2, n_4, \ldots, n_N\}$, $n_p \in \{0, 1, \ldots, N/p\}$, such that $N = \sum(n_p p)$ — and associating with each partition the appropriate gauge-invariant factor and coefficient:

$$\Gamma_N = \frac{(ig)^N}{(4\pi)^2} \frac{(N-2)!}{m^{2(N-2)}} \sum_{\Pi(N)} \prod_{p=1}^{N} \frac{1}{n_p!} \left(\frac{b_p (F_\mu{}^\nu)^p}{2p} \right)^{n_p} . \tag{3.20}$$

Here

$$\begin{aligned} b_{2n} &\equiv \prod_{1}^{2n} \left(\int_0^1 du_i \right) \; \dot{G}_B^{1,2n} \dot{G}_B^{2n,2n-1} \cdots \dot{G}_B^{32} \dot{G}_B^{21} \\ &= \frac{2^{2n+1}}{(2n+1)!} [4n^2 - 4n - 1] + \sum_{r=1}^{n-1} \frac{2^{2r+1} r}{(2r+1)!} b_{2n-2r} \; . \end{aligned} \tag{3.21}$$

For an arbitrary number of dimensions one makes the replacement

$$\frac{(N-2)!}{m^{2(N-2)}(4\pi)^2} \to \frac{(N-D/2)!}{m^{2(N-D/2)}(4\pi)^{D/2}}. \tag{3.22}$$

For spinor loops the result is

$$\Gamma_N = -2\frac{(ig)^N}{(4\pi)^2}\frac{(N-2)!}{m^{2(N-2)}}\sum_{\Pi(N)}\prod_{p=1}^{N}\frac{1}{n_p!}\Big(\frac{(b_p-f_p)(F_\mu{}^\nu)^p}{2p}\Big)^{n_p}. \tag{3.23}$$

where b_p is as above and

$$\begin{aligned} f_{2n} &\equiv \prod_1^{2n}\Big(\int_0^1 du_i\Big)\quad G_F^{1,2n}G_F^{2n,2n-1}\cdots G_F^{32}G_F^{21} \\ &= -\frac{2^{2n-2}}{(2n-1)!}-\sum_{r=1}^{n-1}\frac{2^{2r-1}}{(2r)!}f_{2n-2r}\ . \end{aligned} \tag{3.24}$$

The recursion formulas (3.21) and (3.24), which are straightforward to derive with the help of appendix A, yield

$$\begin{aligned} b_2 = -\frac{1}{3}\ ,\ b_4 = \frac{1}{45}\ ,\ b_6 = -\frac{2}{945}\ ; \\ f_2 = -1\ ;\ f_4 = \frac{1}{3}\ ;\ f_6 = -\frac{2}{15}\ . \end{aligned} \tag{3.25}$$

We may use these results to check that (3.15) and (3.17) are indeed given by the general expressions (3.20) and (3.23).

At order g^6, the scalar hexagon gives

$$\begin{aligned} \Gamma_6 = -\frac{3!g^6}{(4\pi)^2m^8}\Big\{&-\frac{1}{3!\cdot 4^3\cdot 3^3}(F^{\mu\nu}F_{\nu\mu})^3-\frac{1}{4\cdot 8\cdot 135}(F^{\mu\nu}F_{\nu\rho}F^{\rho\sigma}F_{\sigma\mu})(F^{\alpha\beta}F_{\beta\alpha}) \\ &-\frac{2}{12\cdot 945}\ F^{\mu\nu}F_{\nu\rho}F^{\rho\sigma}F_{\sigma\alpha}F^{\alpha\beta}F_{\beta\mu}\Big\} \\ =\frac{g^6}{(4\pi)^2m^8}\Big\{&\frac{31}{8\cdot 315}(E^2-B^2)^3+\frac{11}{2\cdot 315}\ (E^2-B^2)(E\cdot B)^2\Big\} \end{aligned} \tag{3.26}$$

while the Dirac spinor hexagon gives

$$\begin{aligned}\Gamma_6 = &+2\frac{3!g^6}{(4\pi)^2m^8}\left\{\frac{2^3}{3!\cdot 4^3\cdot 3^3}(F^{\mu\nu}F_{\nu\mu})^3 - \frac{28}{4\cdot 8\cdot 135}\,(F^{\mu\nu}F_{\nu\rho}F^{\rho\sigma}F_{\sigma\mu})(F^{\alpha\beta}F_{\beta\alpha})\right.\\ &\left.+\frac{124}{12\cdot 945}\,F^{\mu\nu}F_{\nu\rho}F^{\rho\sigma}F_{\sigma\alpha}F^{\alpha\beta}F_{\beta\mu}\right\}\\ =&-\frac{g^6}{(4\pi)^2m^8}\left\{\frac{8}{315}(E^2-B^2)^3+\frac{52}{315}\,(E^2-B^2)(E\cdot B)^2\right\}\end{aligned} \tag{3.27}$$

At this order, Feynman diagrams generate spurious terms at $1/m^2$, $1/m^4$, and $1/m^6$, all of which cancel in intermediate steps.

The above results of course agree with those derived by Schwinger [12] for constant background fields. Indeed, Schwinger's exact result may be derived from the path integrals of Paper A. Alternatively, one may expand the path integral in powers of the background field; one arrives at the same products of $\dot{G}_B$ and G_F chains as in (3.20) and (3.23).

In summary, the first-quantized approach to calculations allows many of the steps of Feynman diagrams to be circumvented. In this approach it is easy to identify the gauge invariant structures of the effective action and the integrals which must be computed to find their coefficients. The regularity of these structures and their associated integrands makes it possible to analyze the one-loop effective action completely. While this was done years ago by Schwinger [12], it is not straightforward to do this using Feynman diagrams because many spurious terms are generated and must be calculated and cancelled, and because many terms related by gauge invariance multiply different integrands which nonetheless give in the end the same coefficient. These problems are solved by the Bern-Kosower formalism, independently of the $k^2/m^2 \to 0$ limit which was used in this section as an illustration. Thus, the power of the Bern-Kosower approach lies not in its expression in terms of Feynman parameter integrals, which could be simply accomplished from Feynman diagrams [6,7,9], but in its special organization which allows terms to be easily analyzed and computed and unnecessary cancellations to be avoided. This power is evident *even in abelian gauge theory.*

4. The Non-Abelian Case

In this section I treat particle loops in non-abelian background fields. There are several new complications. In a non-abelian theory the expansion in powers of k^2/m^2 at fixed power of g is not useful; for example, there are logarithmically divergent terms not only at order-g^2 but also at g^3 and g^4. We should therefore expand the *full* one-loop effective action in powers of $1/m^2$. At order-g^N we will compute terms up to $1/m^{2(N-2)}$, which is as far as we would go in the abelian case. A second complication is that if the IBP procedure is used, there are extra terms, called pinch terms, which must be computed when working at any given order in g. (The rules for calculating them are closely related to the pinch rules presented by Bern and Kosower [1,2] for the computation of scattering amplitudes; they are discussed in Paper A and presented in appendix C. As of now they have not been proven correct to all orders.) These terms are organized in a way which reflects the underlying gauge invariance. In fact, these pinch terms never need be computed; they always contribute parts of gauge invariant structures at the same or lower order in $1/m^2$ whose coefficients are already known. Another important change concerns the addition of color traces and the associated alteration of the integration regions. Instead of integrating all gauge bosons freely around the loop, we must now restrict the integrations in accordance with the path-ordering associated with each color-ordering, as explained by Bern and Kosower [1,2] or in Paper A. As a consequence the total derivatives which vanished in the abelian case will not do so here — implying that structures other than $(F_{\mu\nu})^{2n}$ will occur in the effective action — and in addition the argument for the vanishing of odd-point amplitudes fails.

At the four-point level and beyond, the discussion is further complicated by the different possibilities for performing the IBP and by ambiguities inherent in the expression for the effective action itself. For example, one may write

$$F^{\mu\nu}D^{\rho}D^{2}D_{\rho}F_{\mu\nu} = -\frac{1}{2}F^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}F_{\mu\nu} + F^{\mu\nu}D^{\sigma}D^{\rho}D_{\sigma}D_{\rho}F_{\mu\nu} \tag{4.1}$$

showing that any given expression for the effective action is not unique. Furthermore, by integrating by parts with respect to x (not proper time, as in the IBP used in the Bern-Kosower formalism) one may change the form of terms in the effective action. Fortunately for the previous chapter, these ambiguities do not arise in the abelian theory.

The vacuum polarization due to massive scalars or Dirac fermions is the same in QCD as in QED,

$$\Gamma_2 = \frac{g^2}{(4\pi)^2}\log\big(\frac{M^2}{m^2}\big)\big(\frac{1}{4}\mathrm{Tr}F^{\mu\nu}F_{\nu\mu}\big)|_{\mathrm{order}-A^2} \times \begin{cases} \frac{1}{3}, & \text{complex scalar loop;} \\ \frac{4}{3}, & \text{Dirac spinor loop.} \end{cases} \tag{4.2}$$

so let us turn our attention to the 3-gluon effective action of a massive particle. We may expect terms of the form $(F_{\mu}{}^{\nu})^3$, but we should not forget that the term $(F_{\mu}{}^{\nu})^2$, which appears at order g^2, has an order g^3 piece which *must* appear in this calculation. We will see that this happens in a very beautiful way.

The expression for the three-gluon amplitude is

$$\Gamma_3(k_1,k_2,k_3) = \frac{(ig)^3}{(4\pi)^2}\Big\{\mathrm{Tr}(T^{a_3}T^{a_2}T^{a_1})\int_0^1 du_3\int_0^{u_3} du_2 + \mathrm{Tr}(T^{a_2}T^{a_3}T^{a_1})\int_0^1 du_3\int_{u_3}^1 du_2\Big\}\,\mathrm{GKF} \tag{4.3}$$

where the generating kinematic factor is

$$\mathrm{GKF} = \Big[(-i)^3\sum_{p,q,r=1}^{3}\epsilon_1\cdot k_p\epsilon_2\cdot k_q\epsilon_3\cdot k_r\dot{G}_B^{1p}\dot{G}_B^{2q}\dot{G}_B^{3r} + \big(-i\epsilon_1\cdot\epsilon_2\ddot{G}_B^{21}\sum_{p=1}^{3}\epsilon_3\cdot k_p\dot{G}_B^{3p} + \text{ cyclic permutations }.\big)\Big] \tag{4.4}$$

(Recall that $\dot{G}_B^{ii} = 0$.) Integrating by parts yields

$$\begin{aligned}\mathrm{IGKF} = -i\sum_{p=1}^{3}\Big[\;&\epsilon_3\cdot k_pD_2(2,1)[\dot{G}_B^{21}]^2\dot{G}_B^{3p}\\ &+\epsilon_1\cdot k_pD_2(3,2)[\dot{G}_B^{32}]^2\dot{G}_B^{1p}\\ &+\epsilon_2\cdot k_pD_2(1,3)[\dot{G}_B^{13}]^2\dot{G}_B^{2p}\;\Big]\\ &-D_3(3,2,1)\dot{G}_B^{32}\dot{G}_B^{21}\dot{G}_B^{13}\end{aligned} \tag{4.5}$$

and the integrations yield

$$\begin{aligned}\Gamma_3(k_1,k_2,k_3) = -\frac{g^3}{30m^2(4\pi)^2}\Big\{&\mathrm{Tr}(T^{a_3}T^{a_2}T^{a_1})\\ &\times\Big(\Big[\epsilon_3\cdot(k_2-k_1)D_2(1,2)+\epsilon_2\cdot(k_1-k_3)D_2(3,1)\\ &\quad+\epsilon_1\cdot(k_3-k_2)D_2(2,3)\Big]-2D_3(3,2,1)\Big)\\ &+\;(2\leftrightarrow 3)\;\Big\}\end{aligned} \tag{4.6}$$

Of course, this is not the entire order-g^3 effective action; there is still a pinch term to be evaluated below.

Clearly, we should identify $D_3(3,2,1)$ as stemming from a term $\mathrm{Tr}(F_\mu{}^\nu)^3$ in the effective action. The other terms contain $(F_\mu{}^\nu)^2$, and in order to be gauge invariant should also contain two covariant derivatives. Because of the Bianchi identity, the term $\mathrm{Tr}(D^\rho F^{\mu\nu})^2$ is the unique term involving two field strengths and two derivatives; the equation of motion is not needed for this conclusion. By writing out these gauge invariant structures, we may easily use (4.6) to identify their coefficients in the effective action:

$$\frac{g^2}{(4\pi)^2}\left\{-\frac{1}{120m^2}\mathrm{Tr}\Big[(D^\rho F_\mu{}^\nu)(D_\rho F_\nu{}^\mu)\Big]+ig\frac{1}{45m^2}\mathrm{Tr}\Big[F_\mu{}^\nu F_\nu{}^\rho F_\rho{}^\mu\Big]\right\} \tag{4.7}$$

In particular, (4.6) contains the part of (4.7) which is order g^3 and which involves the abelian pieces of $F_\mu{}^\nu$. Other terms in (4.7) come from pinches or from expanding (3.4) in k^2/m^2, as will be shown below.

For example, it is interesting to see how the term $(\partial^\rho F^{\mu\nu})^2$, part of $(D^\rho F^{\mu\nu})^2$, appears in this approach. Since it is an order-g^2 term and is higher order in k^2/m^2 than the term $(F_\mu{}^\nu)^2$, it must appear in the next-to-leading term in (3.4). Expanding (3.4) we find

$$\begin{aligned}\Gamma_2(k_1,k_2)|_{1/m^2} &= \frac{(ig)^2}{(4\pi)^2}\Big[\epsilon_1\cdot\epsilon_2\; k_1\cdot k_2-\epsilon_1\cdot k_2\epsilon_2\cdot k_1\Big]\; k_1\cdot k_2\\ &\times\int_0^1 du\;[\dot G_B(u)]^2\int_0^\infty\frac{dT}{T^3}G_B(u,T)e^{-m^2T}\end{aligned} \tag{4.8}$$

which has the correct Lorentz structure but not obviously the correct coefficient. However, since

$$G_B^{ij}=T\int_{u_j}^{u_i} du_k\;\dot G_B^{kj} \tag{4.9}$$

the coefficient in (4.8) is equal to the coefficient of $\epsilon_1\cdot(k_3-k_2)D_2(2,3)$ in (4.3).

Now let us move on to the application of the pinch rules, as described in Paper A and presented in Appendix C, to the IGKF of (4.5). For the color trace $\mathrm{Tr}(T^{a_3}T^{a_2}T^{a_1})$ and the diagram given in figure 3, the pinch rules tell us to extract a contribution from the term $D_3(3,2,1)$ in (4.3) of the form

$$
\begin{aligned}
& i\int_0^\infty dT \int_0^1 du_3 \int_0^{u_3} du_2 \; \exp\Big(\sum_{i<j=1}^{N} k_i\cdot k_j G_B^{ji} - m^2T\Big) \\
& \qquad\qquad \Big[(\epsilon_3\cdot k_1\epsilon_2\cdot\epsilon_1 - \epsilon_2\cdot k_1\epsilon_3\cdot\epsilon_1)\dot{G}_B^{21}\dot{G}_B^{13}\;\delta(u_3-u_2)/T\Big] \\
& = -i\,(\epsilon_3\cdot k_1\epsilon_2\cdot\epsilon_1 - \epsilon_2\cdot k_1\epsilon_3\cdot\epsilon_1) \\
& \qquad\qquad \times\int_0^\infty \frac{dT}{T}e^{-m^2T}\int_0^1 du\;[\dot{G}_B(u)]^2\{1+\mathcal{O}(k^2/m^2)\}
\end{aligned}
\tag{4.10}
$$

where I have left off the overall factor of $(ig)^3(4\pi)^{-2}$. Symmetrizing in $2\leftrightarrow 3$ and summing over cyclic permutations, we get the full pinch contribution, which is plainly the cubic term in $(F_\mu{}^\nu)^2$, as expected from (4.2). Notice it has the same dependence on the Feynman parameters as does the quadratic term (eq. (3.4)); thus it has the correct coefficient. Furthermore, the antisymmetry in exchange of any two indices show that this term is proportional to the structure constant $\mathrm{Tr}([T^{a_3},T^{a_2}]T^{a_1})$, as it should be. We could have guessed this would be the case; every pinch removes a factor of $k_i\cdot k_j/m^2$ from the result without changing the power of g in the coefficient, indicating that pinch terms stem from operators which first appear at order $g^k/m^{2(k-2)}$ but which have pieces at the same order in m and higher order in g.

Eq. (4.10) also demonstrates that pinching a closed chain of $\dot{G}_B$ functions leaves a closed chain of $\dot{G}_B$ functions (though the *kinematic* structure acquires a gap);

similarly, pinching a tail leaves a tail of $\dot{G}_B$ functions. (This is a trivial consequence of the pinch rules.) Of course the number of $\dot{G}_B$ factors in a closed chain or tail decreases by one when it is pinched.

If we now expand (4.10) to the next order in k^2/m^2, we will find a contribution that is at the same order in g, k and m as those in (4.6). However, as we will see, the Lorentz structure of these terms cannot include any of the D_k functions as defined in (2.8) and (2.9), and therefore must involve non-abelian pieces of $F_\mu{}^\nu$ in (4.7). Since the commutator in $F_\mu{}^\nu$ carries a power of g, the only term in (4.7) which fits this description is the order-g^3 part of $(\partial^\rho F^{\mu\nu})^2$. Indeed, the expansion of (4.10) yields

$$\begin{aligned} & i\int_0^\infty dT \int_0^1 du_3 \int_0^{u_3} du_2\, e^{-m^2T} \sum_{i<j=1}^{N} k_i\cdot k_j G_B^{ji} \\ & \qquad \times\Big[(\epsilon_3\cdot k_1\epsilon_2\cdot\epsilon_1 - \epsilon_2\cdot k_1\epsilon_3\cdot\epsilon_1)\dot{G}_B^{21}\dot{G}_B^{13}\ \delta(u_3-u_2)/T\Big] \\ & = -i(\epsilon_3\cdot k_1\epsilon_2\cdot\epsilon_1 - \epsilon_2\cdot k_1\epsilon_3\cdot\epsilon_1)\ k_1\cdot(k_2+k_3) \\ & \qquad \times\int_0^\infty \frac{dT}{T}e^{-m^2T}\int_0^1 du\ [\dot{G}_B(u)]^2 G_B(u)\{1+\mathcal{O}(k^2/m^2)\} \end{aligned} \tag{4.11}$$

which from (4.9) has the correct coefficient.

Repeating all of this for quark loops is simple, using world-line supersymmetry; the same gauge invariant structures appear but with new coefficients. The new terms in the effective action are

$$-\frac{g^2}{(4\pi)^2}\left\{\frac{1}{15m^2}\mathrm{Tr}\Big[(D^\rho F_\mu{}^\nu)(D_\rho F_\nu{}^\mu)\Big] + ig\frac{13}{45m^2}\mathrm{Tr}\Big[F_\mu{}^\nu F_\nu{}^\rho F_\rho{}^\mu\Big]\right\} \tag{4.12}$$

I will now compute the order-g^4 terms in the effective action. If I were merely interested in calculating the effective action, it would be sufficient to use the theorem (2.14) to write down enough of the IGKF to compute the coefficients of all new gauge invariant structures appearing at this order. My interest, however, is in showing that the Bern-Kosower formalism has a special organization which makes it easy to understand how and where individual terms in the effective action arise. For the purpose of illustrating this feature, I will compute the full IGKF in detail, use it to find the coefficients of the new structures in the effective action, and then study the pinches of the IGKF to find explicitly all remaining order-g^4 terms expected from the new structures and from (4.2) and (4.7).

The first step is to compute the full IGKF, which requires choosing an algorithm for integrating by parts. The ideal method for carrying out the IBP has not yet been found. It is useful, for the purposes of this paper, to maintain as much symmetry as possible under cyclic permutations of the indices and under the replacement $\epsilon_i \leftrightarrow k_i$. However, this still does not completely specify exactly how to perform the IBP, nor is complete cyclic invariance possible. Perhaps by systematically analyzing the gauge invariant structures produced at arbitrary order it will be possible to design a particularly elegant algorithm, or perhaps many equivalent techniques can be found, but I will not discuss this issue further in this work.

At order-g^4,

$$\Gamma_4(k_1,k_2,k_3,k_4) =$$

$$\frac{2g^4}{(4\pi)^2 m^4}\Big\{\mathrm{tr}(T^{a_4}T^{a_3}T^{a_2}T^{a_1})\int_0^1 du_4\int_0^{u_4} du_3\int_0^{u_3} du_2 \tag{4.13}$$

$$+\ \text{all non}-\text{cyclic permutations}\Big\}[\mathrm{IGKF}]$$

To find the IGKF, we start from the Master Formula, which gives

$$\begin{aligned}\mathrm{GKF} \ = \Big\{(-i)^4 \sum_{p,q,r,s=1}^{4} & \epsilon_1\cdot k_p\epsilon_2\cdot k_q\epsilon_3\cdot k_r\epsilon_4\cdot k_s\dot G_B^{1p}\dot G_B^{2q}\dot G_B^{3r}\dot G_B^{4s} \\ & + \Big[(-i)^2\epsilon_1\cdot\epsilon_2\ddot G_B^{21}\sum_{r,s=1}^{4}\epsilon_3\cdot k_r\epsilon_4\cdot k_s\dot G_B^{3r}\dot G_B^{4s} \\ & \quad + \text{all non}-\text{equivalent cyclic permutations}\ \Big] \\ & + \Big[\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4\ddot G_B^{21}\ddot G_B^{34}+\epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4\ddot G_B^{31}\ddot G_B^{24}+\epsilon_1\cdot\epsilon_4\epsilon_2\cdot\epsilon_3\ddot G_B^{41}\ddot G_B^{23}\Big]\Big\}\end{aligned} \tag{4.14}$$

At this order one may proceed by applying the IBP first to the terms with the most factors of $\ddot G_B$, though there is an ambiguity in the choice of index.* As a specific algorithm, I choose to IBP with respect to the largest available index whenever the requirements of gauge invariance and cyclicity are not sufficient to determine the next step. With this arbitrary choice, the GKF becomes

$$\begin{aligned}\mathrm{GKF} = \Big\{ \sum_{p,q,r,s=1}^{4} & \epsilon_1\cdot k_p\epsilon_2\cdot k_q\epsilon_3\cdot k_r\epsilon_4\cdot k_s\dot G_B^{1p}\dot G_B^{2q}\dot G_B^{3r}\dot G_B^{4s} \\ & - \Big[\epsilon_1\cdot\epsilon_2\sum_{r,s=1}^{4}\epsilon_3\cdot k_r\epsilon_4\cdot k_s\ddot G_B^{21}\dot G_B^{3r}\dot G_B^{4s} \\ & \quad + \text{all non}-\text{equivalent cyclic permutations}\ \Big] \\ & - \sum_{s=1}^{4}\Big(\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4 k_4\cdot k_s\ddot G_B^{21}\dot G_B^{43}\dot G_B^{4s} \\ & \quad + \ \epsilon_4\cdot\epsilon_1\epsilon_2\cdot\epsilon_3 k_3\cdot k_s\ddot G_B^{41}\dot G_B^{32}\dot G_B^{3s} \\ & \quad + \ \epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4 k_4\cdot k_s\ddot G_B^{31}\dot G_B^{42}\dot G_B^{4s}\Big)\Big\}\end{aligned} \tag{4.15}$$

One may now recognize that the new terms generated in the previous step are related by gauge invariance to other terms in the GKF, and the remainder of the

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⋆ I thank Bern and Kosower for suggesting this method to me.

IBP should be carried out in such a way as to preserve that relationship. The related terms are

$$
\begin{aligned}
&(\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4 k_4\cdot k_s \leftrightarrow \epsilon_1\cdot\epsilon_2\epsilon_3\cdot k_4\epsilon_4\cdot k_s)\ddot{G}_B^{21}\dot{G}_B^{43}\dot{G}_B^{4s}\ ;\\
&(\epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4 k_4\cdot k_s \leftrightarrow \epsilon_1\cdot\epsilon_3\epsilon_2\cdot k_4\epsilon_4\cdot k_s)\ddot{G}_B^{31}\dot{G}_B^{42}\dot{G}_B^{4s}\ ;\\
&(\epsilon_2\cdot\epsilon_3\epsilon_1\cdot\epsilon_4 k_4\cdot k_s \leftrightarrow \epsilon_2\cdot\epsilon_3\epsilon_1\cdot k_4\epsilon_4\cdot k_s)\ddot{G}_B^{32}\dot{G}_B^{41}\dot{G}_B^{4s}\ .
\end{aligned}
\tag{4.16}
$$

The IBP is now applied to all the remaining terms with a $\ddot{G}_B$ function. In most terms, the choice of index for the IBP is irrelevant. For example, if we integrate $\ddot{G}_B^{34}\dot{G}_B^{31}\dot{G}_B^{21}$ by parts with respect to t_4, we find

$$
-\sum_{s=1}^{4}\dot{G}_B^{43}\dot{G}_B^{31}\dot{G}_B^{21}k_4\cdot k_s\dot{G}_B^{4s}
\tag{4.17}
$$

whereas if we IBP with respect to t_3, we get

$$
-\sum_{s=1}^{4}\dot{G}_B^{34}\dot{G}_B^{31}\dot{G}_B^{21}(k_3\cdot k_s\dot{G}_B^{3s}+k_1\cdot k_s\dot{G}_B^{1s}+k_2\cdot k_s\dot{G}_B^{2s})\ .
\tag{4.18}
$$

Since $\dot{G}_B^{34} = -\dot{G}_B^{43}$, *etc.*, (4.17) and (4.18) are equal. Similarly, the IBP of $\ddot{G}_B^{34}\dot{G}_B^{31}\dot{G}_B^{24}$ is unambiguous because

$$
\sum_{s=1}^{4}(k_3\cdot k_s\dot{G}_B^{3s}+k_1\cdot k_s\dot{G}_B^{1s}) = -\sum_{s=1}^{4}(k_4\cdot k_s\dot{G}_B^{4s}+k_2\cdot k_s\dot{G}_B^{2s})
\tag{4.19}
$$

However, in the case of $\ddot{G}_B^{34}[\dot{G}_B^{21}]^2$, using t_3 for the IBP is just as valid as but is not equal to using t_4. The reason for this ambiguity is unclear. Notice, however, that in either case the term $[\dot{G}_B^{34}]^2[\dot{G}_B^{21}]^2$ will occur; this is in accordance with the theorem of Sec. 2. (See also lemma 3 in Appendix D.)

Again I use the prescription that when in doubt the largest available index is used for the IBP. This results in

$$
\begin{aligned}
\mathrm{IGKF} = \Bigg\{ \Big[& \epsilon_1 \cdot k_3 \epsilon_2 \cdot k_3 \dot{G}_B^{13} \dot{G}_B^{23} + \epsilon_1 \cdot k_4 \epsilon_2 \cdot k_4 \dot{G}_B^{14} \dot{G}_B^{24} \\
& + \epsilon_1 \cdot k_3 \epsilon_2 \cdot k_1 \dot{G}_B^{13} \dot{G}_B^{21} + \epsilon_1 \cdot k_3 \epsilon_2 \cdot k_4 \dot{G}_B^{13} \dot{G}_B^{24} \\
& + \epsilon_1 \cdot k_4 \epsilon_2 \cdot k_1 \dot{G}_B^{14} \dot{G}_B^{21} + \epsilon_1 \cdot k_4 \epsilon_2 \cdot k_3 \dot{G}_B^{14} \dot{G}_B^{23} \\
& + (\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_4 - \epsilon_1 \cdot \epsilon_2 k_2 \cdot k_4) \dot{G}_B^{12} \dot{G}_B^{24} \\
& + (\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_3 - \epsilon_1 \cdot \epsilon_2 k_2 \cdot k_3) \dot{G}_B^{12} \dot{G}_B^{23} \Big] D_2(4,3) \dot{G}_B^{34} \dot{G}_B^{43} \\
& \quad + \text{three cyclic permutations} \Bigg\} \\
+ \Bigg\{ \Big[& \epsilon_1 \cdot k_2 \epsilon_3 \cdot k_2 \dot{G}_B^{12} \dot{G}_B^{32} + \epsilon_1 \cdot k_4 \epsilon_3 \cdot k_4 \dot{G}_B^{14} \dot{G}_B^{34} \\
& + \epsilon_1 \cdot k_2 \epsilon_3 \cdot k_4 \dot{G}_B^{12} \dot{G}_B^{34} + \epsilon_1 \cdot k_4 \epsilon_3 \cdot k_2 \dot{G}_B^{14} \dot{G}_B^{32} \\
& + \epsilon_1 \cdot k_2 \epsilon_3 \cdot k_1 \dot{G}_B^{12} \dot{G}_B^{31} + \epsilon_1 \cdot k_4 \epsilon_3 \cdot k_1 \dot{G}_B^{14} \dot{G}_B^{31} \\
& + (\epsilon_1 \cdot k_3 \epsilon_3 \cdot k_2 - \epsilon_1 \cdot \epsilon_3 k_3 \cdot k_2) \dot{G}_B^{13} \dot{G}_B^{32} \\
& + (\epsilon_1 \cdot k_3 \epsilon_3 \cdot k_4 - \epsilon_1 \cdot \epsilon_3 k_3 \cdot k_4) \dot{G}_B^{13} \dot{G}_B^{34} \Big] D_2(4,2) \dot{G}_B^{24} \dot{G}_B^{42} \\
& \quad + \{1 \to 2 \to 3 \to 4 \to 1\} \Bigg\} \\
+ \Bigg\{ & D_2(2,1) D_2(4,3) [\dot{G}_B^{21}]^2 [\dot{G}_B^{43}]^2 + D_2(1,4) D_2(3,2) [\dot{G}_B^{14}]^2 [\dot{G}_B^{32}]^2 \\
& \quad + D_2(3,1) D_2(4,2) [\dot{G}_B^{31}]^2 [\dot{G}_B^{42}]^2 \Bigg\} \\
+ \Bigg\{ & \Big[\sum_{p=2}^{4} \epsilon_1 \cdot k_p \dot{G}_B^{1p} \Big] D_3(4,3,2) \dot{G}_B^{43} \dot{G}_B^{32} \dot{G}_B^{24} \\
& \quad + \text{three cyclic permutations} \Bigg\} \\
+ \Bigg\{ & D_4(4,3,2,1) \dot{G}_B^{43} \dot{G}_B^{32} \dot{G}_B^{21} \dot{G}_B^{14} \\
& + D_4(4,2,3,1) \dot{G}_B^{42} \dot{G}_B^{23} \dot{G}_B^{31} \dot{G}_B^{14} \\
& + D_4(3,1,2,4) \dot{G}_B^{31} \dot{G}_B^{12} \dot{G}_B^{24} \dot{G}_B^{43} \Bigg\}
\end{aligned}
\tag{4.20}
$$

where the D_k are defined in (2.8) and (2.9). In a non-abelian theory the structures

$\mathrm{Tr}(F_\mu{}^\nu F_\nu{}^\rho F_\rho{}^\sigma F_\sigma{}^\mu)$ and $\mathrm{Tr}(F_\mu{}^\nu F_\nu{}^\sigma F_\rho{}^\mu F_\sigma{}^\rho)$ are different; the order of indices in D_p, up to cyclic permutations, is therefore important. Note also that the terms multiplying $D_2(4,2)$ are not cyclically symmetric.

The result of the integrations in the limit $k^2/m^2 \to 0$ is

$$
\begin{aligned}
\Gamma_4^{(0)} = \frac{g^4}{(4\pi)^2 m^4}\frac{1}{630}\Bigg\{ & \mathrm{Tr}[T^{a_4}T^{a_3}T^{a_2}T^{a_1}] \\
& \Bigg[\Bigg\{\Big[-3\epsilon_1\cdot k_3\epsilon_2\cdot k_3 - 3\epsilon_1\cdot k_4\epsilon_2\cdot k_4 + 3\epsilon_1\cdot k_3\epsilon_2\cdot k_1 \\
& \quad +3(\epsilon_1\cdot k_2\epsilon_2\cdot k_4 - \epsilon_1\cdot\epsilon_2 k_2\cdot k_4) - 2\epsilon_1\cdot k_3\epsilon_2\cdot k_4 \\
& \quad -8\epsilon_1\cdot k_4\epsilon_2\cdot k_1 + 8\epsilon_1\cdot k_4\epsilon_2\cdot k_3 \\
& \quad -8(\epsilon_1\cdot k_2\epsilon_2\cdot k_3 - \epsilon_1\cdot\epsilon_2 k_2\cdot k_3)\Big] D_2(4,3) \\
& \qquad + \text{three cyclic permutations}\Bigg\} \\
& +\Bigg\{\Big[6\epsilon_1\cdot k_2\epsilon_3\cdot k_2 + 6\epsilon_1\cdot k_4\epsilon_3\cdot k_4 - 6\epsilon_1\cdot k_2\epsilon_3\cdot k_4 \\
& \quad -6\epsilon_1\cdot k_4\epsilon_3\cdot k_2 + 5\epsilon_1\cdot k_2\epsilon_3\cdot k_1 + 5\epsilon_1\cdot k_4\epsilon_3\cdot k_1 \\
& \quad +5(\epsilon_1\cdot k_3\epsilon_3\cdot k_2 - \epsilon_1\cdot\epsilon_3 k_3\cdot k_2) \\
& \quad +5(\epsilon_1\cdot k_3\epsilon_3\cdot k_4 - \epsilon_1\cdot\epsilon_3 k_3\cdot k_4)\Big] D_2(4,2) \\
& \qquad + \{1\to 2\to 3\to 4\to 1\}\Bigg\} \\
& \quad +30\big[D_2(2,1)D_2(4,3) + D_2(1,4)D_2(3,2)\big] + 10 D_2(3,1)D_2(4,2) \\
& +\Big\{9\epsilon_1\cdot(k_2-k_4)D_3(4,3,2) + \text{three cyclic permutations}\Big\} \\
& \quad -10 D_4(4,3,2,1) + 12\big[D_4(4,2,3,1) + D_4(3,1,2,4)\big]\Bigg] \\
& \qquad + \text{all non}-\text{cyclic permutations},\Bigg\}
\end{aligned}
\tag{4.21}
$$

where the superscript zero is to remind the reader that the pinch contributions are not included in this expression. Note that most of this result could have been written down using the theorem (2.14); only the terms of the form $\epsilon_i \cdot \epsilon_j k_j \cdot k_m D_2(m,n)$ depend on the algorithm used for the IBP.

This expression is far more organized than similar expressions coming from Feynman diagrams. Certain pieces of the above result are easy to identify, while others are more subtle. Terms made entirely of closed chains can be identified as powers of $F_\mu{}^\nu$; the terms with D_3 and a factor $\epsilon_i \cdot k_j$ are evidently part of $F_\mu{}^\nu(D_\alpha F_{\nu\rho})(D^\alpha F^{\rho\mu})$. The terms with a single D_2 are obviously more complicated, and we must enumerate the possible gauge invariant operators. We may note the absence of factors like $\epsilon_i \cdot k_i$ or $\epsilon_i \cdot k_j \epsilon_j \cdot k_i$ in the tails of the $D_2(m,n)$ chains to eliminate structures like $(D^2 F^{\mu\nu})\cdots$ and $(D_\rho D_\sigma F_{\mu\nu})(D^\sigma D^\rho F^{\nu\mu})$ respectively. (Not that these structures are disallowed — they just do not give the simplest representation of (4.21), and since they can be reexpressed in terms of other operators using (4.1) and similar identities there is no need to use them.) The only remaining structures are

$$\mathrm{Tr}[(D_\rho D_\sigma F_{\mu\nu})(D^\rho D^\sigma F^{\nu\mu})]\ , \mathrm{Tr}[(F^{\rho\sigma}(D_\rho F_{\mu\nu})(D_\sigma F^{\nu\mu})]\ . \qquad (4.22)$$

Next, merely by studying the terms in (4.22) which are a product of a term in a D_2 factor and a tail of $\epsilon_i \cdot k_j$ factors, one can easily identify which linear combination of these structures appears in the effective action. The effective action contains the terms

$$
\begin{aligned}
\frac{g^2}{(4\pi)^2 m^4}\mathrm{Tr}\Bigg[& \frac{2ig}{315}[F^{\rho\sigma}(D_\rho F_{\mu\nu})(D_\sigma F^{\nu\mu})] + \frac{1}{840}[(D_\rho D_\sigma F_{\mu\nu})(D^\rho D^\sigma F^{\nu\mu})] \\
& + \frac{ig}{70}[F_\mu{}^\nu(D_\alpha F_\nu{}^\rho)(D^\alpha F_\rho{}^\mu)] + \frac{g^2}{168}(F_\mu{}^\nu F_\nu{}^\mu F_\rho{}^\sigma F_\sigma{}^\rho) \\
& + \frac{g^2}{1008}(F_\mu{}^\nu F_\rho{}^\sigma F_\nu{}^\mu F_\sigma{}^\rho) - \frac{g^2}{252}(F_\mu{}^\nu F_\nu{}^\rho F_\rho{}^\sigma F_\sigma{}^\mu) \\
& + \frac{g^2}{210}(F_\mu{}^\nu F_\rho{}^\sigma F_\nu{}^\rho F_\sigma{}^\mu) + \frac{g^2}{210}(F_\mu{}^\nu F_\rho{}^\sigma F_\sigma{}^\mu F_\nu{}^\rho) \Bigg] .
\end{aligned}
\tag{4.23}
$$

This shows that, using the theorem (2.14), one can write down enough of the IGKF to find the order-$1/m^4$ contribution to the effective action without actually going through the IBP to find (4.20). Whether this is true for any number of gluons is not known.

However, the astute reader will notice that although (4.23) contains many terms of the form $\epsilon_i \cdot \epsilon_j k_j \cdot k_m D(m,n)$ from the structures (4.22), not all of these terms appear in (4.21). The reason for this is as follows: because each pinch *removes* a $k_i \cdot k_j$ factor, while expanding in k^2/m^2 *adds* factors of $k_r \cdot k_s$ with r, s not equal to i, j, it is possible for terms of this type to hide in the k^2/m^2 expansion of pinch contributions. I will show this, as well as other aspects of the pinching process, in the remainder of this section.

First, I should write the result for four gluons and a Dirac spinor loop, which is given by supersymmetrizing (4.20) and repeating the integrals. The new terms in the effective action are

$$
\begin{aligned}
\frac{g^2}{(4\pi)^2 m^4}\mathrm{Tr}\Bigg[& \frac{17ig}{315}[F^{\rho\sigma}(D_\rho F_{\mu\nu})(D_\sigma F^{\nu\mu})] + \frac{1}{70}[(D_\rho D_\sigma F_{\mu\nu})(D^\rho D^\sigma F^{\nu\mu})] \\
& + \frac{32ig}{105}[F_\mu{}^\nu(D_\alpha F_\nu{}^\rho)(D^\alpha F_\rho{}^\mu)] - \frac{g^2}{35}(F_\mu{}^\nu F_\nu{}^\mu F_\rho{}^\sigma F_\sigma{}^\rho) \\
& - \frac{17g^2}{630}(F_\mu{}^\nu F_\rho{}^\sigma F_\nu{}^\mu F_\sigma{}^\rho) - \frac{10g^2}{63}(F_\mu{}^\nu F_\nu{}^\rho F_\rho{}^\sigma F_\sigma{}^\mu) \\
& + \frac{11g^2}{70}(F_\mu{}^\nu F_\rho{}^\sigma F_\nu{}^\rho F_\sigma{}^\mu) + \frac{11g^2}{70}(F_\mu{}^\nu F_\rho{}^\sigma F_\sigma{}^\mu F_\nu{}^\rho)\Bigg]
\end{aligned}
\tag{4.24}
$$

Most of the remainder of this section will be devoted to studying the pinches of (4.20) and their expansions to $1/m^4$, so as to find all order-g^4 terms in (4.2) and (4.7), as well as those in (4.23) which did not appear in (4.21). There are a number of features which we may guess on simple grounds. For example, the double pinches, which come from diagrams like fig. 4, are order $g^4 \log(M^2/m^2)$, and are therefore the g^4 term in (4.2); the single pinches, from fig. 5 and related diagrams, are order g^4/m^2 and therefore come from the g^4 piece of (4.7).

Before going further, it is useful to define some shorthand notation: let $F^2|_2$, $F^2|_3$, $F^2|_4$ be the pieces of $F^{\mu\nu}F_{\nu\mu}$ which are quadratic, cubic and quartic in the gauge field; let $F^3|_3$, $F^3|_4$, *etc.* be similarly defined. We will also write $[OFO'F]_{m,n}$, where O and O' are gauge covariant operators coming from the tail(s) of an $(F_\mu{}^\nu)^2$ closed chain, to indicate the part of this structure involving m factors of the gauge field of which n factors come from $(F_\mu{}^\nu)^2$.

Now I turn to the explicit computation of the pinch contributions. To begin with, the only g^4 term at order $\log(M^2/m^2)$ is $F^2|_4$ from (4.2). Since it is divergent, it must come from a Bern-Kosower diagram whose loop has two legs, as in figure 4. Only D_4 factors from (4.20) contribute to double pinches; if we double-pinch

$D_4(4,3,2,1)\dot{G}_B^{43}\dot{G}_B^{32}\dot{G}_B^{21}\dot{G}_B^{14}$ using the diagram in figure 4, whose associated color trace is $\mathrm{Tr}[T^{a_4}T^{a_3}T^{a_2}T^{a_1}]$, we get

$$\begin{aligned}
&\int_0^\infty dT \int_0^1 du_4 \int_0^{u_4} du_3 \int_0^{u_3} du_2 \; \exp\Big(\sum_{i<j=1}^{N} k_i \cdot k_j G_B^{ji} - m^2 T \Big) \\
&\qquad\qquad \Big[\epsilon_3 \cdot \epsilon_2 \epsilon_4 \cdot \epsilon_1 \dot{G}_B^{14} \dot{G}_B^{32} \; \delta(u_4 - u_3)\delta(u_2 - u_1)/T^2 \Big] \qquad (4.25) \\
&= -\epsilon_3 \cdot \epsilon_2 \epsilon_4 \cdot \epsilon_1 \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_0^1 du \; [\dot{G}_B(u)]^2 \{1 + \mathcal{O}(k^2/m^2)\}
\end{aligned}$$

which has the correct coefficient and Lorentz structure to be part of $F^2|_4$. (In this and subsequent formulas I ignore the overall factor of $(ig)^4/(4\pi)^2$.) By permuting indices, one may easily show that the full set of double pinches gives the full structure of $F^2|_4$.

Eq. (4.7) implies the existence of several terms at order g^4/m^2. These are $F^3|_4$, $[DFDF]_{4,3}$, and $[DFDF]_{4,4} = [(\partial_\rho F_{\mu\nu})^2]|_4$, which all involve the non-abelian part of $F_\mu{}^\nu$, and $[DFDF]_{4,2} = g^2[A_\rho, F_{\mu\nu}][A^\rho, F^{\nu\mu}]|_{4,2}$, which contains factors of the form $D_2(m,n)$. On the other hand, only single pinches of (4.20) and the expansion of the double pinch (4.25) can give contributions at order $1/m^2$. If we pinch a factor $D_4(i,j,m,n)$, we get a term with a closed chain of three $\dot{G}_B$ functions, which can only come from $F^3|_4$; for example, pinching $D_4(4,3,2,1)$ with figure 5 gives

$$\int_0^\infty dT \int_0^1 du_4 \int_0^{u_4} du_3 \int_0^{u_3} du_2 \, \exp\Big(\sum_{i<j=1}^{N} k_i \cdot k_j G_B^{ji} - m^2 T\Big)$$
$$\times \Big[(\epsilon_1 \cdot k_4 \epsilon_3 \cdot \epsilon_4 \epsilon_2 \cdot k_1 - \epsilon_1 \cdot \epsilon_4 \epsilon_3 \cdot k_4 \epsilon_2 \cdot k_1$$
$$+ k_1 \cdot \epsilon_4 \epsilon_3 \cdot k_4 \epsilon_2 \cdot \epsilon_1 - k_1 \cdot k_4 \epsilon_3 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_1)\dot G_B^{14} \dot G_B^{43} \dot G_B^{21} \delta(u_3 - u_2)/T\Big]$$
$$=(\epsilon_1 \cdot k_4 \epsilon_3 \cdot \epsilon_4 \epsilon_2 \cdot k_1 - \epsilon_1 \cdot \epsilon_4 \epsilon_3 \cdot k_4 \epsilon_2 \cdot k_1$$
$$+ k_1 \cdot \epsilon_4 \epsilon_3 \cdot k_4 \epsilon_2 \cdot \epsilon_1 - k_1 \cdot k_4 \epsilon_3 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_1)$$
$$\times \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_0^1 du_4 \int_0^{u_4} du_2 \, \dot G_B^{14} \dot G_B^{42} \dot G_B^{21} \tag{4.26}$$

which is one contribution to $F^3|_4$ in (4.7).

If we pinch a term of the form $\epsilon_i \cdot k_j D_3(j, m, n)$, then we get a closed chain of length 2 which is not proportional to D_2; the $\epsilon_i \cdot k_j$ tail is unchanged. Such a term must contribute to $[DFDF]_{4,3}$, as may easily be checked.

The only other terms in (4.20) which contribute to pinches are terms like $\epsilon_i \cdot \epsilon_j k_j \cdot k_m D_2(m, n)$; the pinch leaves the D_2 factor unchanged, allowing us to identify the result as part of $[DFDF]_{4,2} = g^2 [A_\rho, F_{\mu\nu}][A^\rho, F^{\nu\mu}]|_{4,2}$. For example, pinching the $D_2(4,3)$ and $D_2(4,2)$ terms in (4.20) using figure 5 leaves

$$\int_0^\infty dT \int_0^1 du_4 \int_0^{u_4} du_3 \int_0^{u_3} du_2 \, \exp\Big(\sum_{i<j=1}^{N} k_i \cdot k_j G_B^{ji} - m^2 T\Big)$$
$$\times \Big\{ \Big[\epsilon_1 \cdot \epsilon_2 D_2(4,3) \dot G_B^{12} \dot G_B^{34} \dot G_B^{43} - \epsilon_1 \cdot \epsilon_3 D_2(4,2) \dot G_B^{13} \dot G_B^{24} \dot G_B^{42}\Big] \delta(u_3 - u_2)/T$$
$$= \Big[\epsilon_1 \cdot \epsilon_2 D_2(4,3) - \epsilon_1 \cdot \epsilon_3 D_2(4,2)\Big]$$
$$\times \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_0^1 du_4 \int_0^{u_4} du_2 \, \dot G_B^{24} \dot G_B^{42} \dot G_B^{21} \tag{4.27}$$

which is clearly part of $[A_\rho, F_{\mu\nu}][A^\rho, F^{\nu\mu}]$ with the coefficient function appropriate to $(D^\rho F^{\mu\nu})^2$ (see eqs. (4.5)-(4.9).)

The only remaining $1/m^2$ term, $[DFDF]_{4,4} = [(\partial_\rho F_{\mu\nu})^2]_4$, contains the quartic part of $(F_\mu{}^\nu)^2$, which requires a double pinch; it must therefore stem from the expansion of (4.25) to the next order in k^2/m^2, just as in (4.11) where $[(\partial_\rho F_{\mu\nu})^2]_3$ stems from the expansion of (4.10).

At order $1/m^4$, the expansion of the single pinches of the D_4 and $\epsilon_i \cdot k_j D_3$ terms clearly give $[FD_\alpha FD^\alpha F]_{4,4} = [F_\mu{}^\nu \partial_\alpha F_{\nu\rho} \partial^\alpha F^{\rho\mu}]_{4,4}$ and $[(D^\rho D^\sigma F)^2]_{4,3}$, while expansion of the double pinches of D_4 gives $[(D^\rho D^\sigma F)^2]_{4,4} = [(\partial^\rho \partial^\sigma F^{\mu\nu})^2]_{4,4}$.

However, the expansion of the pinches of $\epsilon_i \cdot \epsilon_j k_j \cdot k_m D_2(m,n)$ is complicated. Before pinching, these terms contribute to the structures $F^{\rho\sigma} D_\rho F_{\mu\nu} D_\sigma F^{\nu\mu}$ and $D_\rho D_\sigma F_{\mu\nu} D^\rho D^\sigma F^{\nu\mu}$. The pinch removes the factor of $k_j \cdot k_m$, but expanding in k^2/m^2 adds a factor of $k_r \cdot k_s$, with $\{r,s\} \neq \{j,m\}$. It is straightforward if tedious to show that all of the required terms from $F^{\rho\sigma} D_\rho F_{\mu\nu} D_\sigma F^{\nu\mu}$ and $D_\rho D_\sigma F_{\mu\nu} D^\rho D^\sigma F^{\nu\mu}$ are produced, though there is nothing elegant about the way in which this occurs. It is also at this point that ambiguities in the IBP procedure make their presence felt, as I will now demonstrate.

As an illustration, let us focus our attention on all terms in the effective action of the form $\epsilon_1 \cdot \epsilon_2 D_2(4,3) k_i \cdot k_j$ with color trace $\mathrm{Tr}[T^{a_4} T^{a_3} T^{a_2} T^{a_1}]$. The IBP procedure used above led to the following terms in (4.20):

$$-\epsilon_1 \cdot \epsilon_2 \dot{G}_B^{12} D_2(4,3) \dot{G}_B^{34} \dot{G}_B^{43} (k_2 \cdot k_3 \dot{G}_B^{23} + k_2 \cdot k_4 \dot{G}_B^{24}) \ . \tag{4.28}$$

For this color trace, only the $k_2 \cdot k_3$ term can be pinched, since gluons 2 and 4 are not adjacent. Carrying out the pinch, which comes from diagram 5, and expanding to the next order in k^2/m^2, one finds the contributions

$$\begin{aligned}&\epsilon_1\cdot\epsilon_2\dot{G}_B^{12}D_2(4,3)\dot{G}_B^{34}\dot{G}_B^{43}\delta(u_3-u_2)/T\sum_{j=1}^{N}k_i\cdot k_j\dot{G}_B^{ij}\\ =&\epsilon_1\cdot\epsilon_2\dot{G}_B^{13}D_2(4,3)\dot{G}_B^{34}\dot{G}_B^{43}\delta(u_3-u_2)/T\\ &[k_1\cdot(k_2+k_3)G_B^{13}+k_1\cdot k_4G_B^{14}+(k_2+k_3)\cdot k_4G_B^{34}]\ .\end{aligned}\tag{4.29}$$

Adding (4.28) and (4.29) and carrying out the integrals over the Feynman parameters, one finds an overall contribution of

$$\begin{aligned}&\frac{g^4}{(4\pi)^2m^4}\frac{1}{630}\epsilon_1\cdot\epsilon_2D_2(4,3)\\ &\qquad\Big[8(k_2\cdot k_3+k_1\cdot k_4)+3(k_2\cdot k_4+k_1\cdot k_3)+3k_1\cdot k_2+6k_3\cdot k_4\Big]\\ &+\dots\end{aligned}\tag{4.30}$$

which is in agreement with (4.23).

By contrast, if the factor $\ddot{G}_B^{12}$ in (4.15) is integrated-by-parts with respect to the index 1 instead of 2, then the IGKF contains the terms

$$+\epsilon_1\cdot\epsilon_2\dot{G}_B^{12}D_2(4,3)\dot{G}_B^{34}\dot{G}_B^{43}(k_1\cdot k_3\dot{G}_B^{13}+k_1\cdot k_4\dot{G}_B^{14})\ .\tag{4.31}$$

This expression is related to (4.28) *via* the exchange of labels $1\leftrightarrow2$, $3\leftrightarrow4$ and an overall minus sign; this minus sign is cancelled by another sign stemming from the change in the integration direction under the label interchange. Since the label interchange leaves (4.30) invariant, the process of pinching and expanding in k^2/m^2 leads to the same result as before.

Thus, the result (4.23) does not depend on the IBP procedure, even though (4.21) does depend upon it. The pattern with which this occurs is somewhat mysterious, and it would be helpful to understand this point better. The way to avoid running into any confusion is to study first the terms whose tails contain only $\epsilon_i \cdot k_j$ factors, since they are independent of the IBP procedure, are covered by the theorem (2.14), and, up to this order at least, are sufficient to specify the effective action.

The project is now complete: all order-g^4 terms in (4.2), (4.7) and (4.23) have been found in the Bern-Kosower formalism. To do the same using Feynman rules would be much more difficult, as the reader is encouraged to verify. This again suggests that the special organization of the Bern-Kosower Master Formula and the accompanying pinch rules gives the new technique advantages over standard Feynman diagrams.

At higher orders, the proliferation of gauge-invariant structures hinders the process of writing down the non-abelian effective action. In terms of Lorentz invariants, any amplitude may be straightforwardly computed; it is merely a matter of patience to write down all the terms and compute the polynomial integrals. It seems, however, that identifying the gauge-invariant structures being represented becomes rapidly more difficult at higher order in g, and an organizing principle is needed if a general all-orders analysis is to be carried out.

The work of this section shows that while the non-abelian effective action is much more complex than the abelian case, it is still possible to make relatively quick progress in comparison to what would be expected from Feynman diagrams, and to identify terms which do not occur for covariantly constant fields. Without the IBP procedure, this would be much more difficult, for the reason that at a given order in g, the terms at low order in $1/m^2$ are a mixture of spurious gauge-non-invariant terms, all of which cancel in intermediate steps, and terms which are completions of gauge invariant operators from lower order in g. Unless the IBP is performed one cannot tell these two types of terms apart. The pinch rules also make the analysis easier by employing Bern-Kosower diagrams, which separate terms according to their dependence on the particle mass, or, when the $k^2/m^2 \to 0$ limit is not taken, by their dependence on the length of the loop T.

5. Discussion and Conclusion

In this paper I have used some simple calculations to illustrate the power of the Bern-Kosower approach. The calculations can be carried out from Feynman diagrams but are clearly easier in the new technique. In contrast to previous computations using the Bern-Kosower rules, this can be verified by any field theorist in an hour or two, and skeptical readers are encouraged to try these calculations themselves.

One crucial point should be kept in mind. These computations were performed in the simplifying limit of low momentum. However, *the cancellations inherent in the IBP procedure do not depend on this limit.* The advantage that spurious non-gauge-invariant terms are removed and the number of integrands and diagrams is reduced is a general feature of the Bern-Kosower rules. This has importance not only in reducing the amount of work in analytic calculations but also in avoiding dangerous cancellations of large spurious terms which occur in numerical evaluation of Feynman diagrams. We still do not know why the IBP procedure works this way, nor do we have a proof of the off-shell pinch rules given in Appendix C and used in Sec. 4.

Finally, I want to stress that the Bern-Kosower rules are not simply equivalent to writing all Feynman diagrams as Feynman parameter integrals. This is the technique used in [6, 7] and also, for comparative purposes, in [9]. Such a technique does not lend itself to the sort of simple analysis used in this paper, *because the results it leads to are not organized by Lorentz invariants and one-dimensional propagators.* Only with the special organizing principle of first-quantization does the IBP become a natural procedure, and without the IBP none of the results of this paper would have been possible. It should also be stressed that without the IBP the simple Bern-Kosower scattering amplitude rules [1,2] would also be impossible. In conclusion, it is the special gauge invariant organization of the Bern-Kosower Master Formula which distinguishes the Bern-Kosower rules from all previous techniques.

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APPENDIX A: Formulas needed to derive recursion relations

In this appendix I compute the integrals

$$\prod_{i=2}^{m}\Big(\int_0^1 du_i\Big)\dot{G}_B^{1m}\dot{G}_B^{m,m-1}\cdots\dot{G}_B^{32}\dot{G}_B^{21}\ ,\ \prod_{i=2}^{m}\Big(\int_0^1 du_i\Big)G_F^{1m}G_F^{m,m-1}\cdots G_F^{32}G_F^{21}.$$

As always I have set $u_1 = 0$.

For this, I use

$$\int_0^1 du_i\ (\sum_{k=0}^{\infty} a_k u_i^k)\dot{G}_B^{ji} = \int_0^1 du_i\ (\sum_{k=0}^{\infty} a_k u_i^k)[\mathrm{sign}(u_{ji}) - 2u_{ji}] =$$
$$= \sum_{k=0}^{\infty}\Big[2a_k(\frac{u_j^{k+1}}{k+1} + \frac{1}{k+2}) - (1-2u_j)\frac{a_k}{k+1}\Big]$$

where $u_{ji} \equiv u_j - u_i$. If we assume $\sum_0^{\infty}\frac{a_k}{k+1} = 0$ then we find

$$\int_0^1 du_i\ (\sum_0^{\infty} a_k u_i^k)\dot{G}_B^{ji} = \sum_0^{\infty} a'_k u_j^k\ ;$$
$$a'_k = \frac{2a_{k-1}}{k},\ k>0\ ;\ a'_0 = \sum_0^{\infty}\frac{a_k}{k+2}\ .$$

which also satisfies $\sum_0^{\infty}\frac{a'_k}{k+1} = 0$. Using this result, along with $\dot{G}_B^{21} = -2u_2 + 1$, it is easy to derive the recursion relation (3.21).

Similarly,

$$\int_0^1 du_i\,(\sum_{k=0}^{\infty} c_k u_i^k) G_F^{ji} = \int_0^1 du_i\,(\sum_{k=0}^{\infty} c_k u_i^k)\mathrm{sign}(u_{ji}) =$$

$$= \sum_{k=0}^{\infty}\Big[\frac{2c_k u_j^{k+1}}{k+1} - \frac{c_k}{k+1}\Big]$$

$$= \sum_0^{\infty} c'_k u_j^k\ ;$$

$$c'_k = \frac{2c_{k-1}}{k},\ k > 0\ ;\ c'_0 = -\sum_0^{\infty}\frac{c_k}{k+1}\ .$$

Using $G_F^{21} = 1$, this leads to (3.24).

APPENDIX B: Expressing $(F_\mu{}^\nu)^p$ in terms of electromagnetic fields

It is easy to write $(F_\mu{}^\nu)^p$ as a function of $S \equiv E^2 - B^2 = \frac{1}{2}\ F_{\mu\nu}F^{\nu\mu}$ and $P \equiv E\cdot B = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$, using the generating formula

$$[(F_\mu{}^\nu)^4]_\alpha^{\ \beta} \equiv F^{\alpha\lambda}F_{\lambda\rho}F^{\rho\sigma}F_{\sigma\beta} = SF^{\alpha\gamma}F_{\gamma\beta} + P^2 g_\alpha^{\ \beta}. \tag{B.1}$$

If we write

$$[(F_\mu{}^\nu)^p]_\alpha^{\ \beta} \equiv A_p F^{\alpha\gamma}F_{\gamma\beta} + C_p g_\alpha^{\ \beta}. \tag{B.2}$$

so that $(F_\mu{}^\nu)^p = 2SA_p + 4C_p$, then it is easy to see that

$$A_p = SA_{p-2} + C_{p-2}\ ,\ C_p = P^2 A_{p-2}\ . \tag{B.3}$$

Starting from $A_0 = 0$, $C_0 = 1$, one may generate the entire sequence. For example, $\mathrm{Tr}(F_\mu{}^\nu)^4 = 2S^2 + 4P^2$, and $\mathrm{Tr}(F_\mu{}^\nu)^6 = 2S^3 + 6SP^2$.

APPENDIX C: Pinch Rules

In the following paragraphs, taken directly from Paper A, I explain how to compute the pinch terms at order-g^N. (These pinch rules have not yet been proven to all orders.)

Draw all (planar) ϕ^3 graphs with one loop, N external legs and any number N_T of trees, such that although each tree may have several vertices, the total number of tree vertices N_V is at most $N/2$. (Diagrams with trees may seem out of place in the construction of a 1PI object like an effective action, but the trees used here, unlike those for scattering amplitudes, do not contribute the usual propagator poles; they serve only as a mnemonic for ensuring all surface terms are accounted for.) The gluons which flow into a tree before entering the loop are said to be pinched; the number of these is $N_V + N_T$.

Consider a particular graph and a particular color(path)-ordering; label the external legs clockwise from 1 to N following the path-ordering. Each tree vertex, since it is a three-point vertex, is characterized by one line pointing toward the loop and two outward pointing lines I and J, with two sets of external legs $i_1, ..., i_m$ and $j_1, ..., j_n$ that flow into them. Let J be the line lying most clockwise. Now examine the improved generating kinematic factor term by term. If a given term does not contain a factor $k_i \cdot k_j \dot{G}_B^{ji}$ or $k_i \cdot k_j G_F^{ji}$ for *each* tree vertex, where i belongs to the set of gluons flowing into line I and j flows into J, then it vanishes. Even then, it must contain exactly *one* $\dot{G}_B^{ji}$ or G_F^{ji} at each vertex; otherwise it vanishes. If it survives, then replace $\dot{G}_B^{ji}$ or G_F^{ji} by $+1$, replace $t_i \to t_j$ in all Green functions, and eliminate the t_i integral. Finally, for every internal tree line (into which flows momentum from gluons $r, r+1, \ldots, s$), divide by

$$\frac{1}{2}\Big[(\sum_{q=r}^{s} k_q)^2 - \sum_{q=r}^{s}(k_q)^2\Big], \tag{C.1}$$

which becomes the expected intermediate-state pole only when all external gluons are on-shell. The effect of this procedure is to produce contact terms; no actual poles are ever generated.

APPENDIX D: Proof of theorem (2.14).

In this appendix I will prove the theorem (2.14), using several lemmas as stepping stones. A reader finding the presentation too dry may wish to study eqs. (4.14) through (4.20) in which the lemmas are illustrated by example. The reader should note that I have ignored most signs in proving the lemmas; they play no role in most of the proofs and merely clutter the discussion. The skeptic may easily add them in.

In Sec. 2 the definitions of *closed chains*, *tails*, and $D_p(i_p, \cdots, i_2, i_1)$ were given. We will need some further terminology and notation in this appendix.

A chain which contains a single $\epsilon_i \cdot \epsilon_j \ddot{G}_B^{ij}$ factor and does not close on itself is called an open chain. An example is

$$\epsilon_5 \cdot k_1 \epsilon_1 \cdot \epsilon_3 k_3 \cdot \epsilon_2 \dot{G}_B^{51} \ddot{G}_B^{13} \dot{G}_B^{32} \ , \tag{D.1}$$

denoted [5(13)2]. Open chains may also have branches, as in $|51\rangle$ [41(32)67]:

$$\dot{G}_B^{51} \dot{G}_B^{41} \dot{G}_B^{13} \ddot{G}_B^{32} \dot{G}_B^{26} \dot{G}_B^{67} \ . \tag{D.2}$$

which may also be denoted $|41\rangle$ [51(32)67]. We may separate the indices of an open chain into those attaching to one index of the $\ddot{G}_B$ function and those attaching to the other; the two groups may be called the two *wings* $W(i)$ and $W(j)$ of the open chain. The wings in (D.2) are the sets of indices $W(3) = (5,4,1,3)$ and $W(2) = (7,6,2)$.

We will see that closed chains with tails and open chains are the only forms that can arise in the GKF before integration by parts. The IBP procedure eliminates all open chains leaving only closed chains with tails.

To get from the GKF to the IGKF one carries out an IBP of all factors of $\ddot{G}_B^{ij}$. The IBP of an open chain leads to the replacement

$$\ddot{G}_B^{ij} \to -\dot{G}_B^{ij} \sum_{m\in W(i)} \sum_{r=1}^{N} k_m \cdot k_r \dot{G}_B^{mr} \ , \tag{D.3}$$

where $W(i)$ is the set of indices in the wing containing index i, or

$$\ddot{G}_B^{ij} \to +\dot{G}_B^{ij} \sum_{m\in W(j)} \sum_{r=1}^{N} k_m \cdot k_r \dot{G}_B^{mr} \ , \tag{D.4}$$

or a linear combination of the two. (Recall $\dot{G}_B^{mm} = 0$.) This is easily proved by noting

$$\int_0^1 du_i \ddot{G}_B^{ij} \dot{G}_B^{ki} \exp\left[\sum_{r<s}^{N} k_r \cdot k_s G_B^{rs}\right] =$$
$$\int_0^1 du_i \Big(-\dot{G}_B^{ij} \dot{G}_B^{ki} \sum_{r=1}^{N} k_i \cdot k_r \dot{G}_B^{ir} + \dot{G}_B^{ij} \ddot{G}_B^{ki}\Big) \exp\left[\sum_{r<s}^{N} k_r \cdot k_s G_B^{rs}\right] \tag{D.5}$$

Every term $C_0 \in$ GKF contains $0 \le n \le N/2$ factors of $\epsilon_i \cdot \epsilon_j \ddot{G}_B^{ij}$ and $N-2n$ factors of $\epsilon_i \cdot k_j \dot{G}_B^{ij}$; (D.3) and (D.4) show that after the complete IBP every term

$C \in$ IGKF will contain n factors of $\epsilon_i \cdot \epsilon_j \dot{G}_B^{ij}$, n factors of $k_i \cdot k_j \dot{G}_B^{ij}$, and $N - 2n$ factors of $\epsilon_i \cdot k_j \dot{G}_B^{ij}$. C thus contains N $\dot{G}_B$ functions along with N dot products of the N polarization vectors and N not necessarily distinct momentum vectors.

The *descendants* of a term $C_0 \in$ GKF are the terms C formed from C_0 during the IBP; the *ancestor* of the terms C is C_0. The ancestor C_0 of any term $C \in$ IGKF may be found by *disintegrating* C, namely, by removing all factors of $k_i \cdot k_j \dot{G}_B^{ij}$ and replacing all factors of $\epsilon_i \cdot \epsilon_j \dot{G}_B^{ij}$ by $\epsilon_i \cdot \epsilon_j \ddot{G}_B^{ij}$ in C.

Lemma 1:

Any term $C \in$ GKF has the property that every index is either in a unique open chain, a unique closed chain, or a unique tail of a closed chain.

Proof:

Choose an index i and a term $C \in$ GKF. The polarization vector ϵ_i appears once in C, either in a factor $\epsilon_i \cdot \epsilon_j \ddot{G}_B^{ij}$ or in a factor $\epsilon_i \cdot k_j \dot{G}_B^{ij}$. In the latter case one may *move forward* in the chain containing i and j by identifying the term $\epsilon_j \cdot \epsilon_m \ddot{G}_B^{jm}$ or $\epsilon_j \cdot k_m \dot{G}_B^{jm}$ in which ϵ_j appears. I will say that j is the *index ahead* of i, while m is two steps ahead of i. Since each ϵ appears only once in C, the definition of the index ahead is unambiguous. If by moving forward one arrives at a term $\epsilon_m \cdot \epsilon_n \ddot{G}_B^{mn}$ then one cannot move further forward, since ϵ_n has already appeared in C; otherwise one may move forward indefinitely. However, since the number of indices is finite, it is obvious that either index i is in a chain containing an $\epsilon_m \cdot \epsilon_n \ddot{G}_B^{mn}$ factor, or it is in a chain which closes somewhere on itself, in which case i is either in a closed chain $\{ij \cdots n\}$ or in a tail of a closed chain $|\cdots ij \cdots n\rangle \, \{n \cdots s\}$.

The notion of *moving backward* is also useful, although there may be any number of *indices behind.* Given an index i, with $\epsilon_i \cdot \epsilon_j \ddot{G}_B^{ij}$ or $\epsilon_i \cdot k_j \dot{G}_B^{ij}$ in C, one moves backward by identifying all factors in C with $\epsilon_h \cdot k_i \dot{G}_B^{hi}$; the set of indices h are the indices behind i. The number of such indices may be anywhere from zero to $N-1$. If the number is zero, then the chain ends; otherwise the chain continues with one or more branches, and one may continue moving backward. Of course one cannot move backward from $\epsilon_i \cdot k_j \dot{G}_B^{ij}$ into a factor $\epsilon_h \cdot \epsilon_i \ddot{G}_B^{hi}$, since ϵ_i appears only once in C.

Choose an index i with $\epsilon_i \cdot k_j \in C$ and begin creating a chain by moving backward. After p steps in this process, consider an arbitrary index g which is p indices beind i. Let us try to move backward one more step; what may happen? There may be no indices behind g, in which case its branch of the chain stops. There may be one or several indices behind g, none of which have previously appeared in the chain, in which case the chain continues as before; however for sufficiently large p this option becomes impossible since the number of indices is finite. The third option is for there to be an index f behind g which has previously occurred in the chain, which implies $\epsilon_f \cdot k_g \in C$. However, if f is in the chain then either ϵ_f has already appeared or $f = j$, in which case $\epsilon_j \cdot k_g \dot{G}_B^{jg}$ is in C and the chain is closed. We therefore conclude that the chain containing ϵ_i either closes on itself so that i is in the closed chain $\{g \cdots ij\}$, or ends with one or more indices g each with $\epsilon_g \cdot k_m \in C$ for some m and $\epsilon_n \cdot k_g \notin C$ for any n, as in $|g \cdots ij\rangle$ or $|g' \cdots h\rangle\, |g \cdots h \cdots ij\rangle$.

If instead we choose an index i with $\epsilon_i \cdot \epsilon_j \in C$ and begin creating a chain by moving backward, we find the arguments of the previous paragraph are unchanged except that since ϵ_j has already appeared in C the chain cannot close on itself;

thus the chain must end as in $[g \cdots (ij) \cdots]$.

From these considerations it follows easily that any chain in the GKF containing an $\epsilon_i \cdot \epsilon_j \ddot{G}_B^{ij}$ factor is an open chain, and that any other chain closes on itself exactly once, forming a closed chain with one or more tails. This is enough to prove lemma 1.

Lemma 2:

Any term $C \in$ IGKF has the property that every index is either in a unique closed chain or in a unique tail of a closed chain.

Proof:

From lemma 1, a term $C_0 \in$ GKF consists of open chains V, closed chains D and tails of closed chains T. Consider the effect of the replacement (D.3) in an open chain V_0, and for any $m \in V_0$ consider the possibilities for r. If r is an index in V_0, then a closed chain is formed, and every index in the formerly open chain is in the closed chain or in a tail connected to it. If r is in a different open chain V, then V_0 and V join to form a new open chain. If r is in a closed chain D, then V_0 becomes a tail of D. If r is in a tail T of a closed chain D, then V_0 becomes a branch of T. In all four cases the resulting term has the property that every index is either in a unique open chain, a unique closed chain, or a unique tail of a closed chain. We can now repeat the process until all open chains have been removed, at which point lemma 2 is obvious.

Lemma 3:

When new closed chains are formed in the IBP of an open chain $V = [\cdots (ij) \cdots]$ they always include the indices i and j, and the same closed chains are formed what-ever the choice of index for the IBP. Furthermore, if $V = [\cdots a \cdots b \cdots (ij) \cdots]$, then any closed chain formed containing index a also contains b.

Proof:

Let us IBP using (D.3) (for definiteness) and try to make a closed chain by using $r \in V$. If $r = n \in W(i)$, no closed chains are formed, because both $k_m \cdot k_n \dot{G}_B^{mn}$ and $k_n \cdot k_m \dot{G}_B^{nm}$ appear in (D.3), and they cancel. If $r \in W(j)$, however, a closed chain $\{m \cdots ij \cdots r\}$ is formed which includes indices i and j. The full set of chains is

$$\sum_{m \in W(i)} \sum_{r \in W(j)} \{m \cdots (ij) \cdots r\} \ . \tag{D.6}$$

If instead we use (D.4), or even a linear combination of (D.3) and (D.4), we get the same set of closed chains. The last part of the lemma is obvious from (D.6).

Lemma 4:

If, at any stage of the IBP, a term C contains an open chain $V = [\cdots (mn) \cdots]$, and $i, j \in W(m)$ or $i, j \in W(n)$, then any closed chain in any descendant of C containing the indices i, j must also contain the indices m, n.

Proof:

Let us assume the contrary, and try to construct a closed chain d containing i, j but not m, n. Let us take $i, j \in W(m)$. Lemma 3 prevents us from forming a chain of this type from V alone, but perhaps we may do so by combining V with another open chain $V' = [\cdots (ab) \cdots]$. Let us consider various choices for the IBP. If we first IBP using index a (or b), we fail, since a term $k_r \cdot k_p$, $r \in W(a)$ and $p \in V = W(i) + W(j)$ creates an open chain $[\cdots (mn) \cdots]$ with $i, j \in W(m)$, to which lemma 3 applies. If instead we IBP using index n, then from the term $k_p \cdot k_r$, $p \in W(n)$ and $r \in V'$ we get an open chain $[\cdots(ab)\cdots]$ with i, j, m, n in the same wing but with m, n lying between i and (ab) and between j and (ab); from lemma 3 any closed chain with i and/or j contains m, n as well.

If we IBP with respect to m, however, the term $k_i \cdot k_r$, $r \in W(a)$ creates an open chain $[\cdots mn \cdots ir \cdots (ab) \cdots]$ with i, j, m, n in $W(a)$ but with m, n further from (ab) than i; the IBP with repect to a or b permits a factor $k_s \cdot k_j$, where $s \in W(b)$, to close the chain, potentially leaving out indices m, n. However, there is another term generated by the same IBP with the indices i, r, a exchanged with j, s, b. The factor $\dot{G}_B^{ab}$ changes sign under this replacement, and so the two occurrences of this chain cancel; thus the lemma holds in this case.

The argument may be continued essentially unchanged for cases involving several open chains.

Lemma 5:

In the IGKF, the kinematic factor in every closed chain of length p contains p dot products of the p polarization vectors and the p momentum vectors which carry its indices, and is therefore a term in D_p, as defined in (2.8) and (2.9). Any other closed chain formed in an intermediate step of the IBP will not occur in the IGKF.

Proof:

If a closed chain is formed only out of factors of $\epsilon_i \cdot k_j$, then the lemma is self-evident. If not, lemma 4 ensures that for every factor of $k_r \cdot k_s$ inserted to help close a chain, a factor $\epsilon_i \cdot \epsilon_j$ occurs as well; since the remainder of the chain is made of $\epsilon_m \cdot k_n$ factors, the lemma holds.

Lemma 6:

Suppose a term C consists of a product of closed chains

$$C = \prod d_p(i_p, \ldots, i_2, i_1)\delta_p(i_p, \ldots, i_2, i_1) \tag{D.7}$$

where d_p is a term in D_p as defined in (2.8) or (2.9), and $\delta_p(n, m \ldots, j, i)$ is defined in (2.10), such that C contains N $\dot{G}_B$ functions and all N polarization vectors appear once in C. Then the ancestor of C is a term $C_0 \in$ GKF, and C appears exactly once in the IGKF. As a corollary, if any such term appears in the IGKF, then all of $\prod D_p$ occurs in the IGKF.

Proof:

The assumptions about C imply that C has N Lorentz dot products of $2N$ vectors of which N are the polarization vectors and N are momentum vectors. Disintegrating C leads therefore to a term C_0 with $0 \leq n \leq N/2$ factors of $\epsilon_i \cdot \epsilon_j \ddot{G}_B^{ij}$ and $N - 2n$ factors of $\epsilon_i \cdot k_j \dot{G}_B^{ij}$; all such terms are in the GKF.

Now we must show that C is a descendant of C_0.

Consider first the case when C consists of a single chain $d_N = \{m_1 m_2 \ldots m_N\}$. If C contains n factors of $\epsilon_i \cdot \epsilon_j \dot{G}_B^{ij}$, then C_0 will consist of a set of n open chains whose wings span the closed chain except for n gaps:

$$\begin{gathered}
[m_1 m_2 \ldots (m_i m_{i+1}) \ldots m_r] \\
[m_{r+1} m_{r+2} \ldots (m_j m_{j+1}) \ldots m_s] \\
[m_{s+1} m_{s+2} \ldots (m_k m_{k+1}) \ldots] \\
\ldots [\ldots m_N]
\end{gathered} \tag{D.8}$$

If we IBP with respect to index m_{j+1}, then $\ddot{G}_B^{j,j+1}$ is replaced with a sum of terms as in (D.4); the factor $k_{m_s} \cdot k_{m_{s+1}}$ appearing in the sum is special in that it closes the gap between m_s and m_{s+1}, leaving a term with $n-1$ open chains and $n-1$ gaps:

$$
\begin{aligned}
&[m_1 m_2 \ldots (m_i m_{i+1}) \ldots m_r] \\
&[m_{r+1} m_{r+2} \ldots m_j m_{j+1} \ldots m_s m_{s+1} m_{s+2} \ldots (m_k m_{k+1}) \ldots] \qquad \text{(D.9)} \\
&\ldots [\ldots m_N]
\end{aligned}
$$

Similarly, if we IBP with respect to index m_j there is a unique term in the sum which closes the gap between m_r and m_{r+1}:

$$
\begin{aligned}
&[m_1 m_2 \ldots (m_i m_{i+1}) \ldots m_r m_{r+1} m_{r+2} \ldots m_j m_{j+1} \ldots m_s] \\
&[m_{s+1} m_{s+2} \ldots (m_k m_{k+1}) \ldots] \qquad \text{(D.10)} \\
&\ldots [\ldots m_N]
\end{aligned}
$$

Thus, the terms stemming from the IBP of $\epsilon_i \cdot \epsilon_j \ddot{G}_B^{ij} \in C_0$ with respect to i include a unique term in which the gap at the end of the wing $W(i)$ is closed by the same $k_r \cdot k_s$ factor that was removed in the disintegration of C. Each of the n integrations-by-parts closes one of the n gaps in C_0; after the last step the factor d_N remains, with the correct closed chain of $\dot{G}_B$ factors, and thus no matter what IBP procedure is used the chain d_N is created in a unique way from C_0.

If C consists of a product of chains, then, since no index appears in more than one chain, the same arguments may be applied to each chain independently. This proves the lemma.

Lemma 7:

Suppose a term C consists of a product of closed chains, as in the previous lemma, and some tails T, all of which are products of $\epsilon_i \cdot k_j \dot{G}_B^{ij}$ factors, such that C contains N $\dot{G}_B$ functions and all N polarization vectors appear once in C. Then the ancestor of C is a term $C_0 \in$ GKF, and C appears exactly once in the IGKF. As a corollary, if any such term is in the IGKF, then all of $T \prod D_p$ appears in the IGKF.

Proof:

Here $C = dT$, where d is the product of the closed chains in C and T consists of all the tails of d. Since T is unchanged by disintegration of C, it follows that $C_0 = d_0 T$, where d_0 is the ancestor of d. As in lemma 6 it is clear $C_0 \in$ GKF. The arguments of lemma 6 apply to d_0, since the terms used in reconstructing d from d_0 do not involve any part of T; again there is a unique way to construct d from d_0 and hence C from C_0.

Lemma 8:

Take any structure $K = T \prod D_p \delta_p$, where $\prod D_p \delta_p$ is a product of closed chains, and where T is a set of tails, each of which is a product of zero or more $\epsilon_i \cdot k_j \dot{G}_B^{ij}$ factors, such that all N polarization vectors appear once in K and each term in K contains N Lorentz dot products. There is at least one term in K which appears in the IGKF, with an overall factor $\pm i^N$.

Proof:

Consider a term in $d \in \prod D_p \delta_p$ which consists only of $\epsilon_i \cdot k_j \dot{G}_B^{ij}$ factors. (That such a term exists follows from (2.8) and (2.9); note it occurs with a factor ± 1.)

Since the terms (3.19) always appear in the GKF, and are unchanged by the IBP, the term Td is in the IGKF with an overall factor of $\pm i^N$.

The theorem now follows from combining lemmas 5 through 8.

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FIGURE CAPTIONS

1) Feynman diagrams associated with the photon vacuum polarization.

2) The three types of Feynman diagrams contributing to the four-photon effective action.

3) A Bern-Kosower diagram associated with a single pinch in the three-gluon effective action.

4) A Bern-Kosower diagram associated with a double pinch in the four-gluon effective action.

5) A Bern-Kosower diagram associated with a single pinch in the four-gluon effective action.

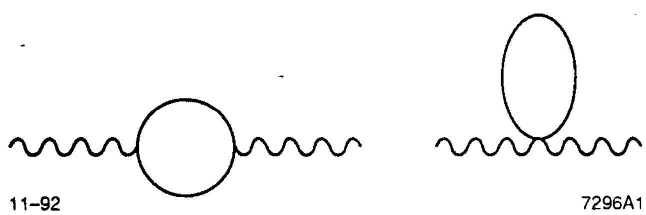

Fig. 1

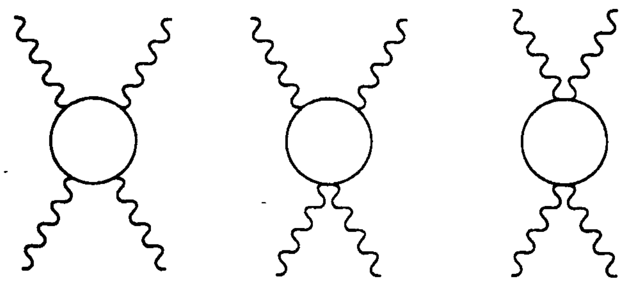

Fig. 2

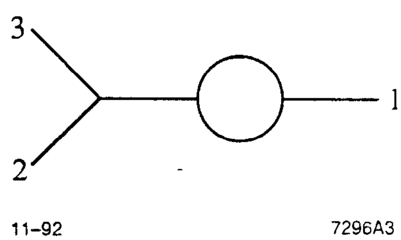

Fig. 3

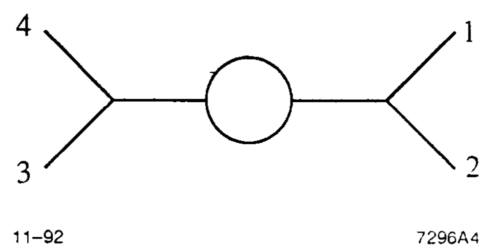

Fig. 4

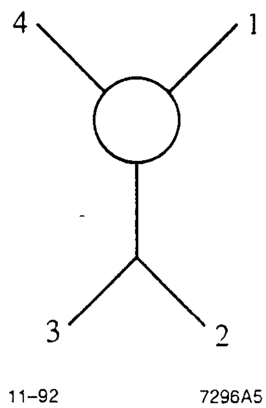

Fig. 5