Analyticity and the Isgur-Wise Function*

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Abstract

We reconsider the recent derivation by de Rafael and Taron of bounds on the slope of the Isgur-Wise function. We argue that one must be careful to include cuts starting below the heavy meson pair production threshold, arising from heavy quark-antiquark bound states, and that if such cuts are properly accounted for then no constraints may be derived.

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It has recently been argued by de Rafael and Taron [1] that one may use analytic properties to derive constraints on the Isgur-Wise function $\xi(w = v \cdot v')$ [2] of the heavy quark effective theory (HQET). These constraints have provoked much interest, because they are in conflict with almost all extractions of $\xi(w)$ from fits to experimental data on semileptonic $B \to D^*$ decays [3], as well as with estimates from QCD sum rules [4] and potential models [5]. Hence it is very important to understand whether the derivation of these bounds is correct. We have reconsidered this issue, and we have found that the careful inclusion of heavy quark-antiquark bound states, lying below the heavy meson pair production threshold, invalidates the argument. In fact, we argue that it is not possible to obtain any such constraints at all.

We begin by reviewing carefully the derivation of de Rafael and Taron, paying particular attention to a number of delicate points which are somewhat glossed over in their analysis. We will then present two counterexamples, namely Isgur-Wise functions which we argue are reasonable in certain physical limits of the theory, but which violate the bounds which are derived. Finally, we will show precisely at which point the argument fails, and argue that the failure can be understood both on technical and physical grounds.

For concreteness, we will follow ref. [1] and take the heavy quark to be the b quark. Then the physical form factor $F(q^2)$ which is of interest is given by the matrix element of the vector current $V^\mu = \bar{\mathrm{b}}\gamma^\mu \mathrm{b}$ between B-meson states:

$$\langle B(p')|\, V^\mu \,|B(p)\rangle = (p+p')^\mu F(q^2)\,. \tag{1}$$

For $q^2 \leq 0$, this is the elastic form factor, corresponding to the kinematic region described in the HQET by the Isgur-Wise function $\xi(w)$. For $q^2 \geq 4m_B^2$, $F(q^2)$ describes s-channel production of $B\bar{B}$ pairs. Conservation of b-number in QCD yields the normalization condition $F(0) = 1$; the question is whether $F(q^2)$, and hence the Isgur-Wise function, can be further constrained in the physically interesting elastic region $q^2 \leq 0$.

One begins the argument by considering the two-point function

$$\Pi(q^2)\,(q^\mu q^\nu - g^{\mu\nu} q^2) = \mathrm{i} \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}q\cdot x}\, \langle 0|\, T\,\{V^\mu(x), V^\nu(0)\}\,|0\rangle\,. \tag{2}$$

The derivative of $\Pi(q^2)$ satifies an unsubtracted dispersion relation ($Q^2 = -q^2$),

$$\chi(Q^2) \equiv -\frac{\partial \Pi(Q^2)}{\partial Q^2} = \frac{1}{2\pi \mathrm{i}} \int_C \frac{\Pi(t)}{(t+Q^2)^2}\,, \tag{3}$$

where the contour C runs below and then above the physical cut on the real axis, closed by a circle at $|t| = \infty$. In the real world, this cut starts at the mass of the lightest state which couples to the current V^μ. Ignoring weak and electromagnetic interactions, this is at $t = 4m_\pi^2$, corresponding to the annihilation of the current into two pions. It is convenient, however, to suppress annihilation diagrams, by imagining that V^μ is actually a flavor-changing current, between two different, but degenerate, heavy quarks. Since these fields are subject to a heavy quark flavor symmetry, none of the low-energy properties of the theory, in particular no properties of the Isgur-Wise function, are affected by this change. But now the cut in the two-point function starts with a pole at the mass of the lightest (and stable) "Υ" state, $t = m^2_{\text{``}\Upsilon\text{''}}$, followed by a series of cuts associated with the production of additional hadrons. There is then a cut starting at $t = 4m_B^2$ corresponding to pair production of $B\bar{B}$ mesons, and so forth.

There is no contribution from the circle at infinity; then the integral may be expressed in terms of the discontinuity across the cut, plus an additional contribution from the "Υ" pole,

$$\chi(Q^2) = \chi_{\rm pole}(Q^2) + \int_{(m_{\text{``}\Upsilon\text{''}}+2m_\pi)^2}^{\infty} {\rm d}t\, \frac{{\rm Im}\,\Pi(t)}{\pi(t+Q^2)^2}\,. \tag{4}$$

Because of the optical theorem, the value of ${\rm Im}\,\Pi(t)$ is proportional to the cross-section for production of on-shell states with invariant mass $\sqrt{t}$. All physical states which couple to the current V^μ contribute to ${\rm Im}\,\Pi(t)$. Dividing these states into those lying above and below the continuum $B\bar{B}$ threshold, we write

$$\chi(Q^2) = \chi_{\rm pole}(Q^2) + \int_{(m_{\text{``}\Upsilon\text{''}}+2m_\pi)^2}^{4m_B^2} {\rm d}t\, \frac{{\rm Im}\,\Pi(t)}{\pi(t+Q^2)^2} + \int_{4m_B^2}^{\infty} {\rm d}t\, \frac{{\rm Im}\,\Pi(t)}{\pi(t+Q^2)^2}\,. \tag{5}$$

The first two terms in eq. (5) are positive; if we estimate them as a sum over perturbative Coulomb bound states, they are proportional to $\alpha_s^3(m_{\rm b})$ They will not play an important role in the analysis. The third term is bounded from below by the contribution of the $B\bar{B}$ final states. (Note that there is *no* relation between ${\rm Im}\,\Pi(t)$ and $|F(t)|^2$ in the physically inaccessible region $t < 4m_B^2$. This is why we must make the decomposition (5).) The first two terms are positive (and small) and so, performing an integral over the available phase space, one then obtains

$$16\pi^2 m_B^2 \chi(Q^2) \geq \frac{1}{12}\int_1^\infty {\rm d}\,y\, \frac{(y-1)^{3/2}}{y^{3/2}(y+Q^2/4m_B^2)^2}\,|F(y)|^2\,, \tag{6}$$

where $y = t/4m_B^2$.

In the strict limit $m_{\rm b} \to \infty$, the form factor $F(y)$ is analytic everywhere except along the cut $y \geq 1$. (For finite $m_{\rm b}$ this is no longer true, and soon we will see that one must treat this issue with care.) If the integral around the end of the cut can be neglected, the integral in (6) may be rewritten as a contour integral, running from $y = \infty$ to $y = 1$ along the bottom of the cut, back to $y = \infty$ along the top, and closing with a circle at $|y| = \infty$. Then the inequality (6), plus the three conditions

(i) $F(y)$ analytic everywhere inside the contour,

(ii) the normalization $F(0) = 1$,

(iii) $-Q^2 < 4m_B^2$ real,

allow one to derive constraints on $F(y)$, in particular in the elastic region $y \leq 0$ [6]. This analysis is performed most conveniently after the change of variables $\sqrt{y-1} = {\rm i}\,\frac{1+z}{1-z}$, under which the interior of the contour is mapped onto the open unit disk, and the elastic region $-\infty < y \leq 0$ onto the real axis $0 \leq z < 1$. The constraints take the form $F_-(z) \leq F(z) \leq F_+(z)$, where

$$
\begin{aligned}
F_\pm(z) =& \frac{1}{(1+z)^2\sqrt{1-z}} \left(\frac{1+z+(1-z)\sqrt{1+Q^2/4m_B^2}}{1+\sqrt{1+Q^2/4m_B^2}} \right)^2 \\
&\times \left\{ 1 \pm \sqrt{\frac{z^2}{1-z^2}} \sqrt{6144\,\pi\, m_B^2\, \chi(Q^2) \left(\frac{1+\sqrt{1+Q^2/4m_B^2}}{2} \right)^4 - 1} \right\} .
\end{aligned}
\tag{7}
$$

If we choose $|Q^2| \gg \Lambda_{\rm QCD}^2$, we may approximate $\chi(Q^2)$ by its value in perturbation theory,

$$
\chi(Q^2) = \frac{3}{4\pi^2} \int_0^1 {\rm d}x\, \frac{2x^2(1-x)^2}{m_{\rm b}^2 + x(1-x)Q^2} + \mathcal{O}(\alpha_s(Q^2)) . \tag{8}
$$

Then expression (7) simplifies to a given function of z and Q^2.

Thus one obtains constraints on $F(z)$ for any value of Q^2 such that the perturbative calculation of $\chi(Q^2)$ is valid. Since at leading order, for $y \leq 0$, the form factor $F(y)$ is equal to the Isgur-Wise function $\xi(w = v \cdot v' = 1 - 2y)$, eq. (7) leads directly to a family of upper and lower limits on the slope $-\rho^2$ of $\xi(w)$ at $w = 1$. The lower limits on ρ^2 (from $F_+(z)$) are not interesting, as they lie below the kinematic bound $\rho^2 \geq \frac{1}{4}$ [7]. As for the upper limit, it is strongest if we take $|Q^2| \ll m_B^2$, which gives the bound quoted ref. [6],

$$
\rho^2 \leq 1.42 . \tag{9}
$$

However, there are reasons to believe that this bound on the slope of the Isgur-Wise function cannot be correct. In particular, one may construct two simple counterexamples. The first involves a hypothetical world in which the light quark in the B meson has a mass m_ℓ such that $m_{\rm b} \gg m_\ell \gg \Lambda_{\rm QCD}$. In this case the nonrelativistic quark model (NRQM) [8] is applicable, as the B meson is analogous to the hydrogen atom, a weakly bound system with coupling strength $\alpha_s(m_\ell)$. The "charge radius" $\xi'(1)$ may be computed using nonrelativistic wavefunctions for the bound states, and one obtains a value which is proportional to $-1/\alpha_s^2(m_\ell)$, in violation of the proposed bound. If such a "B-meson" were really like a hydrogen atom, in particular if it were unconfined, the derivation would have failed because of an anomalous threshold in $F(q^2)$ at the point $q^2 \approx \alpha_s^2 m_{\rm b}^2$. Yet the NRQM should apply arbitrarily well for the confinement scale arbitrarily in the infrared; for such "weakly confined" QCD [9], $F(q^2)$ must be analytic all the way up to the point $q^2 = m^2_{\text{``}\Upsilon\text{''}}$. However in this case there is still a "would-be anomalous threshold" near $q^2 = 0$ controlling the steep behaviour of $\xi(w)$ at zero recoil. In potential models, such would-be anomalous thresholds are typically associated with the presence of many poles in $F(q^2)$, with residues large in magnitude and oscillating in sign, along but near the end of the physical cut [10]. This is a hint that it may be necessary to consider more carefully the behaviour of $F(q^2)$ near the physical threshold.

For our second counterexample, we recall that the techniques of HQET may be used to extract the leading logarithmic dependence on the heavy quark mass $m_{\rm b}$ of the physical form factor $F(y)$. Neglecting terms of order $1/m_{\rm b}$, we find [11]

$$F(y) = [\alpha_s(m_{\rm b})]^{a_L(w)}\, \xi_{\rm ren}(w)\,, \tag{10}$$

where

$$a_L(w) = \frac{8}{33 - 2N_f}\left[\frac{w\,\ln\left(w + \sqrt{w^2-1}\right)}{\sqrt{w^2-1}} - 1\right]\,, \tag{11}$$

$\xi_{\rm ren}(w)$ is a universal $m_{\rm b}$-independent and μ-independent function, and N_f is the number of light flavours. Now for $m_{\rm b}$ arbitrarily large, $\alpha_s(m_{\rm b})$ is arbitrarily small, and the magnitude of the slope of $F(y)$ at $y = 0$ may be made arbitrarily large (note that as $m_{\rm b}$ grows, the inequality (6) is better and better satisfied). Of course, such behaviour is inconsistent with *any* bound on the slope of the form factor which one might hope to derive.

So where does the argument which leads to the constraints (7) fail? The problem is that it is not possible to convert the integral from $y = 1$ to $y = \infty$ of $|F(y)|^2$ (times the weighting function) into a contour integral, *because it is not possible to integrate around* *the end of the physical cut at* $y = 1$*.* This is simple to see when $F(y)$, taken from eq. (10), is expanded about $y = 1$, corresponding to $w = -1$. Writing $w = -1+\varepsilon$ and taking $\varepsilon \ll 1$, we find

$$F(y) \propto [\alpha_s(m_\mathrm{b})]^{a_L(w)} \sim \mathrm{e}^{K/\sqrt{\varepsilon}}\,, \tag{12}$$

for some positive constant K. We see that $F(y)$ has an *essential singularity* at the point $y = 1$. The integral around the end of the cut, taken on a small circle of radius δ, has no well-defined limit as $\delta \to 0$; hence it is not possible to find a closed contour on which the (weighted) integral of $|F(y)|^2$ is bounded, and inside of which $F(y)$ is analytic. Therefore condition (i) cannot be met, and no constraints such as eq. (7) may be derived.

To understand what has happened, we need to look more carefully at the analytic structure of $F(y)$. The integral in the inequality (6) starts at $y = 1$, running to $y = \infty$ along the top of the cut given by continuum production of $B\bar{B}$ pairs. However, there are additional singularities in $F(y)$, corresponding to diagrams in which the current V^μ converts to a heavy quark-antiquark bound state, which then couples directly to the B-meson. These are singularities below $B\bar{B}$ threshold, beginning at the mass of the "Υ" state, which for very large m_b is located at $y = 1 - 4\alpha_s^2(m_\mathrm{b})/9$. To complete the contour integral in such a way that $F(y)$ is analytic everywhere inside, one must close it to the left of $y = 1 - 4\alpha_s^2(m_\mathrm{b})/9$. These additional contributions to the weighted integral of $|F(y)|^2$ need not be negligible, and they are not known. Hence once again we see that we cannot take the crucial step from a bounded integral over $1 \le y < \infty$, to a bounded integral over a closed contour.

In fact, in both of our counterexamples we should expect singular, or at least dramatic, behaviour associated with the "Υ" region below $B\bar{B}$ threshold. In weakly confined QCD, the presence of the would-be anomalous threshold near $y = 0$ is typically associated with rapid variations in $F(y)$ [10], which it is natural to associate with the "Υ" region. In the HQET, the singular leading logarithmic behaviour (10) of $F(y)$ at $y = 0$ arises from the infrared divergence associated with multiple gluon exchange between heavy quarks with nearly the same velocity. The quark and antiquark would like to form a bound state, but cannot in the $m_\mathrm{b} \to \infty$ limit with their velocities absolutely fixed. One may be misled if one fails to account carefully for the contributions of these bound states *below* the threshold for heavy meson pair production.

In conclusion, we wish to stress not simply that the analysis of ref. [6] is flawed, but that it is not likely to be possible to amend this argument so that weaker, but still rigorous, bounds could be derived. Of course one could attempt to model the contributions to $F(q^2)$ of the heavy quark-antiquark bound states, but such an approach would undermine the rigor of the derivation. We have presented two counterexamples in which the magnitude of the slope $\xi'(1)$ may be made arbitrarily large, and in which any universal bounds which one might hope to derive would be violated. We have argued that these are sufficient to preclude the derivation of any such universal constraints on the Isgur-Wise function.

Similar work has also been done by B. Grinstein and P. Mende [12] and by C. E. Carlson, N. Isgur, T. Mannel, J. Milanu and W. Roberts. We are grateful to them for communicating their results to us prior to publication. We also thank D. Kaplan, M. Neubert, M. Peskin and J. Taron for useful conversations. A. F. would like to thank the Department of Physics at UC San Diego, where portions of this work were performed, for their hospitality.

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