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Dimensionally Regulated Pentagon Integrals*

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Abstract

We present methods for evaluating the Feynman parameter integrals associated with the pentagon diagram in $4-2\epsilon$ dimensions, along with explicit results for the integrals with all masses vanishing or with one non-vanishing external mass. The scalar pentagon integral can be expressed as a linear combination of box integrals, up to $\mathcal{O}(\epsilon)$ corrections, a result which is the dimensionally-regulated version of a $D=4$ result of Melrose, and of van Neerven and Vermaseren. We obtain and solve differential equations for various dimensionally-regulated box integrals with massless internal lines, which appear in one-loop n-point calculations in QCD. We give a procedure for constructing the tensor pentagon integrals needed in gauge theory, again through $\mathcal{O}(\epsilon^0)$.

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1. Introduction

The search for new physics at current and future hadron colliders demands that we first refine our understanding of events originating in known physics, most importantly in QCD and QCD-associated processes. To date, the matrix elements for all pure QCD processes with up to seven external legs are known exactly at tree-level [1] allowing the computation of events with up to five jets in the final state. (Various techniques [2] allow one to approximate cross sections with more jets.) Because the perturbation expansion for jet physics in QCD is not an expansion strictly in the coupling constant, but is rather an expansion in the coupling constant times various infrared logarithms, radiative corrections play an important role in matching theoretical expectations to experimental data. The calculation of radiative corrections requires of course the computation of loop corrections to the basic tree-level partonic processes. Thus far, the one-loop corrections are known only for the most basic processes, matrix elements with four external partons [3]. To go beyond these basic processes in the computation of radiative corrections in pure QCD (for example, to calculate the next-to-leading order corrections to three-jet production at hadron colliders), one must calculate five-point one-loop amplitudes in a theory with massless particles; and these in turn require the computation of one-loop Feynman parameter integrals with five external legs, within the dimensional regularization method. To discuss one-loop corrections to five-point amplitudes with external W and Z bosons, at least one of the external legs must be massive. In the present paper we address the computation of such dimensionally-regulated pentagon (and higher-point) integrals. Recently the techniques described in this paper have been used in the calculation of the one-loop helicity amplitudes for five external gluons [4].

Various authors [5,6,7,8] have discussed the computation of pentagon integrals that can be carried out in dimension $D = 4$ (i.e. that have neither soft nor collinear infrared divergences). In particular, Melrose [5] and independently van Neerven and Vermaseren [7] were able to express pentagon integrals as linear combinations of five different loop integrals with four external legs. Such box integrals (which, with external masses but no internal masses, are also required in radiative calculations in QCD) can be calculated readily in dimensional regularization, by direct integration or in terms of hypergeometric functions, if the number of masses is not too large.

The techniques of Melrose and of van Neerven and Vermaseren do not apply directly to dimensionally-regulated integrals, however, and the required pentagon integrals have not yet been presented in a closed and useful form, which is to say with all poles in $\epsilon = (4 - D)/2$ manifest, and with all functions of the kinematic invariants expressed in terms of logarithms and polylogarithms.* Here we will provide such an expression for the basic scalar integral. We employ a set of equations

* We have been informed that R. K. Ellis, W. T. Giele, and E. Yehudai [9] have recently evaluated the pentagon integrals by an independent technique.

derived in a separate paper [10]. These equations actually apply more generally to dimensionally-regulated one-loop n-point integrals; they can be used as a starting point for the reduction of an $(n \geq 5)$-point integral to a linear combination of boxes. For the pentagon integral ($n = 5$), the equations are the dimensionally-regulated analogs of equations derived in references [5,7]. In this paper we will use the equations to obtain explicit expressions for the pentagon with all lines massless, and for the pentagon with one massive external line, up to $\mathcal{O}(\epsilon)$ corrections. Such integrals are of use in the calculation of next-to-leading-order contributions to processes such as $gg \to ggg$ and $Z \to q\bar{q}gg$.

Besides the scalar pentagon integral, in QCD one requires tensor integrals — loop integrals with up to five powers of the loop momentum inserted. In the string-based technique [11,4] for evaluating QCD amplitudes, one obtains directly integrals over Feynman parameters rather than loop momenta. Tensor integrals correspond in this framework to the insertion of polynomials in the Feynman parameters into the numerator of the integrand. In order to construct an integral table that meshes well with this technique, we choose to work in terms of the Feynman-parametrized integrals. This approach also lets us take advantage of an observation that appropriate derivatives of the scalar pentagon insert Feynman parameters into the numerator of the integrand. Thus the scalar pentagon may be used as a generating function for all the tensor integrals.

In the more usual momentum-space approach to tensor integrals, one performs a Brown-Feynman [12] or Passarino-Veltman [13] reduction, solving a system of algebraic equations for the tensor integrals. For example, integrals with just one loop-momentum inserted in the numerator are reduced to a linear combination of scalar integrals [14]. The counterparts of these equations exist for Feynman parameter integrals. In particular, integrals with just one Feynman parameter inserted in the numerator can be expressed as a linear combination of scalar integrals. If one now equates these expressions to the above-mentioned derivative representations of the same one-parameter tensor integrals, one obtains a set of first-order partial differential equations for the scalar integral. Thus an alternate approach to determining the scalar pentagon is to solve a set of differential equations. The differential equations are also an efficient way to obtain various infrared divergent scalar box integrals, with massless internal lines but with 0, 1, 2 or 3 external masses. (Most of these box integrals have been obtained previously by other techniques.) Together with the infrared finite box integral with four external masses [6], for which a compact form has recently been provided by Denner, Nierste, and Scharf [15], these constitute the set of box integrals required for computing one-loop n-point amplitudes in QCD without quark masses, for any n. (These box integrals will appear both in the recursive determination of higher-point diagrams [5,7,10], and as diagrams in their own right.)

The partial differential equation approach just described is reminiscent of similar procedures for performing two-loop and higher-loop integrals (usually with fewer external legs) [16]. However, the latter manipulations have generally been carried out in terms of either a momentum-space or a configuration-space representation of the integrals, in contrast to the Feynman parameter representation used here.

The rest of the paper is organized as follows: in section 2, we introduce the Feynman-parametrized n-point integrals, in particular the pentagon and box integrals, and we make a change of integration variables and kinematic variables that allows the tensor integrals to be expressed as derivatives of the basic scalar integral. In section 3 we present an alternative derivation of the set of algebraic equations derived in ref. [10]. One of these equations can be used to determine the general n-point scalar one-loop integral recursively, as a linear combination of $(n-1)$-point integrals. (For $n \geq 7$ there are some subtleties, as explained in appendix VI.) The other two equations are useful in the calculation of tensor integrals, given the scalar integral. Also, in combination with the results of section 2 they give partial differential equations for the scalar integral. In section 4 we begin by illustrating the general derivation of the partial differential equation in section 3, using the simple example of the box integral with all massless external legs. We then solve the differential equations for box integrals with 0, 1, 2 or 3 massive external legs. In section 5 we use one of the algebraic equations derived in section 3 to obtain explicit formulae for the pentagon with all massless external legs, and with one massive external leg. In section 6, we describe how to obtain the (tensor) pentagon integrals with Feynman parameters in the numerator, through $\mathcal{O}(\epsilon^0)$.

For the reader's convenience, we have collected our results for the scalar box integrals and for the scalar and tensor massless pentagon integrals in appendix I. In appendix II we show that when the integrals are infrared finite, our results for the scalar pentagon integral reduce to the non dimensionally-regulated result of van Neerven and Vermaseren [7]. Appendix III presents an argument (verifying an observation of Ellis, Giele and Yehudai) which shows that the approach of section 6 generates all tensor pentagon integrals needed in gauge theory calculations. In appendix IV, we compute an integration constant for two- and three-mass boxes. In appendix V, as another illustration of the partial differential equation technique, we obtain a manifestly symmetric expression for the triangle integral with all three external legs massive, to all orders in ϵ. (To $\mathcal{O}(\epsilon^0)$, such a formula has been obtained in ref. [17].) In appendix VI, we discuss subtleties that arise in obtaining scalar integrals for $n \geq 7$, and in appendix VII, we derive and discuss formulæ for tensor integrals for both the pentagon and hexagon diagrams.

2. Properties of Feynman Parameter Integrals

In this section, we shall show that Feynman parameter integrals with Feynman parameters inserted in the numerator of the integrand (which arise from tensor integrals) are given by appropriate derivatives of the basic scalar integral. We present the particular cases of the massless box and pentagon integrals in more detail.

For convenience, we assume here that the masses for all internal lines vanish. (The extension to nonvanishing internal masses is entirely straightforward [10].) Then the n-point scalar one-loop integral in $4-2\epsilon$ dimensions is

$$\mathcal{I}_n \;=\; \mu^{2\epsilon}\int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}}\frac{1}{p^2(p-k_1)^2(p-k_1-k_2)^2\cdots(p-k_1-k_2-\cdots-k_{n-1})^2}\;, \tag{2.1}$$

where k_i, $i=1,\ldots,n$, are the external momenta and μ is the usual dimensional regularization scale parameter. Performing the usual Feynman parametrization, and integrating out the loop momentum, we obtain

$$I_n\,[1] \;=\; \Gamma(n-2+\epsilon)\int_0^1 d^n a_i\;\delta(1-\textstyle\sum_i a_i)\frac{1}{\left[\sum_{i,j=1}^n S_{ij}a_ia_j - i\varepsilon\right]^{n-2+\epsilon}}\;, \tag{2.2}$$

where

$$I_n\,[1] \;\equiv\; (-1)^{n+1}\,i\;(4\pi)^{2-\epsilon}\,\mu^{-2\epsilon}\,\mathcal{I}_n \tag{2.3}$$

is the basic n-point parameter integral, the symmetric matrix S_{ij} is defined by

$$S_{ij} \;=\; -\tfrac{1}{2}(k_i+\cdots+k_{j-1})^2, \qquad i\neq j; \qquad S_{ii} \;=\; 0; \qquad (i,j \text{ are mod } n); \tag{2.4}$$

and where we have put in the $i\varepsilon$ explicitly. The poles in I_n produced by the Γ function prefactor are ultraviolet ones; the remaining poles represent infrared divergences. In explicit calculations of cross-sections, they will ultimately cancel corresponding poles arising from soft and collinear emission of particles in $(n+1)$-point tree-level processes.

We shall use the notation $I_n\,[P(\{a_i\})]$ to denote an integral in which the polynomial P appears in the numerator of the integrand,

$$I_n\,[P(\{a_i\})] \;=\; \Gamma(n-2+\epsilon)\int_0^1 d^n a_i\;\delta(1-\textstyle\sum_i a_i)\frac{P(\{a_i\})}{\left[\sum_{i,j=1}^n S_{ij}a_ia_j - i\varepsilon\right]^{n-2+\epsilon}}\;. \tag{2.5}$$

In QCD calculations, one encounters integrals of this form, where the degree of P is less than or equal to n.

For the box (four-point) integral, the "scalar denominator" is

$$\sum_{i,j=1}^{4} S_{ij}a_ia_j \;=\; -sa_1a_3 - ta_2a_4 - m_1^2a_1a_2 - m_2^2a_2a_3 - m_3^2a_3a_4 - m_4^2a_4a_1\;, \tag{2.6}$$

where $s \equiv (k_1 + k_2)^2$ and $t \equiv (k_2 + k_3)^2$, and m_i^2 are the masses of the external legs (some or all of which may vanish). For the all-massless pentagon integrals,

$$\sum_{i,j=1}^{5} S_{ij} a_i a_j \;=\; -s_{12} a_1 a_3 - s_{23} a_2 a_4 - s_{34} a_3 a_5 - s_{45} a_4 a_1 - s_{51} a_5 a_2 \;, \tag{2.7}$$

where $s_{i,i+1} \equiv (k_i + k_{i+1})^2$. A nonzero mass for external leg 5 would add a term $-m_5^2 a_5 a_1$ to (2.7). All external indices are understood to be taken mod n for the n-point function. We will present our results for kinematics in the Euclidean region, where all momentum invariants s_{ij}, m_i^2 are negative. In this region, the scalar denominator is always positive, and the integrals are purely real, which simplifies the resulting expressions. We define the integral for physical values by analytic continuation from the Euclidean region; the analytic continuation back to the physical region should be understood implicitly in all formulæ presented below, and we shall henceforth leave the $i\varepsilon$ implicit.

Following 't Hooft and Veltman [6], we make the change of integration variables in (2.2),

$$\begin{aligned} a_i &= \frac{\alpha_i u_i}{\sum_{j=1}^{n} \alpha_j u_j} \,, \qquad \text{no sum on } i, \\ a_n &= \frac{\alpha_n \left(1 - \sum_{j=1}^{n-1} u_j\right)}{\sum_{j=1}^{n} \alpha_j u_j} \,. \end{aligned} \tag{2.8}$$

Assuming that all α_i are real and positive, the integral becomes

$$I_n[1] \;=\; \Gamma(n-2+\epsilon) \int_0^1 d^n u \, \frac{\delta\left(1 - \sum u_i\right) \left(\prod_{j=1}^{n} \alpha_j\right) \left(\sum_{j=1}^{n} \alpha_j u_j\right)^{n-4+2\epsilon}}{\left[\sum_{i,j} S_{ij} \alpha_i \alpha_j u_i u_j\right]^{n-2+\epsilon}} \,. \tag{2.9}$$

This form for the integral is most useful if we can also define the α_i in such a way that all of the dependence on the α_i-variables is scaled out of the denominator. Let us define the α_i, and simultaneously a matrix ρ, through

$$S_{ij} \;=\; \frac{\rho_{ij}}{\alpha_i \alpha_j} \,. \tag{2.10}$$

The elements of the matrix ρ_{ij} are to be thought of as additional kinematic variables, independent of the α_i. (In specific cases many of the elements ρ_{ij} may be taken to be pure numbers.)

For the four-point integral described by the denominator (2.6), we can choose

$$s = -\frac{1}{\alpha_1 \alpha_3} \,, \quad t = -\frac{1}{\alpha_2 \alpha_4} \,, \quad m_1^2 = -\frac{\hat{m}_1^2}{\alpha_1 \alpha_2}, \quad m_2^2 = -\frac{\hat{m}_2^2}{\alpha_2 \alpha_3}, \quad m_3^2 = -\frac{\hat{m}_3^2}{\alpha_3 \alpha_4}, \quad m_4^2 = -\frac{\hat{m}_4^2}{\alpha_4 \alpha_1}. \tag{2.11}$$

(Other choices are also possible; see section 4.) Equations (2.11) do not have a unique solution in terms of the α_i. One simple solution is $\alpha_1 = \alpha_3 = 1/\sqrt{-s}$, $\alpha_2 = \alpha_4 = 1/\sqrt{-t}$. However, we would like all four α_i variables to be independent of each other, so that we can use the scalar integral as a generating function for integrals with insertions of all four Feynman parameters a_i. Therefore we consider the α_i to be general solutions to equations (2.11), with no other constraints on them. The set of independent kinematic variables corresponding to the choice (2.11) is then $\{\alpha_i; \hat{m}_i^2\}$.

For the all-massless pentagon integral, the unique solution to

$$s_{i,i+1} \;=\; -\frac{1}{\alpha_i \alpha_{i+2}} \tag{2.12}$$

is

$$\begin{aligned} \alpha_1 &= \sqrt{-\frac{s_{23}s_{34}}{s_{45}s_{51}s_{12}}}\,, \qquad \alpha_2 = \sqrt{-\frac{s_{34}s_{45}}{s_{51}s_{12}s_{23}}}\,, \qquad \alpha_3 = \sqrt{-\frac{s_{45}s_{51}}{s_{12}s_{23}s_{34}}}\,, \\ \alpha_4 &= \sqrt{-\frac{s_{51}s_{12}}{s_{23}s_{34}s_{45}}}\,, \qquad \alpha_5 = \sqrt{-\frac{s_{12}s_{23}}{s_{34}s_{45}s_{51}}}\,. \end{aligned} \tag{2.13}$$

Because we have taken the s_{ij} to be negative, the α_i are real. No additional kinematic variables are necessary for the massless pentagon.

With these choices of α_i, the four and five point scalar integrals become

$$\begin{aligned} I_4\,[1] &= \Gamma(2+\epsilon)\left(\prod_{j=1}^{4}\alpha_j\right)\int_0^1 d^4u\; \frac{\delta\left(1-\sum u_i\right)\left(\sum_{j=1}^{4}\alpha_j u_j\right)^{2\epsilon}}{\left[u_1u_3+u_2u_4+\hat{m}_1^2u_1u_2+\hat{m}_2^2u_2u_3+\hat{m}_3^2u_3u_4+\hat{m}_4^2u_4u_1\right]^{2+\epsilon}}\,, \\ I_5\,[1] &= \Gamma(3+\epsilon)\left(\prod_{j=1}^{5}\alpha_j\right)\int_0^1 d^5u\; \frac{\delta\left(1-\sum u_i\right)\left(\sum_{j=1}^{5}\alpha_j u_j\right)^{1+2\epsilon}}{\left[u_1u_3+u_2u_4+u_3u_5+u_4u_1+u_5u_2\right]^{3+\epsilon}}\,. \end{aligned} \tag{2.14}$$

Further examples of the $\{\alpha_i; \rho_{ij}\}$ change of variables are to be found in sections 4,5 and appendix V.

It will be helpful to define the *reduced integrals*

$$\hat{I}_n\,[P(\{a_i\})] \;=\; \left(\prod_{j=1}^{n}\alpha_j\right)^{-1} I_n\,[P(\{a_i/\alpha_i\})]\;. \tag{2.15}$$

As we will see, dividing out the factors of α_i connects the tensor integrals more simply to the scalar integral. For the scalar integrals (that is when the polynomial is simply 1), we will use the abbreviated notation $\hat{I}_n \equiv \hat{I}_n\,[1]$.

We can use the 't Hooft-Veltman form of the tensor integrals (2.15) to obtain derivative relations for them. Let $P_m(\{a_i\})$ denote a homogeneous polynomial of degree m. Then for the massless box integral, the change of variables (2.8) gives

$$\hat{I}_4\,[P_m(\{a_i\})] \;=\; \Gamma(2+\epsilon)\int_0^1 d^4u\;\delta(1-\textstyle\sum_i u_i)\; \frac{P_m(\{u_i\})\left(\sum_{j=1}^{4}\alpha_j u_j\right)^{-m+2\epsilon}}{\left[u_1u_3+u_2u_4\right]^{2+\epsilon}}\,, \tag{2.16}$$

which we can rewrite in terms of derivatives acting on $\hat{I}_4$,

$$\hat{I}_4\left[P_m(\{a_i\})\right] \;=\; \frac{\Gamma(1-m+2\epsilon)}{\Gamma(1+2\epsilon)}\; P_m\left(\left\{\frac{\partial}{\partial\alpha_i}\right\}\right)\,\hat{I}_4\left[1\right]\;. \tag{2.17}$$

Similarly, the change of variables (2.8) leads in the five-point case to

$$\hat{I}_5\left[P_m(\{a_i\})\right] \;=\; \Gamma(3+\epsilon)\int_0^1 d^5u\;\delta(1-\textstyle\sum_i u_i)\;\frac{P_m(\{u_i\})\left(\sum_{j=1}^5 \alpha_j u_j\right)^{1-m+2\epsilon}}{\left[u_1u_3+u_2u_4+u_3u_5+u_4u_1+u_5u_2\right]^{3+\epsilon}}\;, \tag{2.18}$$

which we can write as follows,

$$\hat{I}_5\left[P_m(\{a_i\})\right] \;=\; \frac{\Gamma(2-m+2\epsilon)}{\Gamma(2+2\epsilon)}\,P_m\left(\left\{\frac{\partial}{\partial\alpha_i}\right\}\right)\,\hat{I}_5\left[1\right]\;. \tag{2.19}$$

These equations hold when there are external masses as well, provided that one holds fixed the matrix ρ defined in (2.10) when differentiating with respect to α_i. The result for the general n-point integral is

$$\hat{I}_n\left[P_m(\{a_i\})\right] \;=\; \frac{\Gamma(n-3-m+2\epsilon)}{\Gamma(n-3+2\epsilon)}\,P_m\left(\left\{\frac{\partial}{\partial\alpha_i}\right\}\right)\,\hat{I}_n\left[1\right]\;. \tag{2.20}$$

Equation (2.20) allows one to obtain tensor integrals by differentating the basic scalar integral. Certain subtleties do arise in this approach; they will be dealt with in section 6 and in appendix VII.

Using equations such as (2.17), (2.19), and (2.20), one can translate an algebraic system of equations for integrals with Feynman parameters inserted, into a system of partial differential equations for the basic scalar integral; in principle one can then solve the equations for the latter quantity. This effectively turns a problem of definite integration into one of indefinite integration (in a different set of variables). We shall use this approach to give concrete expressions for all the box integrals. It is also possible, as we shall see in the next section, to derive a purely algebraic set of equations for the n-point integrals $\hat{I}_n$, in which a new unknown quantity enters only at $\mathcal{O}(\epsilon)$.

3. Algebraic Equations for n-Point One-Loop Integrals

In this section we will derive a set of algebraic equations for the general n-point one-loop integrals. Some of the equations are of use in the partial differential equation approach of section 4; others can be used to determine the n-point scalar integrals for $n \geq 5$ in terms of box integrals, in an entirely algebraic fashion (subject to some subtleties for $n \geq 7$, which are explained in appendix VI). The equations have been derived in ref. [10] using a momentum-space representation of the loop integrals. Here we will derive the same general equations using the Feynman parameter representation; in this derivation the equations arise from the consideration of integrals of total derivatives of the Feynman parameters.* For a specific, simple example of the following general derivation, we refer the reader to the beginning of section 4.

The total derivatives we will consider are

$$
\begin{aligned}
J_{n;m} \;\equiv\; \Gamma(n-3+\epsilon) \int_0^1 da_{n-1} \int_0^{1-a_{n-1}} da_{n-2} \cdots \int_0^{1-a_1-a_2-\cdots-\widehat{a_m}-\cdots-a_{n-1}} da_m \\
\times \frac{d}{da_m} \frac{1}{\left[\sum_{i,j=1}^n S_{ij} a_i a_j\right]^{n-3+\epsilon}} \Bigg|_{a_n = 1-a_1-a_2-\cdots-a_{n-1}} .
\end{aligned}
\tag{3.1}
$$

There are two ways to evaluate $J_{n;m}$. First, one can carry out the differentiation with respect to a_m, to get

$$
\begin{aligned}
J_{n;m} &= -2\,\Gamma(n-2+\epsilon) \int d^n a_i \;\delta(1-\textstyle\sum_i a_i) \frac{\sum_{j=1}^n (S_{mj} a_j - S_{nj} a_j)}{\left[\sum_{i,j=1}^n S_{ij} a_i a_j\right]^{n-2+\epsilon}} \\
&= -2 \left(\prod_{\ell=1}^n \alpha_\ell \right) \sum_{j=1}^n \hat{I}_n \left[(S_{mj} - S_{nj})\, \alpha_j \, a_j \right] .
\end{aligned}
\tag{3.2}
$$

Second, one can perform the integral over a_m. At the lower integration endpoint, a_m is set to 0. The remaining $(n-1)$-point integral corresponds to removing the propagator parametrized by a_m — i.e., the propagator between lines $(m-1)$ and m — from the original n-point (scalar) integral; we denote such a "daughter" integral of I_n ($\hat{I}_n$) by $I_{n-1}^{(m)}$ ($\hat{I}_{n-1}^{(m)}$).† Similarly, at the upper integration endpoint a_n is set to 0, yielding the $(n-1)$-point integral $I_{n-1}^{(n)}$. It is always possible to choose the α_i variables for the integrals $\hat{I}_{n-1}^{(j)}$ so that they are the same as those for the parent integral $\hat{I}_n$. Having made this choice, the second evaluation of $J_{n;m}$ gives

$$
J_{n;m} \;=\; I_{n-1}^{(n)}[1] - I_{n-1}^{(m)}[1] \;=\; \left(\prod_{\ell=1}^n \alpha_\ell \right) \left[\frac{\hat{I}_{n-1}^{(n)}}{\alpha_n} - \frac{\hat{I}_{n-1}^{(m)}}{\alpha_m} \right] .
\tag{3.3}
$$

Equating (3.2) and (3.3), using $S_{ij} = \rho_{ij}/(\alpha_i \alpha_j)$ and the definitions (2.15) of the reduced integrals, and relabelling the index $m \to i$, we have

$$
\sum_{j=1}^n \left(\frac{\rho_{ij}}{\alpha_i} - \frac{\rho_{nj}}{\alpha_n} \right) \hat{I}_n[a_j] \;=\; \frac{1}{2} \left[\frac{\hat{I}_{n-1}^{(i)}}{\alpha_i} - \frac{\hat{I}_{n-1}^{(n)}}{\alpha_n} \right] , \qquad i = 1, 2, \ldots, n-1.
\tag{3.4}
$$

---

* The motivation for considering such objects arose from the observation that the field-theory limit of integrals of total derivatives in string theory yields expressions that are sums of loop integrals with differing numbers of external legs (multiplied by various coefficients); these sums must necessarily vanish because the world-sheets in the string loop expansion have no boundaries, when appropriate analytic continuations of the external momenta are used.

† For more explicit examples of this notation, see the beginning of subsection 4.2.

We would like to solve for the n one-parameter integrals $\hat{I}_n[a_j]$. To do so we supplement the $n-1$ equations (3.4) with the equation that follows from the constraint on the Feynman parameters, $\sum_{j=1} a_j = 1$, namely

$$\sum_{j=1}^{n} \alpha_j \, \hat{I}_n[a_j] \;=\; \hat{I}_n[1] \;=\; \hat{I}_n. \tag{3.5}$$

Before solving equations (3.4), (3.5), we introduce a little more notation and some "kinematic" results from ref. [10]. We define the Gram determinant of the $(n-1)$-vector system associated with the n-point integral by

$$\Delta_n \;\equiv\; \det{}'(2k_i \cdot k_j), \tag{3.6}$$

where the prime signifies that one of the n vectors k_i is to be omitted before taking the determinant; due to momentum conservation, $\sum k_i = 0$, any one of the vectors may be omitted.[‡] Next we introduce the *rescaled* Gram determinant,

$$\hat{\Delta}_n \;\equiv\; \Bigl(\prod_{\ell=1}^{n} \alpha_\ell^2\Bigr) \Delta_n, \tag{3.7}$$

which has a simple bilinear representation in terms of the variables α_i:

$$\hat{\Delta}_n \;=\; \sum_{i,j=1}^{n} \eta_{ij} \alpha_i \alpha_j. \tag{3.8}$$

Here η_{ij} is independent of the α_i; in fact η is proportional [10] to the inverse of the matrix ρ defined in equation (2.10):

$$\rho \;=\; N_n \, \eta^{-1}, \qquad \eta \;=\; N_n \, \rho^{-1}, \qquad N_n \;\equiv\; 2^{n-1} \det \rho. \tag{3.9}$$

We also define the variables γ_i by

$$\gamma_i \;\equiv\; \sum_{j=1}^{n} \eta_{ij} \alpha_j \;=\; \frac{1}{2} \frac{\partial \hat{\Delta}_n}{\partial \alpha_i} \bigg|_{\rho_{ij} \text{ fixed}}. \tag{3.10}$$

They are in a sense conjugate to the α_i variables:

$$\sum_{j=1}^{n} \rho_{ij} \gamma_j \;=\; N_n \alpha_i \,. \tag{3.11}$$

If we define

$$R_{ki} \;=\; \eta_{ki} - \frac{\gamma_k \gamma_i}{\hat{\Delta}_n} \,, \tag{3.12}$$

‡ The notation for, and normalization of, the Gram determinant in equation (3.6) differ from other conventions in the literature, e.g. references [5,18].

then we may note the following identity,

$$\sum_{i=1}^{n} \alpha_i R_{ki} \;=\; 0 \;, \tag{3.13}$$

which follows from equations (3.8) and (3.10). Thus if we multiply both sides of equation (3.4) by $\alpha_i R_{ki}$, and then sum over i, the terms not containing α_i will drop out, leaving us with

$$\sum_{i,j=1}^{n} \left(\eta_{ki}\rho_{ij} - \frac{\gamma_k}{\hat{\Delta}_n}\gamma_i\rho_{ij} \right) \hat{I}_n[a_j] \;=\; \frac{1}{2}\sum_{i=1}^{n} R_{ki}\hat{I}_{n-1}^{(i)} \;, \tag{3.14}$$

or

$$N_n \hat{I}_n[a_k] \;=\; \frac{1}{2}\sum_{i=1}^{n} R_{ki}\hat{I}_{n-1}^{(i)} + \frac{\gamma_k}{\hat{\Delta}_n} N_n \sum_{j=1}^{n} \alpha_j \hat{I}_n[a_j] \;. \tag{3.15}$$

Performing the sum on the right-hand side with the help of (3.5), dividing by N_n, and writing out the definition of R_{ki}, we obtain

$$\hat{I}_n[a_i] \;=\; \frac{1}{2N_n}\sum_{j=1}^{n}\Big(\eta_{ij} - \frac{\gamma_i\gamma_j}{\hat{\Delta}_n} \Big)\hat{I}_{n-1}^{(j)} \;+\; \frac{\gamma_i}{\hat{\Delta}_n}\hat{I}_n \;. \tag{3.16}$$

Combining this set of equations with the derivative representation (2.20) for $m = 1$, we obtain a system of partial differential equations for the n-point scalar integral,

$$\frac{1}{n-4+2\epsilon}\frac{\partial \hat{I}_n}{\partial \alpha_i} \;=\; \frac{1}{2N_n}\sum_{j=1}^{n}\Big(\eta_{ij} - \frac{\gamma_i\gamma_j}{\hat{\Delta}_n} \Big)\hat{I}_{n-1}^{(j)} \;+\; \frac{\gamma_i}{\hat{\Delta}_n}\hat{I}_n \;. \tag{3.17}$$

Section 4 is devoted to solving these equations for various scalar box integrals.

In ref. [10] a momentum-space representation was used to derive an algebraic equation that involved only scalar integrals, at the expense of introducing a new object, $\hat{I}_n^{D=6-2\epsilon}$. The object $\hat{I}_n^{D=6-2\epsilon}$ comes from an integral in $D = 4-2\epsilon$ with two loop-momenta inserted in the numerator, but it can also be interpreted as the n-point scalar integral in two higher dimensions. The latter interpretation is helpful for understanding the properties of $\hat{I}_n^{D=6-2\epsilon}$ as $\epsilon \to 0$, which are needed in order to use the "dimension-changing" equation to obtain $D = 4-2\epsilon$ scalar integrals through $\mathcal{O}(\epsilon^0)$. We shall now re-derive this equation using Feynman parameter representations of the integrals.

The integral $I_n^{D=6-2\epsilon}[1]$ is most easily obtained from the $D = 4-2\epsilon$ equation (2.2) by letting $\epsilon \to \epsilon - 1$,

$$I_n^{D=6-2\epsilon}[1] \;=\; \Gamma(n-3+\epsilon)\int_0^1 d^n a_i \;\delta(1-\textstyle\sum_i a_i)\displaystyle\frac{1}{\Big[\sum_{i,j=1}^{n} S_{ij}a_i a_j\Big]^{n-3+\epsilon}} \;, \tag{3.18}$$

It may also be obtained by inserting one power of the scalar denominator of the $D = 4-2\epsilon$ integral into the numerator (summations are implicit in the following derivation):

$$I_n^{D=6-2\epsilon}[1] \;=\; \frac{\Gamma(n-3+\epsilon)}{\Gamma(n-2+\epsilon)} I_n^{D=4-2\epsilon}\left[S_{ij}a_i a_j\right] \;=\; \frac{1}{n-3+\epsilon}\;\rho_{ij}\; I_n^{D=4-2\epsilon}\left[(a_i/\alpha_i)(a_j/\alpha_j)\right] \;. \tag{3.19}$$

In terms of the reduced integrals (2.15), and using the derivative representation (2.20), we have

$$\hat{I}_n^{D=6-2\epsilon} \;=\; \frac{1}{(n-3+\epsilon)(n-4+2\epsilon)(n-5+2\epsilon)}\;\rho_{ij}\frac{\partial^2 \hat{I}_n}{\partial\alpha_i\partial\alpha_j}\;. \tag{3.20}$$

We may now evaluate the right-hand-side of (3.20) using equation (3.17) to replace the derivatives, and also using the relations between ρ, η and γ to simplify the expressions:

$$\begin{aligned}
\frac{1}{n-4+2\epsilon}\rho_{ij}\frac{\partial^2 \hat{I}_n}{\partial\alpha_i\partial\alpha_j} \;&=\; \rho_{ij}\frac{\partial}{\partial\alpha_i}\left(\frac{1}{2N_n}\left[\eta_{jk}-\frac{\gamma_j\gamma_k}{\hat{\Delta}_n}\right]\,\hat{I}_{n-1}^{(k)}+\frac{\gamma_j}{\hat{\Delta}_n}\hat{I}_n\right)\\
&=\;\frac{1}{2N_n}\rho_{ij}\left[2\frac{\gamma_i\gamma_j\gamma_k}{\hat{\Delta}_n^2}-\frac{(\eta_{ij}\gamma_k+\eta_{ik}\gamma_j)}{\hat{\Delta}_n}\right]\,\hat{I}_{n-1}^{(k)}+\frac{1}{2N_n}\left[\eta_{jk}-\frac{\gamma_j\gamma_k}{\hat{\Delta}_n}\right]\,\rho_{ij}\frac{\partial\hat{I}_{n-1}^{(k)}}{\partial\alpha_i}\\
&\quad+\frac{\rho_{ij}}{\hat{\Delta}_n}\left[\eta_{ij}-2\frac{\gamma_i\gamma_j}{\hat{\Delta}_n}\right]\,\hat{I}_n+(n-4+2\epsilon)\frac{\rho_{ij}\gamma_j}{\hat{\Delta}_n}\left(\frac{1}{2N_n}\left[\eta_{ik}-\frac{\gamma_i\gamma_k}{\hat{\Delta}_n}\right]\,\hat{I}_{n-1}^{(k)}+\frac{\gamma_i}{\hat{\Delta}_n}\hat{I}_n\right)\\
&=\;-\frac{n-1}{2}\frac{\gamma_k}{\hat{\Delta}_n}\,\hat{I}_{n-1}^{(k)}+\frac{1}{2}\left[\frac{\partial}{\partial\alpha_k}-\frac{\gamma_k}{\hat{\Delta}_n}\alpha_i\frac{\partial}{\partial\alpha_i}\right]\hat{I}_{n-1}^{(k)}+\Big((n-2)+(n-4+2\epsilon)\Big)\frac{N_n}{\hat{\Delta}_n}\,\hat{I}_n\;.
\end{aligned}$$

Now $\hat{I}_{n-1}^{(k)}$ is actually independent of α_k (since a_k has been set to 0 in $\hat{I}_{n-1}^{(k)}$); also

$$\alpha_i\frac{\partial}{\partial\alpha_i}\hat{I}_{n-1}^{(k)} \;=\; (n-5+2\epsilon)\;\hat{I}_{n-1}^{(k)}\,. \tag{3.21}$$

So we obtain

$$\frac{1}{n-4+2\epsilon}\rho_{ij}\frac{\partial^2 \hat{I}_n}{\partial\alpha_i\partial\alpha_j} \;=\; (n-3+\epsilon)\left[-\sum_{k=1}^{n}\frac{\gamma_k}{\hat{\Delta}_n}\hat{I}_{n-1}^{(k)}+\frac{2N_n}{\hat{\Delta}_n}\hat{I}_n\right]\;, \tag{3.22}$$

which can be solved for $\hat{I}_n$ using equation (3.20),

$$\hat{I}_n \;=\; \frac{1}{2N_n}\left[\sum_{i=1}^{n}\gamma_i\,\hat{I}_{n-1}^{(i)}\;+\;(n-5+2\epsilon)\,\hat{\Delta}_n\,\hat{I}_n^{D=6-2\epsilon}\right]\;. \tag{3.23}$$

In ref. [10] it is shown how to use this equation to obtain n-point integrals with $n \geq 6$. However, for $n \geq 7$ there are some complications, which are discussed in appendix VI. In this paper our main interest is the pentagon integral ($n = 5$). For the scalar pentagon integral it suffices to note that the integral $\hat{I}_5^{D=6-2\epsilon}$ is finite as $\epsilon \to 0$, because the $D = 6$ scalar pentagon integral possesses neither ultraviolet nor infrared divergences (soft or collinear), and also that the coefficient of $\hat{I}_5^{D=6-2\epsilon}$ in equation (3.23) is of order ϵ. Therefore to $\mathcal{O}(\epsilon^0)$ the general scalar pentagon integral is given by the sum of five scalar box integrals,

$$\hat{I}_5 \;=\; \frac{1}{2N_5}\sum_{i=1}^{5}\gamma_i\,\hat{I}_4^{(i)}\;+\;\mathcal{O}(\epsilon). \tag{3.24}$$

A schematic depiction of this equation, with the coefficients suppressed, is given in fig. 1. For the tensor pentagon integrals we have to keep the $\hat{I}_5^{D=6-2\epsilon}$ term around a while longer (see section 6).

One further equation for all n can be obtained by eliminating $\hat{I}_n$ from equation (3.16) using equation (3.23), with the result

$$\frac{1}{n-4+2\epsilon}\,\frac{\partial \hat{I}_n}{\partial \alpha_i} \;=\; \hat{I}_n[a_i] \;=\; \frac{1}{2N_n}\left[\sum_{j=1}^{n} \eta_{ij}\;\hat{I}_{n-1}^{(j)} \;+\; (n-5+2\epsilon)\,\gamma_i\;\hat{I}_n^{D=6-2\epsilon}\right]. \tag{3.25}$$

Since the $D=6$ scalar box is also finite, setting $n=4$ in equation (3.25) yields a simple set of partial differential equations for the box integrals, through $\mathcal{O}(\epsilon)$:

$$\frac{\partial \hat{I}_4}{\partial \alpha_i} \;=\; \frac{\epsilon}{N_4}\left[\sum_{j=1}^{4} \eta_{ij}\;\hat{I}_3^{(j)} \;+\; (-1+2\epsilon)\,\gamma_i\;\hat{I}_4^{D=6-2\epsilon}\right] \;=\; \frac{\epsilon}{N_4}\sum_{j=1}^{4} \eta_{ij}\;\hat{I}_3^{(j)} \;+\; \mathcal{O}(\epsilon)\;. \tag{3.26}$$

The right-hand-side depends only on the infrared singular pieces of the triangle integrals.

This completes our re-derivation of general all-n results presented in ref. [10]; we now apply these results to various box and pentagon integrals.

4. Partial Differential Equation Technique

In this section, we solve the partial differential equations (3.17), (3.26) for scalar box integrals with all internal lines massless, but with 0, 1, 2 or 3 massive external lines.

4.1 The Massless Box Integral

We begin with the box integral with all external lines massless,

$$I_4\,[1] \;=\; \Gamma(2+\epsilon)\int_0^1 d^4a_i\;\delta(1-\textstyle\sum_i a_i)\dfrac{1}{\left[-sa_1a_3-ta_2a_4\right]^{2+\epsilon}}\;. \tag{4.1}$$

This integral is simple enough to perform directly after the following change of variables [19] which factorizes the integrand[†]

$$a_1 \;=\; y(1-x), \qquad a_2 \;=\; z(1-y), \qquad a_3 \;=\; (1-y)(1-z), \qquad a_4 \;=\; xy. \tag{4.2}$$

However, our purpose here is to illustrate the partial differential equation technique, including the derivation of the equations, via this simple example.

† J. Vermaseren has pointed out to us that the factorization of the integrand in terms of x, y, z arises naturally if one combines pairs of propagators using Feynman parameters, and then combines the two resulting factors using another Feynman parameter. See also ref. [19].

As noted above, algebraic equations for Feynman parameter integrals can be obtained by considering integrals of total derivatives. Here we consider the box integral $I_4[1]$, with the parameter a_4 eliminated, and with the integrand differentiated with respect to a_1:

$$J_{4;1} \equiv \Gamma(1+\epsilon)\int_0^1 da_3 \int_0^{1-a_3} da_2 \int_0^{1-a_2-a_3} da_1 \frac{\partial}{\partial a_1} \frac{1}{[-sa_1a_3 - ta_2(1-a_1-a_2-a_3)]^{1+\epsilon}} \ . \tag{4.3}$$

Observe that $J_{4;1}$ can be evaluated in two ways, either by explicit differentiation, or by evaluating the integrand at the boundaries $a_4 = 1 - a_1 - a_2 - a_3 = 0$ and $a_1 = 0$. The boundary terms yield

$$\Gamma(1+\epsilon)\int_0^1 da_1 da_2 da_3 \frac{\delta(1-\sum_{i=1}^3 a_i)}{[-sa_1a_3]^{1+\epsilon}} \ - \ \Gamma(1+\epsilon)\int_0^1 da_2 da_3 da_4 \frac{\delta(1-\sum_{i=2}^4 a_i)}{[-ta_2a_4]^{1+\epsilon}} \ , \tag{4.4}$$

which is the difference of two triangle integrals, each with one massive external leg, as depicted in fig. 2. These integrals are easily evaluated,

$$I_3^{\rm 1m}(s) \equiv \Gamma(1+\epsilon)\int_0^1 d^3a_i \frac{\delta(1-\sum_{i=1}^3 a_i)}{[-sa_1a_3]^{1+\epsilon}} = \frac{r_\Gamma}{\epsilon^2}(-s)^{-1-\epsilon} \ , \tag{4.5}$$

where

$$r_\Gamma \equiv \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}$$

is a ubiquitous prefactor. Thus

$$J_{4;1} = \frac{r_\Gamma}{\epsilon^2}\Big((-s)^{-1-\epsilon} - (-t)^{-1-\epsilon}\Big) \ . \tag{4.6}$$

The other way of evaluating equation (4.3), explicit differentiation, yields

$$\begin{aligned} J_{4;1} &= -\Gamma(2+\epsilon)\int_0^1 d^4a \ \delta(1-\textstyle\sum_i^4 a_i)\displaystyle\frac{-sa_3 + ta_2}{[-sa_1a_3 - ta_2a_4]^{2+\epsilon}} \\ &= I_4[sa_3 - ta_2] = \frac{1}{2\epsilon}\left(\prod_{i=1}^4 \alpha_i\right)\left[-\frac{1}{\alpha_1}\frac{\partial}{\partial\alpha_3} + \frac{1}{\alpha_4}\frac{\partial}{\partial\alpha_2}\right]\hat{I}_4 \ , \end{aligned} \tag{4.7}$$

where we have used equations (2.11), (2.15) and (2.17) in the last step.

Equations (4.6) and (4.7) together constitute one differential equation for $\hat{I}_4$. In fact, due to the symmetries of the original integral, total derivatives in other Feynman parameters do not yield independent equations. Instead, we recognize at this stage that $\hat{I}_4$ is really a function of s and t alone, not of all four α_i,

$$\hat{I}_4 = \hat{I}_4(s,t) = \hat{I}_4\big(-(\alpha_1\alpha_3)^{-1}, -(\alpha_2\alpha_4)^{-1}\big), \tag{4.8}$$

so that

$$\frac{1}{\alpha_1}\frac{\partial\hat{I}_4}{\partial\alpha_3} = s^2\frac{\partial\hat{I}_4}{\partial s} \ , \qquad \frac{1}{\alpha_4}\frac{\partial\hat{I}_4}{\partial\alpha_2} = t^2\frac{\partial\hat{I}_4}{\partial t} \ . \tag{4.9}$$

Combining equations (4.6), (4.7) and (4.9), we see that $\hat{I}_4(s,t)$ satisfies the partial differential equation

$$s^2 \frac{\partial \hat{I}_4}{\partial s} \;-\; t^2 \frac{\partial \hat{I}_4}{\partial t} \;=\; -\frac{2r_\Gamma}{\epsilon} st \left[(-s)^{-1-\epsilon} \;-\; (-t)^{-1-\epsilon} \right] \;. \tag{4.10}$$

We still need one additional equation, which comes from the fact that the dimension of $\hat{I}_4$ is equal to $-\epsilon \times$ dimension(s,t), so that

$$s \frac{\partial \hat{I}_4}{\partial s} \;+\; t \frac{\partial \hat{I}_4}{\partial t} \;=\; -\epsilon\, \hat{I}_4 \;. \tag{4.11}$$

Equations (4.10) and (4.11) form a complete set of partial differential equations.

If we consider instead of $\hat{I}_4$ the dimensionless quantity $\hat{I}_4^0$, defined by

$$\hat{I}_4^0(s,t) \equiv \left(-\frac{s+t}{st} \right)^{-\epsilon} \hat{I}_4(s,t) \;, \tag{4.12}$$

we see that it is a function only of the ratio $\chi \equiv t/s$, and that

$$s^2 \frac{\partial \hat{I}_4^0}{\partial s} \;-\; t^2 \frac{\partial \hat{I}_4^0}{\partial t} \;=\; -t(1+\chi) \frac{d\hat{I}_4^0}{d\chi} \;. \tag{4.13}$$

In terms of χ, the first equation (4.10) becomes

$$\frac{d\hat{I}_4^0}{d\chi} \;=\; -\frac{2r_\Gamma}{\epsilon} \frac{(\chi^\epsilon - \chi^{-1})}{(1+\chi)^{1+\epsilon}} \;. \tag{4.14}$$

One can solve this differential equation to all orders in ϵ as follows. We observe that the transformation $\chi \to \chi^{-1}$ interchanges the two terms on the right-hand side. Taking the second term, shifting $\chi \to \chi - 1$, using the hypergeometric function formulæ

$$\int dz\, z^c \;\, {}_pF_q\left(\{a_i\};\{b_i\};z\right) \;=\; \frac{z^{c+1}}{c+1} \;\, {}_{p+1}F_{q+1}\left(\{a_i\}, c+1; \{b_i\}, c+2; z\right) \tag{4.15}$$

and

$${}_1F_0(\xi; z) \;=\; (1-z)^{-\xi} \;, \tag{4.16}$$

the hypergeometric function identity

$${}_2F_1(1,-\epsilon;1-\epsilon;1+\chi) \;=\; (-\chi)^\epsilon\, {}_2F_1(-\epsilon,-\epsilon;1-\epsilon;1+\chi^{-1}) \;, \tag{4.17}$$

and using the interchange of χ and χ^{-1} to furnish the first term, we obtain (note that χ should be thought of as having a small imaginary part in order to avoid difficulties with branch cuts)

$$\begin{aligned} \hat{I}_4^0 &= \frac{2r_\Gamma}{\epsilon^2} \left[\left(1+\chi^{-1}\right)^{-\epsilon} {}_2F_1\left(1,-\epsilon;1-\epsilon;1+\chi^{-1}\right) + (1+\chi)^{-\epsilon}\, {}_2F_1\left(1,-\epsilon;1-\epsilon;1+\chi\right) \right] \\ &= \frac{2r_\Gamma}{\epsilon^2} \left(1+\chi^{-1}\right)^{-\epsilon} \left[(-\chi^{-1})^\epsilon\, {}_2F_1\left(-\epsilon,-\epsilon;1-\epsilon;1+\chi\right) + (-\chi)^\epsilon \chi^{-\epsilon}\, {}_2F_1\left(-\epsilon,-\epsilon;1-\epsilon;1+\chi^{-1}\right) \right] \;. \end{aligned} \tag{4.18}$$

The constant of integration may be determined by evaluating the integral (4.1) directly at some convenient value of χ, say $\chi = t/s = 1$ $(s = t = -1)$,

$$\begin{aligned}\hat{I}_4^0(\chi = 1) &= 2^{-\epsilon} I_4[1] = 2^{-\epsilon}\Gamma(2+\epsilon)\int_0^1 d^4a_i \, \frac{\delta(1-\sum a_i)}{[a_1a_3 + a_2a_4]^{2+\epsilon}} \\ &= 2^{-\epsilon}\Gamma(2+\epsilon)\int_0^1 dx \int_0^1 dy \int_0^1 dz \; y^{-1-\epsilon}(1-y)^{-1-\epsilon}\big[(1-x)(1-z)+xz\big]^{-2-\epsilon} ,\end{aligned} \tag{4.19}$$

where we have made the change of variables (4.2). The y and z integrals are elementary and leave us with a standard hypergeometric integral,

$$\begin{aligned}\hat{I}_4^0(\chi = 1) &= -\frac{2^{1-\epsilon} r_\Gamma}{\epsilon}\int_0^1 dx \, \frac{x^{-1-\epsilon} - (1-x)^{-1-\epsilon}}{1-2x} \\ &= -\frac{2^{1-\epsilon} r_\Gamma}{\epsilon}\frac{\Gamma(-\epsilon)\Gamma(1)}{\Gamma(1-\epsilon)}\lim_{\delta\to 0}\Big({}_2F_1(1,-\epsilon;1-\epsilon;2+i\delta) - {}_2F_1(1,1;1-\epsilon;2+i\delta)\Big) \\ &= \frac{2^{1-\epsilon} r_\Gamma}{\epsilon^2}\lim_{\delta\to 0}\Big({}_2F_1(1,-\epsilon;1-\epsilon;2+i\delta) + {}_2F_1(1,-\epsilon;1-\epsilon;2-i\delta)\Big) .\end{aligned} \tag{4.20}$$

Comparing with the first line of equation (4.18), we see that the constant of integration vanishes.

Alternatively, we may solve equation (4.14) order by order in ϵ. Observe that $\hat{I}_4^0$ must contain $1/\epsilon^2$ poles from the overlap of collinear and soft singularities. As the right-hand side of the differential equation only contains a single power of $1/\epsilon$, this leading pole should be multiplied by something to the $\pm\epsilon$ power, so that one power of ϵ is cancelled upon differentiation. Through $\mathcal{O}(\epsilon^0)$, we then have

$$\begin{aligned}\hat{I}_4^0 &= r_\Gamma\left\{\frac{2}{\epsilon^2}\left[(1+\chi)^{-\epsilon} + (1+\chi^{-1})^{-\epsilon}\right] - \ln^2\chi - \pi^2\right\} + \mathcal{O}(\epsilon) \\ &= r_\Gamma\left\{\frac{2}{\epsilon^2}\left[\Big(\frac{s+t}{s}\Big)^{-\epsilon} + \Big(\frac{s+t}{t}\Big)^{-\epsilon}\right] - \ln^2\Big(\frac{t}{s}\Big) - \pi^2\right\} + \mathcal{O}(\epsilon) ,\end{aligned} \tag{4.21}$$

where the constant of integration can be fixed as in the all-orders solution.

Restoring the prefactor $(-(s+t)/st)^\epsilon$, and expressing the result in terms of the α_i, we have

$$\begin{aligned}\hat{I}_4[1] &= \frac{2r_\Gamma}{\epsilon^2}\left[(-\alpha_2\alpha_4)^\epsilon \, {}_2F_1\left(-\epsilon,-\epsilon;1-\epsilon;1+\frac{\alpha_1\alpha_3}{\alpha_2\alpha_4}\right) + (-\alpha_1\alpha_3)^\epsilon \, {}_2F_1\left(-\epsilon,-\epsilon;1-\epsilon;1+\frac{\alpha_2\alpha_4}{\alpha_1\alpha_3}\right)\right] \\ &= r_\Gamma\left[\frac{2}{\epsilon^2}\left((\alpha_1\alpha_3)^\epsilon + (\alpha_2\alpha_4)^\epsilon\right) - \ln^2\left(\frac{\alpha_1\alpha_3}{\alpha_2\alpha_4}\right) - \pi^2\right] + \mathcal{O}(\epsilon) .\end{aligned} \tag{4.22}$$

In this form, the differentiation formula (2.17) may be applied to the scalar integral $\hat{I}_4$ to obtain the integrals with arbitrary Feynman parameter polynomials inserted. Because of the $\Gamma(1-m+2\epsilon)$ prefactor in (2.17), the $\mathcal{O}(\epsilon)$ terms in $\hat{I}_4$ contribute to the polynomial integrals at $\mathcal{O}(\epsilon^0)$. Instead of displaying the $\mathcal{O}(\epsilon)$ terms in $\hat{I}_4$ explicitly, we quote the reduced integrals with one parameter inserted, $\hat{I}_4[a_i]$, to $\mathcal{O}(\epsilon^0)$:

$$\begin{aligned}
\alpha_1\,\hat{I}_4[a_1] \;=\; \alpha_3\,\hat{I}_4[a_3] \;&=\; r_\Gamma\left\{\frac{1}{\epsilon^2}(\alpha_2\alpha_4)^\epsilon - \frac{1}{2}\left(\frac{\alpha_1\alpha_3}{\alpha_1\alpha_3+\alpha_2\alpha_4}\right)\left[\ln^2\left(\frac{\alpha_1\alpha_3}{\alpha_2\alpha_4}\right)+\pi^2\right]\right\} \;+\; \mathcal{O}(\epsilon),\\
\alpha_2\,\hat{I}_4[a_2] \;=\; \alpha_4\,\hat{I}_4[a_4] \;&=\; r_\Gamma\left\{\frac{1}{\epsilon^2}(\alpha_1\alpha_3)^\epsilon - \frac{1}{2}\left(\frac{\alpha_2\alpha_4}{\alpha_1\alpha_3+\alpha_2\alpha_4}\right)\left[\ln^2\left(\frac{\alpha_1\alpha_3}{\alpha_2\alpha_4}\right)+\pi^2\right]\right\} \;+\; \mathcal{O}(\epsilon).
\end{aligned} \tag{4.23}$$

The latter integrals may be differentiated further to obtain through $\mathcal{O}(\epsilon^0)$ the integral with any polynomial of the Feynman parameters inserted.

As mentioned previously, the branch cuts can be obtained by inserting the $i\varepsilon$ associated with each kinematic variable,

$$\begin{aligned}
(-s)^{-\epsilon} \;&\to\; |s|^{-\epsilon}e^{+i\pi\epsilon\Theta(s)}\ ,\\
\ln(-s) \;&\to\; \ln|s| - i\pi\Theta(s)\ ,
\end{aligned} \tag{4.24}$$

where $\Theta(x)$ is the usual Heavyside function: $\Theta(x)=1$ for $x>0$ and $\Theta(x)=0$ for $x<0$. For the massless scalar box we therefore obtain

$$\begin{aligned}
\mathcal{I}_4[1] \;=\; i\frac{r_\Gamma}{(4\pi)^2}\,\frac{1}{st}\Bigg\{&\frac{2}{\epsilon^2}\left[\left(\frac{|s|}{4\pi\mu^2}\right)^{-\epsilon}e^{i\pi\epsilon\Theta(s)} + \left(\frac{|t|}{4\pi\mu^2}\right)^{-\epsilon}e^{i\pi\epsilon\Theta(t)}\right]\\
&-\ln^2\left|\frac{s}{t}\right| + 2\pi i\,(\Theta(s)-\Theta(t))\ln\left|\frac{s}{t}\right| - \pi^2\left[1-(\Theta(s)-\Theta(t))^2\right]\Bigg\} + \mathcal{O}(\epsilon)\ ,
\end{aligned} \tag{4.25}$$

where s and t are the Mandelstam variables defined below equation (2.6).

4.2 The Box Integral with One External Mass

Following the same techniques, we can obtain partial differential equations for boxes with one external massive leg (or equivalently, one external leg off-shell),

$$I_4^{1\mathrm{m}}(s_1,s_2,m_4^2) \;=\; \Gamma(2+\epsilon)\int_0^1 d^4a\ \delta(1-\textstyle\sum_i a_i)\frac{1}{\left[-sa_1a_3 - ta_2a_4 - m_4^2a_4a_1\right]^{2+\epsilon}}\ . \tag{4.26}$$

(This integral could also be evaluated using the same change of variables (4.2) as for the massless box.) By analogy with equation (4.4), such integrals will clearly arise in the consideration of massless pentagon integrals. Following the conventions of section 3, we label these boxes by $I_4^{(i)}$ when the momentum invariant $s_{i-1,i}$ for the adjacent legs $(i-1)$ and i of the pentagon diagram serves as the "mass" of the massive leg of the box. For example,

$$\begin{aligned}
I_4^{(5)}[1] \;&=\; I_4^{1\mathrm{m}}(s_{12},s_{23},s_{45})\\
&=\; \Gamma(2+\epsilon)\int_0^1 d^4a\,\delta(1-\textstyle\sum_i a_i)\,\frac{1}{\left[-s_{12}a_1a_3 - s_{23}a_2a_4 - s_{45}a_4a_1\right]^{2+\epsilon}}
\end{aligned} \tag{4.27}$$

is the box integral arising from the diagram depicted in fig. 3, in which a tree with external legs 4 and 5 is attached to a four-point loop. Note that the scalar denominator for the integral (4.27) can be obtained from the massless pentagon denominator by setting the parameter a_5 to zero. Similarly, $I_4^{(i)}$ can be obtained by setting $a_i \to 0$ in the massless pentagon.

From these remarks it is clear that the change of integration variables described earlier for the pentagon can be used here to remove the kinematic factors from the denominator of the box integral,

$$I_4^{(5)}[1] \;=\; \Gamma(2+\epsilon)\left(\prod_{j=1}^{4}\alpha_j\right)\int_0^1 d^4u\,\frac{\delta\left(1-\sum u_i\right)\left(\sum_{j=1}^{4}\alpha_j u_j\right)^{2\epsilon}}{\left[u_1u_3+u_2u_4+u_4u_1\right]^{2+\epsilon}}\;, \tag{4.28}$$

where α_i are given by equation (2.13). (These variables α_i should not be confused with the corresponding α_i for the massless box.) The other integrals that will arise,

$$\begin{aligned} I_4^{(1)} &= I_4^{1\mathrm{m}}(s_{23},s_{34},s_{51}), \qquad & I_4^{(2)} &= I_4^{1\mathrm{m}}(s_{34},s_{45},s_{12}),\\ I_4^{(3)} &= I_4^{1\mathrm{m}}(s_{45},s_{51},s_{23}), \qquad & I_4^{(4)} &= I_4^{1\mathrm{m}}(s_{51},s_{12},s_{34}), \end{aligned} \tag{4.29}$$

can be obtained from $I_4^{(5)}$ by cyclic permutation of the α_i. We define the reduced integral, $\hat{I}_4^{(i)}$ or $\hat{I}_4^{1\mathrm{m}}$, via equation (2.15).

We can now apply the general results of section 3 to the example of equation (4.27). The matrix ρ defined in equation (2.10) is now given by

$$\rho^{1\mathrm{m}} \;=\; \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 \end{pmatrix}\;, \tag{4.30}$$

so that $N_4^{1\mathrm{m}} = \frac{1}{2}$, and the rescaled Gram determinant is given, using eqs. (3.8) and (3.9), by

$$\hat{\Delta}_4^{1\mathrm{m}} = 2(\alpha_1\alpha_3+\alpha_2\alpha_4-\alpha_2\alpha_3)\,. \tag{4.31}$$

Using equations (3.8) and (3.10), the explicit values of the quantities $\gamma_i^{1\mathrm{m}}$ and $\eta_{ij}^{1\mathrm{m}}$ can be read off from (4.31):

$$\begin{aligned} &\gamma_1^{1\mathrm{m}} \;=\; \alpha_3, \quad \gamma_2^{1\mathrm{m}} \;=\; \alpha_4-\alpha_3, \quad \gamma_3^{1\mathrm{m}} \;=\; \alpha_1-\alpha_2, \quad \gamma_4^{1\mathrm{m}} \;=\; \alpha_2,\\ &\eta_{13}^{1\mathrm{m}} \;=\; \eta_{24}^{1\mathrm{m}} \;=\; -\eta_{23}^{1\mathrm{m}} \;=\; \eta_{31}^{1\mathrm{m}} \;=\; \eta_{42}^{1\mathrm{m}} \;=\; -\eta_{32}^{1\mathrm{m}} \;=\; 1, \qquad \text{remaining } \eta_{ij}^{1\mathrm{m}} \;=\; 0. \end{aligned} \tag{4.32}$$

In terms of these quantities, the differential equations (3.16) read

$$\begin{aligned} \frac{\partial \hat{I}_4^{1\mathrm{m}}}{\partial\alpha_i} &= 2\epsilon\left[\sum_{j=1}^{4}\left(\eta_{ij}^{1\mathrm{m}}-\frac{\gamma_i^{1\mathrm{m}}\gamma_j^{1\mathrm{m}}}{\hat{\Delta}_4^{1\mathrm{m}}}\right)\hat{I}_3^{(j)}+\frac{\gamma_i^{1\mathrm{m}}}{\hat{\Delta}_4^{1\mathrm{m}}}\hat{I}_4^{1\mathrm{m}}\right]\\ &= 2\epsilon\left[\sum_{j=1}^{4}\sqrt{\hat{\Delta}_4^{1\mathrm{m}}}\frac{\partial^2\sqrt{\hat{\Delta}_4^{1\mathrm{m}}}}{\partial\alpha_i\partial\alpha_j}\,\hat{I}_3^{(j)}+\frac{1}{\sqrt{\hat{\Delta}_4^{1\mathrm{m}}}}\frac{\partial\sqrt{\hat{\Delta}_4^{1\mathrm{m}}}}{\partial\alpha_i}\,\hat{I}_4^{1\mathrm{m}}\right]\,. \end{aligned} \tag{4.33}$$

The triangle integrals appearing on the right-hand-side of (4.33) include both the triangle integral with one external massive leg, I_3^{1m}, defined in equation (4.5), and the triangle with two external masses,

$$I_3^{2m}(s_1, s_2) \;=\; \frac{r_\Gamma}{\epsilon^2} \frac{(-s_1)^{-\epsilon} - (-s_2)^{-\epsilon}}{(-s_1) - (-s_2)} \;. \tag{4.34}$$

Explicitly, the following reduced triangle integrals appear (defined again via (2.15)):

$$\begin{aligned}
\hat{I}_3^{(1)} &= \frac{1}{\alpha_2\alpha_3\alpha_4} I_3^{1m}\left(\frac{-1}{\alpha_2\alpha_4}\right) = \frac{r_\Gamma}{\epsilon^2}\frac{\alpha_4^\epsilon \alpha_2^\epsilon}{\alpha_3} \,, \\
\hat{I}_3^{(2)} &= \frac{1}{\alpha_1\alpha_3\alpha_4} I_3^{2m}\left(\frac{-1}{\alpha_1\alpha_3}, \frac{-1}{\alpha_4\alpha_1}\right) = -\frac{r_\Gamma}{\epsilon^2}\alpha_1^\epsilon \frac{\alpha_4^\epsilon - \alpha_3^\epsilon}{\alpha_4 - \alpha_3} \,, \\
\hat{I}_3^{(3)} &= \frac{1}{\alpha_1\alpha_2\alpha_4} I_3^{2m}\left(\frac{-1}{\alpha_2\alpha_4}, \frac{-1}{\alpha_4\alpha_1}\right) = -\frac{r_\Gamma}{\epsilon^2}\alpha_4^\epsilon \frac{\alpha_1^\epsilon - \alpha_2^\epsilon}{\alpha_1 - \alpha_2} \,, \\
\hat{I}_3^{(4)} &= \frac{1}{\alpha_1\alpha_2\alpha_3} I_3^{1m}\left(\frac{-1}{\alpha_1\alpha_3}\right) = \frac{r_\Gamma}{\epsilon^2}\frac{\alpha_1^\epsilon \alpha_3^\epsilon}{\alpha_2} \,.
\end{aligned} \tag{4.35}$$

The differential equations (4.33) have the solution,

$$\begin{aligned}
\hat{I}_4^{1m} &= \frac{2r_\Gamma}{\epsilon^2}\Bigg[(-\alpha_3(\alpha_1-\alpha_2))^\epsilon \;{}_2F_1\left(-\epsilon,-\epsilon;1-\epsilon;\frac{\alpha_1\alpha_3+\alpha_2\alpha_4-\alpha_2\alpha_3}{\alpha_3(\alpha_1-\alpha_2)}\right) \\
&\quad + (-\alpha_2(\alpha_4-\alpha_3))^\epsilon \;{}_2F_1\left(-\epsilon,-\epsilon;1-\epsilon;\frac{\alpha_1\alpha_3+\alpha_2\alpha_4-\alpha_2\alpha_3}{\alpha_2(\alpha_4-\alpha_3)}\right) \\
&\quad - ((\alpha_1-\alpha_2)(\alpha_4-\alpha_3))^\epsilon \;{}_2F_1\left(-\epsilon,-\epsilon;1-\epsilon;-\frac{\alpha_1\alpha_3+\alpha_2\alpha_4-\alpha_2\alpha_3}{(\alpha_1-\alpha_2)(\alpha_4-\alpha_3)}\right)\Bigg] \\
&= \frac{2r_\Gamma}{\epsilon^2}\Bigg[\left(-\gamma_1^{1m}\gamma_3^{1m}\right)^\epsilon \;{}_2F_1\left(-\epsilon,-\epsilon;1-\epsilon;\frac{\hat{\Delta}_4^{1m}}{2\gamma_1^{1m}\gamma_3^{1m}}\right) \\
&\quad + \left(-\gamma_2^{1m}\gamma_4^{1m}\right)^\epsilon \;{}_2F_1\left(-\epsilon,-\epsilon;1-\epsilon;\frac{\hat{\Delta}_4^{1m}}{2\gamma_2^{1m}\gamma_4^{1m}}\right) \\
&\quad - \left(\gamma_2^{1m}\gamma_3^{1m}\right)^\epsilon \;{}_2F_1\left(-\epsilon,-\epsilon;1-\epsilon;-\frac{\hat{\Delta}_4^{1m}}{2\gamma_2^{1m}\gamma_3^{1m}}\right)\Bigg] \\
&= 2r_\Gamma\left[\frac{(\alpha_2\alpha_3)^\epsilon}{\epsilon^2} + \mathrm{Li}_2\left(1-\frac{\alpha_1}{\alpha_2}\right) + \mathrm{Li}_2\left(1-\frac{\alpha_4}{\alpha_3}\right) - \frac{\pi^2}{6}\right] + \mathcal{O}(\epsilon) \,,
\end{aligned} \tag{4.36}$$

where Li_2 is the dilogarithm [20], which satisfies

$$\frac{d}{dx}\,\mathrm{Li}_2(1-x) \;=\; \frac{\ln(x)}{1-x} \,, \tag{4.37}$$

and also the identity

$$\mathrm{Li}_2(1-x) + \mathrm{Li}_2(1-x^{-1}) \;=\; -\frac{1}{2}\ln^2(x), \qquad\qquad x > 0. \tag{4.38}$$

In principle, one could also add any solution of the homogeneous equations, (4.33) with $\hat{I}_3^{(j)}$ set to zero. The coefficient of such a solution vanishes, as may again be demonstrated by evaluation at a special kinematic point.

In terms of momentum invariants, the unreduced integral is

$$I_4^{(5)} = r_\Gamma \frac{2}{s_{12}s_{23}} \frac{(-s_{12})^{-\epsilon}(-s_{23})^{-\epsilon}}{(-s_{45})^{-\epsilon}} \left[\frac{1}{\epsilon^2} + \mathrm{Li}_2\Big(1 - \frac{s_{12}}{s_{45}}\Big) + \mathrm{Li}_2\Big(1 - \frac{s_{23}}{s_{45}}\Big) - \frac{\pi^2}{6}\right] + \mathcal{O}(\epsilon), \qquad (4.39)$$

or alternatively, after using the dilogarithm identity (4.38) and rearranging the terms

$$\begin{aligned} I_4^{(5)} = & \frac{r_\Gamma}{s_{12}s_{23}} \Bigg\{ \frac{2}{\epsilon^2}\Big[(-s_{12})^{-\epsilon} + (-s_{23})^{-\epsilon} - (-s_{45})^{-\epsilon}\Big] \\ & - 2\ \mathrm{Li}_2\left(1 - \frac{s_{45}}{s_{12}}\right) - 2\ \mathrm{Li}_2\left(1 - \frac{s_{45}}{s_{23}}\right) - \ln^2\left(\frac{s_{12}}{s_{23}}\right) - \frac{\pi^2}{3} \Bigg\} + \mathcal{O}(\epsilon). \end{aligned} \qquad (4.40)$$

This second form is appropriate for studying the limit $s_{45} \to 0$ as we do at the end of this section.

Including the overall normalization factors appropriate for the momentum-space integral (2.1) yields

$$\mathcal{I}_4^{(5)} = \frac{ir_\Gamma}{(4\pi)^2} \frac{2}{s_{12}s_{23}} \left(\frac{-s_{12}}{4\pi\mu^2}\right)^{-\epsilon} \left(\frac{-s_{23}}{4\pi\mu^2}\right)^{-\epsilon} \left(\frac{-s_{45}}{4\pi\mu^2}\right)^{\epsilon} \left[\frac{1}{\epsilon^2} + \mathrm{Li}_2\Big(1 - \frac{s_{12}}{s_{45}}\Big) + \mathrm{Li}_2\Big(1 - \frac{s_{23}}{s_{45}}\Big) - \frac{\pi^2}{6}\right] + \mathcal{O}(\epsilon), \qquad (4.41)$$

in agreement with the results of refs. [21]. The correct analytic continuation to the physical region can be obtained from this expression by taking $s_{ij} \to s_{ij} + i\varepsilon$.

As with the massless box, it is useful to quote the integrals $\hat{I}_4^{1\mathrm{m}}[a_i]$ to $\mathcal{O}(\epsilon^0)$; further differentiation of them will give any desired integral to $\mathcal{O}(\epsilon^0)$ as well. The $\hat{I}_4^{1\mathrm{m}}[a_i]$ may be read off from equations (4.33) and (4.36). We rewrite them in terms of a combination of dilogarithms and logarithms that will reappear in the massless pentagon tensor integrals:

$$\begin{aligned} \hat{I}_4^{(5)}[a_1] &= r_\Gamma \left[-\frac{1}{\epsilon^2} \frac{\alpha_4^\epsilon(\alpha_1^\epsilon - \alpha_2^\epsilon)}{\alpha_1 - \alpha_2} + \frac{\alpha_3\, L_5}{\alpha_1\alpha_3 + \alpha_2\alpha_4 - \alpha_2\alpha_3} \right], \\ \hat{I}_4^{(5)}[a_2] &= r_\Gamma \left[\frac{1}{\epsilon^2} \left(\frac{\alpha_1^\epsilon \alpha_3^\epsilon}{\alpha_2} + \frac{\alpha_4^\epsilon(\alpha_1^\epsilon - \alpha_2^\epsilon)}{\alpha_1 - \alpha_2} \right) + \frac{(\alpha_4 - \alpha_3)\, L_5}{\alpha_1\alpha_3 + \alpha_2\alpha_4 - \alpha_2\alpha_3} \right], \\ \hat{I}_4^{(5)}[a_3] &= r_\Gamma \left[\frac{1}{\epsilon^2} \left(\frac{\alpha_4^\epsilon \alpha_2^\epsilon}{\alpha_3} + \frac{\alpha_1^\epsilon(\alpha_4^\epsilon - \alpha_3^\epsilon)}{\alpha_4 - \alpha_3} \right) + \frac{(\alpha_1 - \alpha_2)\, L_5}{\alpha_1\alpha_3 + \alpha_2\alpha_4 - \alpha_2\alpha_3} \right], \\ \hat{I}_4^{(5)}[a_4] &= r_\Gamma \left[-\frac{1}{\epsilon^2} \frac{\alpha_1^\epsilon(\alpha_4^\epsilon - \alpha_3^\epsilon)}{\alpha_4 - \alpha_3} + \frac{\alpha_2\, L_5}{\alpha_1\alpha_3 + \alpha_2\alpha_4 - \alpha_2\alpha_3} \right], \end{aligned} \qquad (4.42)$$

where

$$L_i \equiv \mathrm{Li}_2\left(1 - \frac{\alpha_{i+1}}{\alpha_{i+2}}\right) + \mathrm{Li}_2\left(1 - \frac{\alpha_{i-1}}{\alpha_{i-2}}\right) + \ln\left(\frac{\alpha_{i+1}}{\alpha_{i+2}}\right) \ln\left(\frac{\alpha_{i-1}}{\alpha_{i-2}}\right) - \frac{\pi^2}{6}\,. \qquad (4.43)$$

Note that L_i vanishes as $\alpha_{i+1}\alpha_{i-2} + \alpha_{i+2}\alpha_{i-1} - \alpha_{i+2}\alpha_{i-2} \to 0$, so the $\hat{I}_4^{(i)}[a_j]$ are not singular in that limit.

4.3 Box Integrals with Two External Masses

In order to evaluate the pentagon integral with one external mass, or the all-massless hexagon integral, one needs box integrals with two external masses, of which there are two types, which we will call 'easy' and 'hard'. Both of these integrals have been performed previously [22,9]. The 'easy' box, with external masses at diagonally opposite corners, can be done with the same change of variables (4.2) described in section 4. We will not discuss it further, but merely quote the result,

$$
\begin{aligned}
I_4^{2\mathrm{me}}[1] \ &\equiv \ \Gamma(2+\epsilon)\int_0^1 d^4a_i \ \frac{\delta\big(1-\sum a_i\big)}{\big[-sa_1a_3-ta_2a_4-m_1^2a_1a_2-m_3^2a_3a_4\big]^{2+\epsilon}} \\
&= \ \frac{2\,r_\Gamma}{st-m_1^2m_3^2}\Bigg\{\frac{1}{\epsilon^2}\left((-s)^{-\epsilon}+(-t)^{-\epsilon}-(-m_1^2)^{-\epsilon}-(-m_3^2)^{-\epsilon}\right) \ + \ \mathrm{Li}_2\left(1-\frac{m_1^2m_3^2}{st}\right) \\
&\quad -\mathrm{Li}_2\left(1-\frac{m_1^2}{s}\right)-\mathrm{Li}_2\left(1-\frac{m_1^2}{t}\right)-\mathrm{Li}_2\left(1-\frac{m_3^2}{s}\right)-\mathrm{Li}_2\left(1-\frac{m_3^2}{t}\right)-\frac{1}{2}\ln^2\left(\frac{s}{t}\right)\Bigg\}+\mathcal{O}(\epsilon) \ .
\end{aligned}
\tag{4.44}
$$

The 'hard' box, with external masses at adjacent corners (legs 3 and 4),

$$
I_4^{2\mathrm{m}h}[1] \ = \ \Gamma(2+\epsilon)\int_0^1 d^4a\frac{\delta(1-\sum_i a_i)}{\big[-sa_1a_3-ta_2a_4-m_3^2a_3a_4-m_4^2a_4a_1\big]^{2+\epsilon}} \ , \tag{4.45}
$$

cannot be easily done this way; but it is amenable to the partial differential equation technique. We change to α_i variables defined by

$$
s \ = \ -\frac{1}{\alpha_1\alpha_3} \ , \qquad t \ = \ -\frac{1}{\alpha_2\alpha_4} \ , \qquad m_3^2 \ = \ -\frac{1}{\alpha_3\alpha_4} \ , \qquad m_4^2 \ = \ -\frac{1}{\alpha_4\alpha_1} \ .
$$

Then the matrix ρ is given by

$$
\rho^{2\mathrm{m}h} \ = \ \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \ , \tag{4.46}
$$

we have $N_4^{2\mathrm{m}h}=\frac{1}{2}$, and the rescaled Gram determinant is

$$
\hat\Delta_4^{2\mathrm{m}h} = 2(\alpha_1\alpha_3+\alpha_2\alpha_4-\alpha_1\alpha_2-\alpha_2\alpha_3+\alpha_2^2) \ , \tag{4.47}
$$

from which $\gamma_i^{2\mathrm{m}h}$ and $\eta_{ij}^{2\mathrm{m}h}$ can be obtained via equations (3.8) and (3.10).

To obtain the box integral to $\mathcal{O}(\epsilon^0)$, we can use the simple partial differential equations (3.26), which are sensitive only to the pieces of the triangle integrals that are singular as $\epsilon\to 0$. In particular, the three-mass triangle does not contribute, because it is finite as $\epsilon\to 0$. We find that to $\mathcal{O}(\epsilon^0)$,

$$
\begin{aligned}
\frac{\partial \hat I_4^{2\mathrm{m}h}}{\partial\alpha_i} \ &= \ 2\epsilon\sum_{j=1}^{4}\eta_{ij}^{2\mathrm{m}h}\,\hat I_3^{(j)} \\
&= \ 2\,r_\Gamma\left[\frac{1}{\epsilon}\eta_{i4}^{2\mathrm{m}h}\frac{1}{\alpha_2}-\eta_{i1}^{2\mathrm{m}h}\frac{\ln(\alpha_2/\alpha_3)}{\alpha_2-\alpha_3}-\eta_{i3}^{2\mathrm{m}h}\frac{\ln(\alpha_1/\alpha_2)}{\alpha_1-\alpha_2}+\eta_{i4}^{2\mathrm{m}h}\frac{\ln(\alpha_1\alpha_3)}{\alpha_2}\right] \ .
\end{aligned}
\tag{4.48}
$$

Writing

$$\hat{I}_4^{2\mathrm{m}h} \;=\; r_\Gamma \left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon} X_{-1} + X_0 + c_0\right] \;, \tag{4.49}$$

and solving the differential equations for X_{-1} and X_0, we find

$$\begin{aligned} X_{-1} \;&=\; 2 \ln \alpha_2, \\ X_0 \;&=\; 2\,\mathrm{Li}_2\left(1 - \frac{\alpha_1}{\alpha_2}\right) + 2\,\mathrm{Li}_2\left(1 - \frac{\alpha_3}{\alpha_2}\right) + 2 \ln^2 \alpha_2, \end{aligned} \tag{4.50}$$

or

$$\hat{I}_4^{2\mathrm{m}h} \;=\; r_\Gamma\, \alpha_2^{2\epsilon} \left[\frac{1}{\epsilon^2} + 2\,\mathrm{Li}_2\left(1 - \frac{\alpha_1}{\alpha_2}\right) + 2\,\mathrm{Li}_2\left(1 - \frac{\alpha_3}{\alpha_2}\right) + c_0\right] \;+\; \mathcal{O}(\epsilon). \tag{4.51}$$

The constant c_0 may be determined by computing the function at a specific point, say where all the α_i are equal; the resulting integral is evaluated explicitly in appendix IV, whence we find $c_0 = 0$. Finally, rewriting the result (4.51) back in terms of the conventional kinematic variables yields

$$I_4^{2\mathrm{m}h}[1] = r_\Gamma \frac{(-m_3^2)^\epsilon(-m_4^2)^\epsilon}{(-t)^{1+2\epsilon}(-s)^{1+\epsilon}} \left[\frac{1}{\epsilon^2} + 2\,\mathrm{Li}_2\left(1 - \frac{t}{m_3^2}\right) + 2\,\mathrm{Li}_2\left(1 - \frac{t}{m_4^2}\right)\right] \;+\; \mathcal{O}(\epsilon). \tag{4.52}$$

Using the dilogarithm identity (4.38) and rearranging the terms this can be written in the alternative form

$$\begin{aligned} I_4^{2\mathrm{m}h}[1] \;=\; & \frac{r_\Gamma}{st}\Bigg\{\frac{2}{\epsilon^2}\Big[(-s)^{-\epsilon} + (-t)^{-\epsilon} - (-m_3^2)^{-\epsilon} - (-m_4^2)^{-\epsilon}\Big] \;+\; \frac{1}{\epsilon^2}\frac{(-m_3^2)^{-\epsilon}(-m_4^2)^{-\epsilon}}{(-s)^{-\epsilon}} \\ & -\; 2\;\mathrm{Li}_2\left(1 - \frac{m_3^2}{t}\right) \;-\; 2\;\mathrm{Li}_2\left(1 - \frac{m_4^2}{t}\right) \;-\; \ln^2\left(\frac{s}{t}\right)\Bigg\} \;+\; \mathcal{O}(\epsilon), \end{aligned} \tag{4.53}$$

which is more convenient for studying the massless limit, as we do at the end of this section.

4.4 The Box Integral with Three External Masses

Here we compute the three-mass scalar box integral,

$$I_4^{3\mathrm{m}}[1] \;=\; \Gamma(2+\epsilon) \int d^4 a_i \; \frac{\delta\left(1 - \sum a_i\right)}{\left[-s a_1 a_3 - t a_2 a_4 - m_2^2 a_2 a_3 - m_3^2 a_3 a_4 - m_4^2 a_4 a_1\right]^{2+\epsilon}} \;. \tag{4.54}$$

We again use the partial differential equations (3.26), with the change-of-variables

$$s \;=\; -\frac{1}{\alpha_1\alpha_3}, \quad t \;=\; -\frac{1}{\alpha_2\alpha_4}, \quad m_2^2 \;=\; -\frac{\lambda}{\alpha_2\alpha_3}, \quad m_3^2 \;=\; -\frac{1}{\alpha_3\alpha_4}, \quad m_4^2 \;=\; -\frac{1}{\alpha_4\alpha_1}, \tag{4.55}$$

which is the same as that used for the hard two-mass box, except that now $\lambda \neq 0$. The matrix ρ becomes

$$\rho_4^{3\mathrm{m}} \;=\; \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & \lambda & 1 \\ 1 & \lambda & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \;, \tag{4.56}$$

the normalization factor in (3.26) is $N_4^{3m} = \frac{1}{2}(1-\lambda)^2$, and the matrix η used to construct $\hat{\Delta}_4^{3m}$ is

$$\eta^{3m} = \begin{pmatrix} 2\lambda & -1-\lambda & 1-\lambda & -\lambda(1-\lambda) \\ -1-\lambda & 2 & -(1-\lambda) & 1-\lambda \\ 1-\lambda & -(1-\lambda) & 0 & 0 \\ -\lambda(1-\lambda) & 1-\lambda & 0 & 0 \end{pmatrix} . \tag{4.57}$$

We expand $\hat{I}_4^{3m}$ as

$$\hat{I}_4^{3m} = r_\Gamma \Big[\frac{c_1(\lambda)}{\epsilon} + X_0(\alpha_i, \lambda) + c_0(\lambda) \Big] + \mathcal{O}(\epsilon). \tag{4.58}$$

(We will see below that there is no $1/\epsilon^2$ singularity.) To solve the partial differential equations order-by-order in ϵ we need to first know $c_1(\lambda)$. We know that $c_1(\lambda)$ is independent of the α_i because the daughter triangles here are of the two-mass and three-mass varieties; the two-mass triangle has a $1/\epsilon$ pole, which feeds into $X_0(\alpha_i, \lambda)$, while the three-mass triangle is finite and can be ignored altogether. So we may compute $c_1(\lambda)$ by doing the integral $\hat{I}_4^{3m}$ for the special choice of all $\alpha_i = 1$. We should compute the finite part of the integral while we're at it, since this result will fix the constant of integration $c_0(\lambda)$. This computation is done in appendix IV, where we find

$$c_1(\lambda) = \frac{\ln \lambda}{1-\lambda} . \tag{4.59}$$

Next we solve the partial differential equations (3.26). Plug the expansion of $\hat{I}_4^{3m}$ (equation (4.58)) and the divergent pieces of the 2-mass triangles $\hat{I}_3^{(3)}$ and $\hat{I}_3^{(4)}$,

$$\begin{aligned} \hat{I}_3^{(3)} &= -\frac{r_\Gamma}{\epsilon} \frac{\ln(\alpha_1/\alpha_2)}{\alpha_1 - \alpha_2} + \mathcal{O}(\epsilon^0) , \\ \hat{I}_3^{(4)} &= -\frac{r_\Gamma}{\epsilon} \frac{\ln(\lambda\alpha_1/\alpha_2)}{\lambda\alpha_1 - \alpha_2} + \mathcal{O}(\epsilon^0) , \end{aligned} \tag{4.60}$$

into the far right-hand-side of (3.26) and use the result (4.57) for η_{ij}, to get

$$\begin{aligned} \frac{\partial X_0}{\partial \alpha_1} &= \frac{2}{1-\lambda} \Big[-\frac{\ln(\alpha_1/\alpha_2)}{\alpha_1 - \alpha_2} + \lambda \frac{\ln(\lambda\alpha_1/\alpha_2)}{\lambda\alpha_1 - \alpha_2} \Big] , \\ \frac{\partial X_0}{\partial \alpha_2} &= \frac{2}{1-\lambda} \Big[\frac{\ln(\alpha_1/\alpha_2)}{\alpha_1 - \alpha_2} - \frac{\ln(\lambda\alpha_1/\alpha_2)}{\lambda\alpha_1 - \alpha_2} \Big] , \\ \frac{\partial X_0}{\partial \alpha_3} &= 0 , \\ \frac{\partial X_0}{\partial \alpha_4} &= 0 . \end{aligned} \tag{4.61}$$

Solving these equations for $X_0(\alpha_i, \lambda)$, and fixing the constant $c_0(\lambda)$ using equation (IV.6), yields

$$\begin{aligned} \hat{I}_4^{3m} = \frac{r_\Gamma}{1-\lambda} \Big[& \frac{\ln \lambda}{\epsilon} + 2\ \mathrm{Li}_2\Big(1 - \frac{\alpha_1}{\alpha_2}\Big) - 2\ \mathrm{Li}_2\Big(1 - \frac{\lambda\alpha_1}{\alpha_2}\Big) + 2\ \mathrm{Li}_2(1-\lambda) + 2 \ln\lambda\ \ln\alpha_2 - \frac{1}{2}\ln^2\lambda \Big] \\ & + \mathcal{O}(\epsilon) . \end{aligned} \tag{4.62}$$

Returning to the original kinematic variables, and using dilogarithm identities [20], we get

$$
\begin{aligned}
I_4^{3\mathrm{m}}(s,t,m_i^2) \;=\; & \frac{r_\Gamma}{st - m_2^2 m_4^2}\Bigg\{ \frac{2}{\epsilon^2}\Big[(-s)^{-\epsilon} + (-t)^{-\epsilon} - (-m_2^2)^{-\epsilon} - (-m_3^2)^{-\epsilon} - (-m_4^2)^{-\epsilon}\Big] \\
& + \frac{1}{\epsilon^2}\frac{(-m_2^2)^{-\epsilon}(-m_3^2)^{-\epsilon}}{(-t)^{-\epsilon}} + \frac{1}{\epsilon^2}\frac{(-m_3^2)^{-\epsilon}(-m_4^2)^{-\epsilon}}{(-s)^{-\epsilon}} \\
& - 2\,\mathrm{Li}_2\left(1 - \frac{m_2^2}{s}\right) - 2\,\mathrm{Li}_2\left(1 - \frac{m_4^2}{t}\right) + 2\,\mathrm{Li}_2\left(1 - \frac{m_2^2 m_4^2}{st}\right) - \ln^2\left(\frac{s}{t}\right)\Bigg\} + \mathcal{O}(\epsilon)\ .
\end{aligned}
\tag{4.63}
$$

4.5 The Box Integral with Four External Masses

The four-mass box integral is infrared finite and has been performed in $D = 4$ in ref. [6]; a compact expression is given in ref. [15]. Amusingly, the partial differential equations (3.26) for it are trivial, because the three-mass triangles appearing on the right-hand-side are non-singular as $\epsilon \to 0$. In other words, through $\mathcal{O}(\epsilon)$, the reduced four-mass box cannot depend on the α_i, but only on the two other, dimensionless variables, say λ_1 and λ_2, where we define

$$
s = -\frac{1}{\alpha_1\alpha_3},\quad t = -\frac{1}{\alpha_2\alpha_4},\quad m_1^2 = -\frac{\lambda_1}{\alpha_1\alpha_2},\quad m_2^2 = -\frac{\lambda_2}{\alpha_2\alpha_3},\quad m_3^2 = -\frac{1}{\alpha_3\alpha_4},\quad m_4^2 = -\frac{1}{\alpha_4\alpha_1}.
\tag{4.64}
$$

One can check that the answer $D_0(s,t,m_i^2)$ given in ref. [15] does have this property — when the integral is divided by $\alpha_1\alpha_2\alpha_3\alpha_4$ it depends only on λ_1 and λ_2. Indeed,

$$
\begin{aligned}
\hat{I}_4^{4\mathrm{m}} \;=\; \frac{D_0}{\prod \alpha_i} \;=\; \frac{1}{r}\Bigg\{ & \mathrm{Li}_2\left(\tfrac{1}{2}(1-\lambda_1+\lambda_2+r)\right) - \mathrm{Li}_2\left(\tfrac{1}{2}(1-\lambda_1+\lambda_2-r)\right) \\
& + \mathrm{Li}_2\left(\tfrac{-1}{2\lambda_1}(1-\lambda_1-\lambda_2-r)\right) - \mathrm{Li}_2\left(\tfrac{-1}{2\lambda_1}(1-\lambda_1-\lambda_2+r)\right) \\
& + \frac{1}{2}\ln\left(\frac{\lambda_1}{\lambda_2^2}\right)\ln\left(\frac{1+\lambda_1-\lambda_2+r}{1+\lambda_1-\lambda_2-r}\right)\Bigg\} + \mathcal{O}(\epsilon)\ ,
\end{aligned}
\tag{4.65}
$$

where

$$
r \;\equiv\; \sqrt{1 - 2\lambda_1 - 2\lambda_2 + \lambda_1^2 - 2\lambda_1\lambda_2 + \lambda_2^2}\ .
\tag{4.66}
$$

4.6 The Massless Limit of Massive Boxes

In general there is no reason for the massless limits to be smooth. The limit of taking a mass to zero does not necessarily commute with the $1/\epsilon$ expansion of dimensional regularization, which has been truncated at $\mathcal{O}(\epsilon^0)$. For $\epsilon < 0$ (as is required to regulate the infrared divergences in the box integrals), we see that the single external mass box $I_4^{1\mathrm{m}}(s,t,m_4^2)$ (given in equation (4.40) with $s = s_{12}$, $t = s_{23}$, $m_4^2 = s_{45}$) goes over smoothly to the massless box $I_4^{0\mathrm{m}}(s,t)$ as $m_4 \to 0$, and the easy two-mass box $I_4^{2\mathrm{m}e}(s,t,m_3^2,m_4^2)$ goes over smoothly to the one-mass box $I_4^{1\mathrm{m}}(s,t,m_4^2)$ as $m_3 \to 0$. On the other hand, the limits, $I_4^{2\mathrm{m}h} \to I_4^{1\mathrm{m}}$, $I_4^{3\mathrm{m}} \to I_4^{2\mathrm{m}e}$, and $I_4^{3\mathrm{m}} \to I_4^{2\mathrm{m}h}$, are not smooth: in each of these cases there are “missing” dilogarithms.

The fact that some of the above limits happen to be smooth, with only the exponentiation of the logarithms $(-s)^{-\epsilon}, (-t)^{-\epsilon}, (-m_i^2)^{-\epsilon}$, can be understood from the representation (3.23) (for $n = 4$) of the $D = 4 - 2\epsilon$ box integral as the sum of $D = 4 - 2\epsilon$ triangles and a $D = 6 - 2\epsilon$ box integral. The $D = 6 - 2\epsilon$ box integral is infrared (and ultraviolet) convergent for any choice of mass, so it has a smooth limit as any mass goes to zero. The $D = 4 - 2\epsilon$ triangles appearing in the representations (3.23) for $I_4^{\rm 0m}$, $I_4^{\rm 1m}$ and $I_4^{{\rm 2m}e}$ have either one or two nonvanishing external masses; these integrals can be written in closed form to all order in ϵ merely by exponentiating logarithms. (See equations (4.5) and (4.34).) In contrast, the representations (3.23) of the box integrals $I_4^{{\rm 2m}h}$ and $I_4^{\rm 3m}$ require the triangle with three external masses, $I_3^{\rm 3m}$, whose all-orders-in-ϵ form (V.11) is considerably more complicated, involving hypergeometric functions. One should not expect that these latter box integrals, truncated to $\mathcal{O}(\epsilon^0)$, could be made to have smooth limits simply by exponentiating logarithms.

5. Algebraic Approach to Pentagon Integrals

It is possible to solve the partial differential equations (3.17) for the massless scalar pentagon through $\mathcal{O}(\epsilon^0)$. However, a simpler approach, which works equally well for arbitrary pentagon kinematics, is to use the general algebraic equation (3.23) derived in section 3 to express the scalar pentagon integral $\hat{I}_5$ as a sum of five scalar box integrals, up to $\mathcal{O}(\epsilon)$ corrections:

$$\hat{I}_5 \;=\; \frac{1}{2N_5}\left[\sum_{i=1}^{5}\gamma_i\,\hat{I}_4^{(i)} \;+\; 2\epsilon\,\hat{\Delta}_5\,\hat{I}_5^{D=6-2\epsilon}\right]. \tag{5.1}$$

(See also fig. 1.) To give an explicit expression for the pentagon, we need only collect the relevant scalar box integrals from section 4, and compute the kinematic coefficients N_5, $\hat{\Delta}_5$, γ_i and η_{ij}. (The η_{ij} are relevant for computing tensor integrals.) We now do this for the all-massless pentagon integral, and for the pentagon with one external mass.

5.1 The Massless Pentagon Integral

For the massless pentagon, equation (2.7) for the scalar denominator, with the change of variables (2.12), leads to a matrix ρ given by

$$\rho = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}. \tag{5.2}$$

We find that $N_5 = 1$, and

$$\begin{aligned}\hat{\Delta}_5 &= \sum_{i=1}^{5}\Big(\alpha_i^2 - 2\alpha_i\alpha_{i+1} + 2\alpha_i\alpha_{i+2}\Big) \ , \\ \gamma_i &= \alpha_{i-2} - \alpha_{i-1} + \alpha_i - \alpha_{i+1} + \alpha_{i+2} \ , \\ \eta_{ij} &= 1 - 2\,\delta_{i,j-1} - 2\,\delta_{i,j+1} \ .\end{aligned} \tag{5.3}$$

Plugging the one-mass box integrals (4.36) into equation (3.24), and using the dilogarithm identity (4.38), we obtain

$$\hat{I}_5[1] \ = \ r_\Gamma \sum_{j=1}^{5} \alpha_j^{1+2\epsilon}\left[\frac{1}{\epsilon^2} + 2\,\mathrm{Li}_2\Big(1 - \frac{\alpha_{j+1}}{\alpha_j}\Big) + 2\,\mathrm{Li}_2\Big(1 - \frac{\alpha_{j-1}}{\alpha_j}\Big) - \frac{\pi^2}{6}\right] \ + \ \mathcal{O}(\epsilon). \tag{5.4}$$

In terms of momentum invariants, the unreduced integral is

$$I_5 \ = \ \frac{r_\Gamma \ (-s_{51})^\epsilon(-s_{12})^\epsilon}{(-s_{23})^{1+\epsilon}(-s_{34})^{1+\epsilon}(-s_{45})^{1+\epsilon}}\left[\frac{1}{\epsilon^2} + 2\,\mathrm{Li}_2\Big(1 - \frac{s_{23}}{s_{51}}\Big) + 2\,\mathrm{Li}_2\Big(1 - \frac{s_{45}}{s_{12}}\Big) - \frac{\pi^2}{6}\right] + \text{cyclic} + \mathcal{O}(\epsilon). \tag{5.5}$$

From this expression we can obtain the value in any region by using the usual $i\varepsilon$ prescription and observing that I_5 is manifestly real in the region where all $s_{ij} < 0$.

For the tensor integrals, we do need some information about the $\mathcal{O}(\epsilon)$ parts of the scalar pentagon. It turns out that leaving the six-dimensional pentagon $\hat{I}_5^{D=6-2\epsilon}$ in equation (5.1) leaves us with enough information about these terms that we *can* use the scalar pentagon as a generating function for the tensor integrals to $\mathcal{O}(1)$, without having to evaluate $\hat{I}_5^{D=6-2\epsilon}$ explicitly. (The explicit solution for $\hat{I}_5^{D=6}$ involves a rather long combination of Li_3's, Li_2's, and logarithms whose arguments are complicated solutions of various quadratic equations.) We show how to do so in the next section.

5.2 The Pentagon Integral with One External Mass

For the pentagon with one external mass, $m_5 \neq 0$, we use the same change of variables (2.12), (2.13) as in the massless case, except that we also define the rescaled mass $\hat{m}_5^2 \equiv -\alpha_5\alpha_1 m_5^2$, which is taken to be a variable independent of the α_i. We find that the normalization factor is $N_5 = 1 - \hat{m}_5^2$, while the rescaled Gram determinant is given by

$$\hat{\Delta}_5^{\rm 1m} \ = \ \hat{\Delta}_5^{\rm 0m} \ + \ \hat{m}_5^2(-2\alpha_1\alpha_3 + 2\alpha_2\alpha_3 - 2\alpha_3^2 - 4\alpha_2\alpha_4 + 2\alpha_3\alpha_4 - 2\alpha_3\alpha_5) \ + \ \alpha_3^2(\hat{m}_5^2)^2 \ , \tag{5.6}$$

where $\hat{\Delta}_5^{\rm 0m}$ is given in equation (5.3). Using these values in the general expression for the scalar pentagon (3.24), and collecting the box integrals with one and two external masses from section 4, we get

$$
\begin{aligned}
I_5^{1\mathrm{m}}[1] \;=\;& r_\Gamma\Big(\prod \alpha_\ell\Big)\bigg\{\frac{1}{\epsilon^2}\bigg[\alpha_2^{1+2\epsilon}+\alpha_3^{1+2\epsilon}+\alpha_4^{1+2\epsilon}+\big[(\alpha_1-\hat m_5^2\alpha_2)\alpha_1^{2\epsilon}+(\alpha_5-\hat m_5^2\alpha_4)\alpha_5^{2\epsilon}\big]\,\frac{1-(\hat m_5^2)^{-\epsilon}}{1-\hat m_5^2}\bigg]\\
&+\;2\alpha_2\left[\mathrm{Li}_2\left(1-\frac{\alpha_1}{\alpha_2}\right)+\mathrm{Li}_2\left(1-\frac{\alpha_3}{\alpha_2}\right)\right]+\;2\alpha_4\left[\mathrm{Li}_2\left(1-\frac{\alpha_5}{\alpha_4}\right)+\mathrm{Li}_2\left(1-\frac{\alpha_3}{\alpha_4}\right)\right]\\
&+\;2\alpha_3\left[\mathrm{Li}_2\left(1-\frac{\alpha_2}{\alpha_3}\right)+\mathrm{Li}_2\left(1-\frac{\alpha_4}{\alpha_3}\right)\right]+\;\frac{\alpha_1-\alpha_2+(1-\hat m_5^2)\alpha_3-\alpha_4+\alpha_5}{1-\hat m_5^2}\,\mathrm{Li}_2\left(1-\hat m_5^2\right)\\
&+\;2\frac{\alpha_1-\hat m_5^2\alpha_2}{1-\hat m_5^2}\left[\mathrm{Li}_2\left(1-\frac{\alpha_2}{\alpha_1}\right)-\mathrm{Li}_2\left(1-\frac{\alpha_2}{\alpha_1}\hat m_5^2\right)\right]+\;2\frac{\alpha_5-\hat m_5^2\alpha_4}{1-\hat m_5^2}\left[\mathrm{Li}_2\left(1-\frac{\alpha_4}{\alpha_5}\right)-\mathrm{Li}_2\left(1-\frac{\alpha_4}{\alpha_5}\hat m_5^2\right)\right]\\
&-\;\frac{\pi^2}{3}\,\alpha_3\bigg\}\;+\;\mathcal{O}(\epsilon),
\end{aligned}
\tag{5.7}
$$

or in terms of more conventional kinematic variables

$$
\begin{aligned}
I_5^{1\mathrm{m}}[1] =& -\frac{r_\Gamma}{s_{12}s_{23}s_{34}s_{45}s_{51}}\bigg\{\frac{1}{\epsilon^2}\bigg[\frac{(-s_{34})^{1+\epsilon}(-s_{45})^{1+\epsilon}}{(-s_{51})^\epsilon(-s_{12})^\epsilon(-s_{23})^\epsilon}+\frac{(-s_{45})^{1+\epsilon}(-s_{51})^{1+\epsilon}}{(-s_{12})^\epsilon(-s_{23})^\epsilon(-s_{34})^\epsilon}+\frac{(-s_{51})^{1+\epsilon}(-s_{12})^{1+\epsilon}}{(-s_{23})^\epsilon(-s_{34})^\epsilon(-s_{45})^\epsilon}\\
&\quad+\frac{s_{45}s_{51}}{s_{45}s_{51}-m_5^2s_{23}}\bigg[s_{23}s_{34}\left(1-\frac{m_5^2}{s_{51}}\right)\frac{(-s_{23})^\epsilon(-s_{34})^\epsilon}{(-s_{45})^\epsilon(-s_{51})^\epsilon(-s_{12})^\epsilon}\\
&\qquad+\;s_{12}s_{23}\left(1-\frac{m_5^2}{s_{45}}\right)\frac{(-s_{12})^\epsilon(-s_{23})^\epsilon}{(-s_{34})^\epsilon(-s_{45})^\epsilon(-s_{51})^\epsilon}\bigg]\left(1-\left(\frac{m_5^2s_{23}}{s_{45}s_{51}}\right)^{-\epsilon}\right)\bigg]\\
&+\;2s_{34}s_{45}\left[\mathrm{Li}_2\left(1-\frac{s_{23}}{s_{45}}\right)+\mathrm{Li}_2\left(1-\frac{s_{51}}{s_{34}}\right)\right]+\;2s_{51}s_{12}\left[\mathrm{Li}_2\left(1-\frac{s_{23}}{s_{51}}\right)+\mathrm{Li}_2\left(1-\frac{s_{45}}{s_{12}}\right)\right]\\
&+\;2s_{45}s_{51}\left[\mathrm{Li}_2\left(1-\frac{s_{34}}{s_{51}}\right)+\mathrm{Li}_2\left(1-\frac{s_{12}}{s_{45}}\right)\right]\\
&+\;s_{45}s_{51}\frac{s_{23}s_{34}-s_{34}s_{45}+s_{45}s_{51}-m_5^2s_{23}-s_{51}s_{12}+s_{12}s_{23}}{s_{45}s_{51}-m_5^2s_{23}}\,\mathrm{Li}_2\left(1-\frac{m_5^2s_{23}}{s_{45}s_{51}}\right)\\
&+\;2\frac{(s_{51}-m_5^2)s_{23}s_{34}s_{45}}{s_{45}s_{51}-s_{23}m_5^2}\left[\mathrm{Li}_2\left(1-\frac{s_{45}}{s_{23}}\right)-\mathrm{Li}_2\left(1-\frac{m_5^2}{s_{51}}\right)\right]\\
&+\;2\frac{(s_{45}-m_5^2)s_{51}s_{12}s_{23}}{s_{45}s_{51}-s_{23}m_5^2}\left[\mathrm{Li}_2\left(1-\frac{s_{51}}{s_{23}}\right)-\mathrm{Li}_2\left(1-\frac{m_5^2}{s_{45}}\right)\right]-\frac{\pi^2}{3}\,s_{45}s_{51}\bigg\}\;+\;\mathcal{O}(\epsilon).
\end{aligned}
\tag{5.8}
$$

Observe that it has the expected symmetry under flipping external legs $1 \leftrightarrow 4$ and $2 \leftrightarrow 3$. The limit of the expression (5.8) as $m_5 \to 0$ does not yield the massless pentagon integral (5.5), for similar reasons as explained at the end of section 4 for box integrals. The single mass pentagon $I_5^{1\mathrm{m}}$, which is given through $\mathcal{O}(\epsilon^0)$, should not be expected to have a smooth limit onto the massless pentagon as $m_5 \to 0$, because $I_5^{1\mathrm{m}}$ incorporates the box integral $I_4^{2\mathrm{m}h}$, and through it the triangle integral $I_3^{3\mathrm{m}}$ which does not have a smooth limit.

6. Feynman Parameters in the Numerator

In this section, we explain how to use the scalar pentagon $\hat{I}_5$, when expressed in terms of box integrals and $\hat{I}_5^{D=6-2\epsilon}$ via equation (5.1), as a generating function for the tensor integrals $\hat{I}_5[P(a_i)]$ through $\mathcal{O}(\epsilon^0)$. The general discussion applies to the pentagon integral with any number of external (or internal) masses; we shall also give explicit formulæ for the massless pentagon at the end of the section.

The only complication in applying the differentiation formula (2.19) is the appearance of $\hat{I}_5^{D=6-2\epsilon}$ and its derivatives at $\mathcal{O}(\epsilon^0)$ when the degree of $P(a_i)$ is two or higher. It is easy to eliminate the derivatives of $\hat{I}_5^{D=6-2\epsilon}$ in favor of $\hat{I}_5^{D=6-2\epsilon}$ itself and the $D = 6-2\epsilon$ scalar box integrals $\hat{I}_4^{D=6-2\epsilon\ (j)}$. We just let $\epsilon \to \epsilon - 1$ in equation (3.17), whence

$$\frac{\partial \hat{I}_5^{D=6-2\epsilon}}{\partial \alpha_i} \ = \ (-1+2\epsilon) \left[\sum_{j=1}^{5} \frac{1}{2N_5} \left(\eta_{ij} - \frac{\gamma_i \gamma_j}{\hat{\Delta}_5} \right) \hat{I}_4^{D=6-2\epsilon\ (j)} \ + \ \frac{\gamma_i}{\hat{\Delta}_5} \hat{I}_5^{D=6-2\epsilon} \right] . \tag{6.1}$$

Since each term in this equation is nonsingular as $\epsilon \to 0$, and since we need $\hat{I}_5^{D=6-2\epsilon}$ only to $\mathcal{O}(\epsilon^0)$, we can set $\epsilon = 0$ in $\hat{I}_5^{D=6-2\epsilon}$ and $\hat{I}_4^{D=6-2\epsilon\ (j)}$, and use in place of (6.1) the slightly simpler equation

$$\frac{\partial \hat{I}_5^{D=6}}{\partial \alpha_i} \ = \ -\sum_j \frac{1}{2N_5} \left(\eta_{ij} - \frac{\gamma_i \gamma_j}{\hat{\Delta}_5} \right) \hat{I}_4^{D=6\ (j)} \ - \ \frac{\gamma_i}{\hat{\Delta}_5} \hat{I}_5^{D=6} . \tag{6.2}$$

The $D = 6$ scalar box integrals can be worked out directly, or they can be determined from the $D = 4-2\epsilon$ box integrals and triangle integrals, using equation (3.25) with $n = 4$. For the box with one external mass, needed for the massless pentagon, the explicit result is

$$\hat{I}_4^{D=6\ (j)} \ = \ -\frac{4\, L_j}{\hat{\Delta}_5 - \gamma_j^2} , \tag{6.3}$$

where L_j is defined in equation (4.43).

Having eliminated its derivatives, we still have to deal with the appearance of $\hat{I}_5^{D=6}$ itself in the integrals $\hat{I}_5[P(a_i)]$, for $m \geq 2$. The way to proceed is suggested by an argument due to Ellis, Giele and Yehudai [23]. They work in terms of loop-momentum integrals directly, and use the Brown-Feynman or Passarino-Veltman procedure to solve for the tensor pentagon integrals in terms of lower-order tensor integrals (pentagons and boxes), all evaluated in $D = 4-2\epsilon$. The quantity $\hat{I}_5^{D=6}$ does not appear at $\mathcal{O}(\epsilon^0)$ in any momentum-space tensor integral. This fact suggests that in our approach, $\hat{I}_5^{D=6}$ will cancel out of the integral of any Feynman parameter polynomial that is the Feynman parametrization of some tensor integral in momentum-space. In appendix III, we show explicitly that this is indeed true for integrals with up to three loop-momenta inserted. It is straightforward to extend the argument to five loop-momenta, the maximum number encountered in any gauge theory amplitude. (Beyond five insertions of the loop momentum, ultraviolet divergences of the integrals complicate matters.)

While $\hat{I}_5^{D=6}$ disappears from the final answer in any gauge theory calculation, it is still useful to know with what coefficient it appears in any particular term. One may use the vanishing of its coefficient in the final expression as a check on the complete calculation. Also, it is simple to write recursive formulæ for the integrals of monomials in the Feynman parameters using this information. Let us work out the coefficient of $\hat{I}_5^{D=6}$ in $\hat{I}_5[a_{i_1}\ldots a_{i_m}]$. Define $d_{i_1\ldots i_m}$ by

$$\hat{I}_5[a_{i_1}\ldots a_{i_m}] \;\equiv\; \frac{d_{i_1\ldots i_m}}{N_5\,\hat{\Delta}_5^{m-1}}\;\hat{I}_5^{D=6} \;+\; \ldots, \tag{6.4}$$

where '...' denotes scalar box integrals (in $D=4-2\epsilon$ and in $D=6$) and their derivatives. Notice from equations (6.2), (3.8), and (3.10) that $\sqrt{\hat{\Delta}_5}\hat{I}_5^{D=6}$ satisfies a simple equation,

$$\frac{\partial\left(\sqrt{\hat{\Delta}_5}\hat{I}_5^{D=6}\right)}{\partial\alpha_i} \;=\; -\frac{1}{2N_5}\sum_{j=1}^{5}\left(\eta_{ij}-\frac{\gamma_i\gamma_j}{\hat{\Delta}_5}\right)\sqrt{\hat{\Delta}_5}\,\hat{I}_4^{D=6\ (j)}\ . \tag{6.5}$$

Now write the term $(\epsilon/N_5)\,\hat{\Delta}_5\,\hat{I}_5^{D=6}$ in equation (5.1) as

$$\frac{\epsilon}{N_5}\sqrt{\hat{\Delta}_5}\times\left(\sqrt{\hat{\Delta}_5}\,\hat{I}_5^{D=6}\right)\,, \tag{6.6}$$

and apply the differentiation formula (2.19) to get

$$d_{i_1\ldots i_m} \;=\; \frac{(-1)^m\,\hat{\Delta}_5^{m-1/2}}{2(m-2)!}\;\frac{\partial^m\hat{\Delta}_5^{1/2}}{\partial\alpha_{i_1}\ldots\partial\alpha_{i_m}}\ . \qquad m\geq 2. \tag{6.7}$$

We have taken the limit $\epsilon\to 0$ in the Γ-function prefactor in (2.19), since we are working only to $O(\epsilon^0)$. Carrying out the differentiations explicitly for the cases of interest, $m=2,3,4,5$, we get

$$\begin{aligned}
d_{ij} \;&=\; \tfrac{1}{2}\Big[\eta_{ij}\,\hat{\Delta}_5-\gamma_i\gamma_j\Big]\ ,\\
d_{ijk} \;&=\; \tfrac{1}{2}\Big[(\eta_{ij}\,\gamma_k+\eta_{jk}\,\gamma_i+\eta_{ki}\,\gamma_j)\hat{\Delta}_5-3\,\gamma_i\gamma_j\gamma_k\Big]\ ,\\
d_{ijkl} \;&=\; -\tfrac{1}{4}\Big[(\eta_{ij}\eta_{kl}+\eta_{ik}\eta_{jl}+\eta_{il}\eta_{jk})\hat{\Delta}_5^2\\
&\qquad -3(\eta_{ij}\gamma_k\gamma_l+\eta_{ik}\gamma_j\gamma_l+\eta_{il}\gamma_j\gamma_k+\eta_{jk}\gamma_i\gamma_l+\eta_{jl}\gamma_i\gamma_k+\eta_{kl}\gamma_i\gamma_j)\hat{\Delta}_5+15\,\gamma_i\gamma_j\gamma_k\gamma_l\Big]\ ,\\
d_{ijklm} \;&=\; -\tfrac{1}{4}\Big[\big(\eta_{ij}\eta_{kl}\gamma_m\;+\;\text{perms of } ijklm\ (15\text{ terms})\big)\hat{\Delta}_5^2\\
&\qquad -5\big(\eta_{ij}\gamma_k\gamma_l\gamma_m\;+\;\text{perms of } ijklm\ (10\text{ terms})\big)\hat{\Delta}_5+35\,\gamma_i\gamma_j\gamma_k\gamma_l\gamma_m\Big]\ .
\end{aligned} \tag{6.8}$$

In some calculational schemes for gauge theory amplitudes, the $D=6-2\epsilon$ pentagon integral will itself appear with Feynman parameter polynomials of degree $m\leq 3$ inserted. It is then useful to know the coefficient $d^{D=6}_{i_1\ldots i_m}$ defined by

$$\hat{I}_5^{D=6}[a_{i_1}\ldots a_{i_m}] \;\equiv\; \frac{d^{D=6}_{i_1\ldots i_m}}{\hat{\Delta}_5^{m}}\;\hat{I}_5^{D=6} \;+\; \ldots, \tag{6.9}$$

where again '...' denotes scalar box integrals (in $D=6$) and their derivatives. Writing $\hat{I}_5^{D=6} = \hat{\Delta}_5^{-1/2} \times (\hat{\Delta}_5^{1/2} \hat{I}_5^{D=6})$ and repeating the above steps we find

$$d_{i_1\ldots i_m}^{D=6} \;=\; \frac{(-1)^m\,\hat{\Delta}_5^{m+1/2}}{m!}\;\frac{\partial^m \hat{\Delta}_5^{-1/2}}{\partial\alpha_{i_1}\ldots\partial\alpha_{i_m}}\;. \qquad m \geq 2. \tag{6.10}$$

The explicit values for the cases of interest are

$$\begin{aligned} d_i &= \gamma_i\;, \\ d_{ij} &= -\tfrac{1}{2}\Big[\eta_{ij}\,\hat{\Delta}_5 - 3\gamma_i\gamma_j\Big]\;, \\ d_{ijk} &= -\tfrac{1}{2}\Big[(\eta_{ij}\,\gamma_k + \eta_{jk}\,\gamma_i + \eta_{ki}\,\gamma_j)\hat{\Delta}_5 - 5\,\gamma_i\gamma_j\gamma_k\Big]\;. \end{aligned} \tag{6.11}$$

Now we shall give explicit formulæ for the massless pentagon integrals with up to two Feynman parameters inserted, along with a simple recursion relation for generating the remainder of the integrals.

For a single parameter insertion, equation (2.19) gives

$$\hat{I}_5\,[a_i] \;=\; \frac{1}{1+2\epsilon}\frac{\partial \hat{I}_5}{\partial\alpha_i}\;. \tag{6.12}$$

Thus we may differentiate the $\mathcal{O}(\epsilon^0)$ expression (5.4) for $\hat{I}_5$, using also equation (4.37), to get

$$\hat{I}_5\,[a_i] \;=\; r_\Gamma\,\left[\frac{\alpha_i^{2\epsilon}}{\epsilon^2} \;+\; 2\;\mathrm{Li}_2\left(1-\frac{\alpha_{i+1}}{\alpha_i}\right) \;+\; 2\;\mathrm{Li}_2\left(1-\frac{\alpha_{i-1}}{\alpha_i}\right) - \frac{\pi^2}{6}\right] \;+\; \mathcal{O}(\epsilon). \tag{6.13}$$

In the case of two Feynman parameter insertions, we have

$$\hat{I}_5\,[a_i a_j] \;=\; \frac{1}{2\epsilon(1+2\epsilon)}\,\frac{\partial^2 \hat{I}_5}{\partial\alpha_i\partial\alpha_j}\;, \tag{6.14}$$

which must now be applied to the expression (5.1) for $\hat{I}_5$. The $\mathcal{O}(1)$ terms in $\hat{I}_5\,[a_i a_j]$ receive contributions both from $\hat{I}_5^{D=6}$ and from the $\mathcal{O}(\epsilon)$ terms in the box integrals $\hat{I}_4^{(j)}$. Since γ_i is linear in the α_i, only *derivatives* of the $\hat{I}_4^{(j)}$ appear on the right-hand side of (6.14). The derivatives $\partial\hat{I}_4^{(j)}/\partial\alpha_i$ at $\mathcal{O}(\epsilon)$ are nothing but the single insertions $\hat{I}_4^{(j)}[a_i]$ at $\mathcal{O}(1)$, thanks to the box differentiation formula

$$\hat{I}_4^{(j)}\,[a_i] \;=\; \frac{1}{2\epsilon}\,\frac{\partial \hat{I}_4^{(j)}}{\partial\alpha_i}\;. \tag{6.15}$$

These integrals are tabulated in equation (4.42). Carrying out the differentiations in (6.14), we find

$$\begin{aligned} \hat{I}_5[a_i a_j] \;=\; r_\Gamma\Bigg\{ & \frac{\delta_{i,j}}{\epsilon^2}\left(\frac{\alpha_{i+1}^\epsilon\alpha_{i-1}^\epsilon}{\alpha_i} + \frac{\alpha_{i-2}^\epsilon(\alpha_i^\epsilon - \alpha_{i+1}^\epsilon)}{\alpha_i-\alpha_{i+1}} + \frac{\alpha_{i+2}^\epsilon(\alpha_i^\epsilon-\alpha_{i-1}^\epsilon)}{\alpha_i-\alpha_{i-1}}\right) \\ & - \frac{\delta_{i+1,j}\,\alpha_{i-2}^\epsilon + \delta_{j+1,i}\,\alpha_{j-2}^\epsilon}{\epsilon^2}\,\frac{\alpha_i^\epsilon-\alpha_j^\epsilon}{\alpha_i-\alpha_j} \\ & + \sum_{k=1}^{5}\left[\eta_{ik}\gamma_j \;+\; \eta_{jk}\gamma_i \;-\; \eta_{ik}\eta_{jk}\gamma_k \;-\; \frac{\gamma_i\gamma_j\gamma_k}{\hat{\Delta}_5}\right]\frac{L_k}{\hat{\Delta}_5-\gamma_k^2} \;+\; \frac{d_{ij}}{\hat{\Delta}_5}\;\hat{I}_5^{D=6}\Bigg\} \;+\; \mathcal{O}(\epsilon). \end{aligned} \tag{6.16}$$

For a generalization of this formula to arbitrary pentagon kinematics, and also to hexagon ($n = 6$) integrals, see equation (VII.8) in appendix VII.

For more than two parameters inserted, we can proceed recursively. Define some new quantities $\hat{I}_5^{\rm phys}[a_{i_1} \ldots a_{i_m}]$ to be the "non-$\hat{I}_5^{D=6}$" terms in $\hat{I}_5[a_{i_1} \ldots a_{i_m}]$, i.e.

$$\hat{I}_5[a_{i_1} \ldots a_{i_m}] \;\equiv\; \hat{I}_5^{\rm phys}[a_{i_1} \ldots a_{i_m}] \;+\; \frac{d_{i_1 \ldots i_m}}{\hat{\Delta}_5^{m-1}} \; \hat{I}_5^{D=6} \;. \tag{6.17}$$

Then the differentiation formula (2.19) along with (6.2) generates the following recursion relation for $\hat{I}_5^{\rm phys}[a_{i_1} \ldots a_{i_m}]$:

$$\hat{I}_5^{\rm phys}[a_{i_1} \ldots a_{i_m}] \;=\; \left(\frac{-1}{m-2-2\epsilon} \right) \left[\frac{\partial \hat{I}_5^{\rm phys}[a_{i_1} \ldots a_{i_{m-1}}]}{\partial \alpha_{i_m}} \;+\; 4 \, \frac{d_{i_1 \cdots i_{m-1}}}{\hat{\Delta}_5^{m-1}} \sum_{j=1}^{5} \frac{d_{i_m j} \, L_j}{\hat{\Delta}_5 - \gamma_j^2} \right] \;. \tag{6.18}$$

In applying this formula, it is convenient to have a differentiation formula for the L_j, in terms of logarithms:

$$\begin{aligned} \frac{\partial L_j}{\partial \alpha_i} \;&=\; \frac{1}{\alpha_i} \left(\delta_{i,j+1} - \delta_{i,j+2} \right) \left[- \frac{\alpha_{j+1} \ln(\alpha_{j+1}/\alpha_{j+2})}{\alpha_{j+1} - \alpha_{j+2}} + \ln(\alpha_{j-1}/\alpha_{j-2}) \right] \\ &+\; \frac{1}{\alpha_i} \left(\delta_{i,j-1} - \delta_{i,j-2} \right) \left[- \frac{\alpha_{j-1} \ln(\alpha_{j-1}/\alpha_{j-2})}{\alpha_{j-1} - \alpha_{j-2}} + \ln(\alpha_{j+1}/\alpha_{j+2}) \right] \;. \end{aligned} \tag{6.19}$$

This completes our prescription for evaluating massless pentagon integrals with Feynman parameters inserted, in terms of dilogarithms and logarithms. The same basic procedure also works when external and/or internal masses are present, provided that the relevant box and triangle integrals are known through $\mathcal{O}(\epsilon^0)$. (The triangles appear through equation (6.15) in combination with (3.16).) If all internal lines are massless, then all the requisite boxes and triangles can be found in section 4, except for the three-mass triangle. This triangle may be computed in $D = 4$; see for example refs. [6,17]. In appendix V it is computed in $D = 4 - 2\epsilon$ for arbitrary ϵ, as a further illustration of the partial differential equation approach to scalar integrals.

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Appendix I. Collection of Massless Pentagon and Scalar Box Results

In this appendix we collect those results that are useful in an explicit calculation. The massless pentagon integral of interest is

$$I_5[P_m(\{a_i\})] \;=\; \Gamma(3+\epsilon) \int_0^1 d^5 a_i \; \frac{\delta\left(1 - \sum_i a_i\right) P_m(\{a_i\})}{\left[-s_{12} a_1 a_3 - s_{23} a_2 a_4 - s_{34} a_3 a_5 - s_{45} a_4 a_1 - s_{51} a_5 a_2 - i\varepsilon \right]^{3+\epsilon}} \;, \tag{I.1}$$

where $P_m(\{a_i\})$ is a polynomial in the a_i of degree m. For use in differentiation formulae we define the reduced integrals

$$\hat{I}_n\left[\hat{P}(\{a_i\})\right] \;=\; \left(\prod_{j=1}^{n}\alpha_j\right)^{-1} I_n\left[P(\{a_i/\alpha_i\})\right] \;. \tag{I.2}$$

where $s_{i,i+1} = -1/(\alpha_i\alpha_{i+1})$ mod 5. The basic differentiation formula for the pentagon is given by

$$\hat{I}_5\left[\hat{P}_m(\{a_i\})\right] \;=\; \frac{\Gamma(2-m+2\epsilon)}{\Gamma(2+2\epsilon)}\, P_m\left(\left\{\frac{\partial}{\partial\alpha_i}\right\}\right)\, \hat{I}_5\left[1\right] \;. \tag{I.3}$$

Through $\mathcal{O}(1)$ the scalar pentagon is given by

$$\hat{I}_5[1] \;=\; r_\Gamma \sum_{j=1}^{5} \alpha_j^{1+2\epsilon}\left[\frac{1}{\epsilon^2} + 2\,\mathrm{Li}_2\Big(1-\frac{\alpha_{j+1}}{\alpha_j}\Big) + 2\,\mathrm{Li}_2\Big(1-\frac{\alpha_{j-1}}{\alpha_j}\Big) - \frac{\pi^2}{6}\right] \;+\; \mathcal{O}(\epsilon) \;. \tag{I.4}$$

where $r_\Gamma \;\equiv\; \Gamma(1+\epsilon)\Gamma^2(1-\epsilon)/\Gamma(1-2\epsilon)$. The case of one Feynman parameter in the numerator may be obtained by directly applying the differentiation formula (I.3).

Beyond one Feynman parameter it is best to use the explicit value of the two Feynman parameter integral as a generating function for integrals with three or more Feynman parameters in the numerator. The two parameter integral is given by

$$\begin{aligned} \hat{I}_5[a_i a_j] \;=\; r_\Gamma \Bigg\{ & \frac{\delta_{i,j}}{\epsilon^2}\left(\frac{\alpha_{i+1}^\epsilon\alpha_{i-1}^\epsilon}{\alpha_i} + \frac{\alpha_{i-2}^\epsilon(\alpha_i^\epsilon-\alpha_{i+1}^\epsilon)}{\alpha_i-\alpha_{i+1}} + \frac{\alpha_{i+2}^\epsilon(\alpha_i^\epsilon-\alpha_{i-1}^\epsilon)}{\alpha_i-\alpha_{i-1}}\right) \\ & - \;\frac{\delta_{i+1,j}\,\alpha_{i-2}^\epsilon + \delta_{j+1,i}\,\alpha_{j-2}^\epsilon}{\epsilon^2}\,\frac{\alpha_i^\epsilon-\alpha_j^\epsilon}{\alpha_i-\alpha_j} \\ & + \;\sum_{k=1}^{5}\left[\eta_{ik}\gamma_j \;+\; \eta_{jk}\gamma_i \;-\; \eta_{ik}\eta_{jk}\gamma_k \;-\; \frac{\gamma_i\gamma_j\gamma_k}{\hat{\Delta}_5}\right]\frac{L_k}{\hat{\Delta}_5-\gamma_k^2} \;+\; \frac{d_{ij}}{\hat{\Delta}_5}\,\hat{I}_5^{D=6}\Bigg\} \;+\; \mathcal{O}(\epsilon), \end{aligned} \tag{I.5}$$

where

$$\begin{aligned} \hat{\Delta}_5 \;&\equiv\; \sum_{j=1}^{5}\big(\alpha_j^2 - 2\alpha_j\alpha_{j+1} + 2\alpha_j\alpha_{j+2}\big) \\ \gamma_i \;&\equiv\; \frac{1}{2}\frac{\partial\hat{\Delta}_5}{\partial\alpha_i} \;=\; \alpha_{i-2}-\alpha_{i-1}+\alpha_i-\alpha_{i+1}+\alpha_{i+2}\;, \\ \eta_{ij} \;&\equiv\; \frac{\partial\gamma_i}{\partial\alpha_j} \;=\; \frac{1}{2}\frac{\partial^2\hat{\Delta}_5}{\partial\alpha_i\partial\alpha_j} \;=\; \begin{cases} -1, & i=j\pm 1, \\ +1, & \text{otherwise}, \end{cases} \\ d_{ij} \;&=\; \tfrac{1}{2}\left[\eta_{ij}\,\hat{\Delta}_5 - \gamma_i\gamma_j\right] \end{aligned} \tag{I.6}$$

and

$$L_i \;\equiv\; \mathrm{Li}_2\left(1-\frac{\alpha_{i+1}}{\alpha_{i+2}}\right) + \mathrm{Li}_2\left(1-\frac{\alpha_{i-1}}{\alpha_{i-2}}\right) + \ln\left(\frac{\alpha_{i+1}}{\alpha_{i+2}}\right)\ln\left(\frac{\alpha_{i-1}}{\alpha_{i-2}}\right) - \frac{\pi^2}{6} \;. \tag{I.7}$$

In calculations there is no need to know the explicit value of the six-dimensional pentagon $\hat{I}_5^{D=6}$ since it cancels from all quantities arising from loop momentum integrals. However, when applying the differentiation formula (I.3) the terms containing $\hat{I}_5^{D=6}$ cannot be dropped since they generate $D = 6$ box integrals via the equation

$$\frac{\partial \hat{I}_5^{D=6}}{\partial \alpha_i} = -\sum_{j=1}^{5} \frac{1}{2} \left(\eta_{ij} - \frac{\gamma_i \gamma_j}{\hat{\Delta}_5} \right) \hat{I}_4^{D=6\ (j)} - \frac{\gamma_i}{\hat{\Delta}_5} \hat{I}_5^{D=6} . \tag{I.8}$$

Useful formulæ when applying the differentiation formula (I.3) are

$$\begin{aligned} \frac{\partial L_j}{\partial \alpha_i} &= \frac{1}{\alpha_i} (\delta_{i,j+1} - \delta_{i,j+2}) \left[-\frac{\alpha_{j+1} \ln(\alpha_{j+1}/\alpha_{j+2})}{\alpha_{j+1} - \alpha_{j+2}} + \ln(\alpha_{j-1}/\alpha_{j-2}) \right] \\ &+ \frac{1}{\alpha_i} (\delta_{i,j-1} - \delta_{i,j-2}) \left[-\frac{\alpha_{j-1} \ln(\alpha_{j-1}/\alpha_{j-2})}{\alpha_{j-1} - \alpha_{j-2}} + \ln(\alpha_{j+1}/\alpha_{j+2}) \right] . \end{aligned} \tag{I.9}$$

and

$$\hat{I}_4^{D=6\ (j)} = -\frac{4\, L_j}{\hat{\Delta}_5 - \gamma_j^2} , \tag{I.10}$$

where L_j is defined in equation (I.7).

We collect here the dimensionally-regulated scalar box integrals with massless internal lines, but 0, 1, 2 or 3 nonzero external masses, which appear in the process of evaluating ($n \geq 5$)-point integrals, and in subdiagrams in QCD loop calculations. The integrals are defined through equations (2.5) and (2.6).

$$I_4^{0\mathrm{m}}(s,t) = \frac{r_\Gamma}{st} \left\{ \frac{2}{\epsilon^2} \Big[(-s)^{-\epsilon} + (-t)^{-\epsilon} \Big] - \ln^2\left(\frac{s}{t}\right) - \pi^2 \right\} + \mathcal{O}(\epsilon), \tag{I.11}$$

$$\begin{aligned} I_4^{1\mathrm{m}}(s,t,m_4^2) &= \frac{r_\Gamma}{st} \left\{ \frac{2}{\epsilon^2} \Big[(-s)^{-\epsilon} + (-t)^{-\epsilon} - (-m_4^2)^{-\epsilon} \Big] \right. \\ &\left. - 2\ \mathrm{Li}_2\left(1 - \frac{m_4^2}{s}\right) - 2\ \mathrm{Li}_2\left(1 - \frac{m_4^2}{t}\right) - \ln^2\left(\frac{s}{t}\right) - \frac{\pi^2}{3} \right\} + \mathcal{O}(\epsilon), \end{aligned} \tag{I.12}$$

$$\begin{aligned} I_4^{2\mathrm{me}}(s,t,m_2^2,m_4^2) &= \frac{r_\Gamma}{st - m_2^2 m_4^2} \left\{ \frac{2}{\epsilon^2} \Big[(-s)^{-\epsilon} + (-t)^{-\epsilon} - (-m_2^2)^{-\epsilon} - (-m_4^2)^{-\epsilon} \Big] \right. \\ &- 2\ \mathrm{Li}_2\left(1 - \frac{m_2^2}{s}\right) - 2\ \mathrm{Li}_2\left(1 - \frac{m_2^2}{t}\right) - 2\ \mathrm{Li}_2\left(1 - \frac{m_4^2}{s}\right) - 2\ \mathrm{Li}_2\left(1 - \frac{m_4^2}{t}\right) \\ &\left. + 2\ \mathrm{Li}_2\left(1 - \frac{m_2^2 m_4^2}{st}\right) - \ln^2\left(\frac{s}{t}\right) \right\} + \mathcal{O}(\epsilon), \end{aligned} \tag{I.13}$$

$$\begin{aligned} I_4^{2\mathrm{m}h}(s,t,m_3^2,m_4^2) &= \frac{r_\Gamma}{st} \left\{ \frac{2}{\epsilon^2} \Big[(-s)^{-\epsilon} + (-t)^{-\epsilon} - (-m_3^2)^{-\epsilon} - (-m_4^2)^{-\epsilon} \Big] + \frac{1}{\epsilon^2} \frac{(-m_3^2)^{-\epsilon}(-m_4^2)^{-\epsilon}}{(-s)^{-\epsilon}} \right. \\ &\left. - 2\ \mathrm{Li}_2\left(1 - \frac{m_3^2}{t}\right) - 2\ \mathrm{Li}_2\left(1 - \frac{m_4^2}{t}\right) - \ln^2\left(\frac{s}{t}\right) \right\} + \mathcal{O}(\epsilon), \end{aligned} \tag{I.14}$$

$$
\begin{aligned}
I_4^{3\mathrm{m}}(s,t,m_i^2) \;=\; & \frac{r_\Gamma}{st-m_2^2m_4^2}\left\{\frac{2}{\epsilon^2}\Big[(-s)^{-\epsilon}+(-t)^{-\epsilon}-(-m_2^2)^{-\epsilon}-(-m_3^2)^{-\epsilon}-(-m_4^2)^{-\epsilon}\Big]\right.\\
& +\;\frac{1}{\epsilon^2}\frac{(-m_2^2)^{-\epsilon}(-m_3^2)^{-\epsilon}}{(-t)^{-\epsilon}} \;+\; \frac{1}{\epsilon^2}\frac{(-m_3^2)^{-\epsilon}(-m_4^2)^{-\epsilon}}{(-s)^{-\epsilon}}\\
& -\;2\ \mathrm{Li}_2\left(1-\frac{m_2^2}{s}\right) \;-\; 2\ \mathrm{Li}_2\left(1-\frac{m_4^2}{t}\right)\\
& \left.+\;2\ \mathrm{Li}_2\left(1-\frac{m_2^2m_4^2}{st}\right) \;-\; \ln^2\left(\frac{s}{t}\right)\right\} \;+\; \mathcal{O}(\epsilon).
\end{aligned}
\tag{I.15}
$$

Appendix II. Connection with the Work of van Neerven and Vermaseren

Melrose [5] and Van Neerven and Vermaseren [7] were able to represent the general scalar pentagon integral in $D=4$ as a sum of five $D=4$ box integrals. On the other hand, equation (3.23) expresses the pentagon integral in $D=4-2\epsilon$ as a linear combination of five box integrals (also in $D=4-2\epsilon$), plus the pentagon in $D=6-2\epsilon$ dimensions, so it can be thought of as the dimensionally-regulated version of the equations in refs. [5,7]. Indeed, the $D=4$ equation in ref. [7] was our motivation to find an algebraic $D=4-2\epsilon$ equation. (Similar relations have recently been found using momentum-space, rather than Feynman parameter, techniques by Ellis, Giele and Yehudai [9].) We would like to verify that the $D=4-2\epsilon$ and $D=4$ equations are consistent with each other, or in other words that the (D-independent) coefficients of the box integrals in equation (3.24) are equal to the corresponding coefficients in refs. [5,7] (up to normalization conventions for the integrals). To do this, it is simplest to rewrite equation (3.24) in terms of unreduced integrals as

$$
I_5[1] \;=\; \frac{1}{2}\sum_{i=1}^{5} c_i\ I_4^{(i)}[1] \;+\; \mathcal{O}(\epsilon) \tag{II.1}
$$

where

$$
c_i \;=\; \frac{\alpha_i\gamma_i}{N_5} \;=\; \sum_{j=1}^{5} S_{ij}^{-1}\ . \tag{II.2}
$$

The second form of the c_i, in terms of more conventional kinematic variables (the matrix S is defined in (2.4)), is the form in which the c_i were obtained in ref. [10]. In this form the c_i are manifestly the same as those found by Melrose.

The coefficients found by van Neerven and Vermaseren involve the $D=4$ Levi-Civita tensor, and are not manifestly equal to (II.2). Expressed in our notation, they are given by

$$
c_1 \;=\; -\frac{4\Delta_5 - 2\sum_{i=1}^{4} v_i\cdot w}{w^2-4\Delta_5 M_1^2}\ , \qquad c_{i+1} \;=\; -\frac{2v_i\cdot w}{w^2-4\Delta_5 M_1^2}\ , \qquad i=1,2,3,4. \tag{II.3}
$$

Here the "axial vectors" v_i are the $D = 4$ duals of the vectors p_i appearing in the momentum-space version of the pentagon integral

$$v_1^\mu \equiv \varepsilon^{\mu p_2 p_3 p_4}, \quad v_2^\mu \equiv \varepsilon^{p_1 \mu p_3 p_4}, \quad v_3^\mu \equiv \varepsilon^{p_1 p_2 \mu p_4}, \quad v_4^\mu \equiv \varepsilon^{p_1 p_2 p_3 \mu},$$
$$p_i^\mu \equiv \sum_{j=1}^{i} k_j^\mu, \qquad p_5^\mu = 0, \tag{II.4}$$

where $\varepsilon_{\mu p_2 p_3 p_4}$ is short for $\varepsilon_{\mu\mu_2\mu_3\mu_4} p_2^{\mu_2} p_3^{\mu_3} p_4^{\mu_4}$, etc. The Gram determinant of the vectors p_i is $\Delta_5 \equiv \varepsilon^{p_1 p_2 p_3 p_4} \varepsilon_{p_1 p_2 p_3 p_4}$, and w^μ is defined by

$$w^\mu \equiv \sum_{i=1}^{4} r_i v_i^\mu, \qquad r_i \equiv p_i^2 + M_1^2 - M_{i+1}^2, \quad i = 1,2,3,4, \tag{II.5}$$

where M_i are the masses on the internal lines. The definition of the c_i in (II.3) may seem to be tied to $D = 4$, because of the presence of the axial vectors. However, the inner products $v_i \cdot v_j$ can be eliminated in favor of the inverse of the matrix $t_{ij} \equiv 2p_i \cdot p_j$, according to

$$v_i \cdot v_j = 2\Delta_5 \, (t^{-1})_{ij} \, , \qquad i,j = 1,2,3,4. \tag{II.6}$$

Thus the c_i can be written in a D-independent form:

$$c_1 = \frac{-2 \; + \; 2\sum_{i,j=1}^4 (t^{-1})_{ij} \, r_j}{\sum_{k,l=1}^4 r_k \, (t^{-1})_{kl} \, r_l \; - \; 2M_1^2} \, ,$$
$$c_{i+1} = \frac{-2\sum_{j=1}^4 (t^{-1})_{ij} \, r_j}{\sum_{k,l=1}^4 r_k \, (t^{-1})_{kl} \, r_l \; - \; 2M_1^2}, \qquad i = 1,2,3,4. \tag{II.7}$$

To show that the c_i in (II.7) agree with those in (II.2), it suffices to show that they obey

$$\sum_{j=1}^{5} S_{ij} c_j = 1, \qquad i = 1, \ldots, 5, \tag{II.8}$$

since S is generically invertible for $n = 5$. When internal masses are also present, S is given by

$$S_{ij} = \tfrac{1}{2}(M_i^2 + M_j^2 - (p_{i-1} - p_{j-1})^2) = \tfrac{1}{2}(2M_1^2 - r_{i-1} - r_{j-1} + t_{i-1,j-1}) \, , \tag{II.9}$$

and $r_{j-1} = t_{i-1,j-1} = 0$ for $j = 1$, so that

$$\sum_{j=1}^{5} S_{ij} c_j = \frac{1}{2}(2M_1^2 - r_{i-1}) \left(\sum_{j=1}^{5} c_j \right) + \frac{1}{2} \sum_{j=1}^{4} (-r_j + t_{i-1,j}) c_{j+1}) \, . \tag{II.10}$$

Plugging in the values of c_i from equation (II.7), we get

$$\sum_{j=1}^{5} S_{ij} c_j = \frac{\left(-2M_1^2 \; + \; r_{i-1}\right) \; + \; \left(\sum_{j,k=1}^4 r_j \, (t^{-1})_{jk} \, r_k \; - \; r_{i-1}\right)}{\sum_{k,l=1}^4 r_k \, (t^{-1})_{kl} \, r_l \; - \; 2M_1^2} = 1, \tag{II.11}$$

as required.

In the same fashion, an equation obtained by van Neerven and Vermaseren, relating hexagon integrals to pentagon integrals, can be shown to be equivalent to equation (3.23) for $n = 6$ (and $D = 4$ external kinematics).

Appendix III. Proof That $I_5^{D=6}$ Drops Out

An explanation of why an explicit computation of $I_5^{D=6}$ is not needed for the evaluation of pentagon integrals near $D=4$ comes from the momentum-space representation of tensor integrals; when performing a Passarino-Veltman decomposition $I_5^{D=6}$ never appears [23] and therefore it can be expected to cancel from amplitudes evaluated using the Feynman parameter techniques discussed in this paper. In this appendix, we will demonstrate that $I_5^{D=6}$ cancels when summing over contributions which reconstruct the loop momentum integrals appearing in dimensionally-regulated four-dimensional field theory amplitudes. Thus, there is no need to explicitly evaluate $I_5^{D=6}$. (In this appendix we treat $I_5^{D=6}$ as equivalent to $I_5^{D=6-2\epsilon}$; since $I_5^{D=6}$ is completely finite the difference between the two is of $\mathcal{O}(\epsilon)$.) The argument holds for general kinematics (arbitrary external or internal masses), though here we suppress internal masses.

Define the general pentagon integral by

$$I_5[P(p^\mu)] \equiv i\,(4\pi)^{2-\epsilon} 4! \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \int d^5 a_i \frac{\delta\big(1-\sum a_i\big) P(p^\mu)}{\big(a_1 p^2 + a_2(p-p_1)^2 + a_3(p-p_2)^2 + a_4(p-p_3)^2 + a_5(p-p_4)^2\big)^5} \tag{III.1}$$

where $p_i \equiv \sum_{j=1}^{i} k_j$ and $P(p^\mu)$ is some polynomial in the loop momentum p^μ. The normalization factor in front ensures that the integral, when Feynman-parameterized, is normalized in the same way as the integrals $I_5[P(a_i)]$ defined in section 2.

In order to relate the integral (III.1) to Feynman-parametrized integrals of the form (2.5), we complete the square and integrate out the loop momentum in the usual fashion. To complete the square in the denominator, we shift the loop-momentum variables to

$$p = q + \sum_{i=1}^{4} a_{i+1} p_i \,. \tag{III.2}$$

Integrating out the loop momentum, for up to three powers of loop momentum in the numerator, then gives

$$\begin{aligned}
I_5[p^\mu] &= \sum_{i=1}^{4} I_5[a_{i+1}]\, p_i^\mu, \\
I_5[p^\mu p^\nu] &= -\frac{1}{2} I_5^{D=6}[1]\ \delta^{\mu\nu}_{[4-2\epsilon]} + \sum_{i,j=1}^{4} p_i^\mu p_j^\nu I_5[a_{i+1}a_{j+1}]\ , \\
I_5[p^\mu p^\nu p^\rho] &= -\frac{1}{2}\Big(\delta^{\mu\nu}_{[4-2\epsilon]} \sum_i p_i^\rho I_5^{D=6}[a_{i+1}] + \delta^{\mu\rho}_{[4-2\epsilon]} \sum_i p_i^\nu I_5^{D=6}[a_{i+1}] + \delta^{\nu\rho}_{[4-2\epsilon]} \sum_i p_i^\mu I_5^{D=6}[a_{i+1}]\Big) \\
&\qquad + \sum_{ijk} p_i^\mu p_j^\nu p_k^\rho I_5[a_{i+1}a_{j+1}a_{k+1}]\ .
\end{aligned} \tag{III.3}$$

Here we will explicitly consider only up to three loop momenta; the other cases follow similarly.

For the case with one loop momentum inserted, since the explicit value for $I_5[a_{i+1}]$ given in equation (6.13) does not contain $I_5^{D=6}$, there is nothing to check. Beyond this, we have from section 6 that the coefficient of $I_5^{D=6}$ in the explicit value for $I_5[a_1 \cdots a_k]$ is given by

$$c_{i_1 i_2 \cdots i_m} \;=\; \frac{(-1)^m \hat{\Delta}_5^{1/2}}{2N_5\,(m-2)!} \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_m} \frac{\partial^m \hat{\Delta}_5^{1/2}}{\partial\alpha_{i_1}\partial\alpha_{i_2}\cdots\partial\alpha_{i_m}} \qquad (m \geq 2), \tag{III.4}$$

so that for $m = 2, 3$,

$$\begin{aligned} c_{ij} &= \frac{\alpha_i\alpha_j}{2N_5\,\hat{\Delta}_5}\Big[\eta_{ij}\,\hat{\Delta}_5 - \gamma_i\gamma_j\Big] \ , \\ c_{ijk} &= \frac{\alpha_i\alpha_j\alpha_k}{2N_5\,\hat{\Delta}_5^2}\Big[(\eta_{ij}\,\gamma_k + \eta_{jk}\,\gamma_i + \eta_{ki}\,\gamma_j)\hat{\Delta}_5 - 3\,\gamma_i\gamma_j\gamma_k\Big] \ , \end{aligned} \tag{III.5}$$

where γ_i and η_{ij} are defined in equations (3.8) and (3.10).

The identity that we will use to show that $I_5^{D=6}$ cancels is

$$c_{ij} \;=\; \frac{\alpha_i\alpha_j}{2N_5\,\hat{\Delta}_5}\Big(\eta_{ij}\,\hat{\Delta}_5 - \gamma_i\gamma_j\Big) \;=\; (t^{-1})_{i-1,j-1}\ , \qquad i,j = 2,3,4,5, \tag{III.6}$$

where $t_{ij} = 2p_i \cdot p_j$. To verify the identity, we multiply it on the right by $t_{j-1,k-1}$, which can be written [10] in terms of the matrix $\rho = N_n \eta^{-1}$ using

$$p_{i-1}\cdot p_{j-1} \;=\; \frac{\rho_{ij}}{\alpha_i\alpha_j} - \frac{\rho_{i1}}{\alpha_i\alpha_1} - \frac{\rho_{1j}}{\alpha_1\alpha_j} + \frac{\rho_{11}}{\alpha_1^2}\ , \qquad i,j = 2,3,\ldots,n. \tag{III.7}$$

Thus we have (using the equations (3.9)–(3.13) that relate ρ, η, γ_i and α_i)

$$\begin{aligned} \sum_{j=2}^{5} c_{ij} t_{j-1,k-1} &= \sum_{j=1}^{5} \frac{\alpha_i\alpha_j}{N_5\,\hat{\Delta}_5}\Big(\eta_{ij}\hat{\Delta}_5 - \gamma_i\gamma_j\Big)\left(\frac{\rho_{jk}}{\alpha_j\alpha_k} - \frac{\rho_{j1}}{\alpha_j\alpha_1} - \frac{\rho_{1k}}{\alpha_1\alpha_k} + \frac{\rho_{11}}{\alpha_1^2}\right) \\ &= \delta_{ik}, \qquad i,k = 2,3,4,5. \end{aligned} \tag{III.8}$$

Equation (III.6) implies that

$$\sum_{i,j=1}^{4} p_i^\mu p_j^\nu c_{i+1,j+1} \;=\; \sum_{i,j=1}^{4} p_i^\mu p_j^\nu (t^{-1})_{i,j} \;=\; \frac{1}{2}\delta_{[4]}^{\mu\nu}\ , \tag{III.9}$$

since the four vectors p_i^μ span $D = 4$ Minkowski space.

Using this identity and keeping only the $I_5^{D=6}$ content we then have

$$\begin{aligned} I_5[p_\mu p_\nu] &= -\frac{1}{2} I_5^{D=6}[1]\ \delta_{[4-2\epsilon]}^{\mu\nu} + \sum_{i,j=1}^{4} p_i^\mu p_j^\nu c_{i+1,j+1} I_5^{D=6}[1] + \text{boxes} \\ &= -\frac{1}{2}\delta_{[-2\epsilon]} I_5^{D=6}[1] + \text{boxes} + \mathcal{O}(\epsilon) \\ &= \text{boxes} + \mathcal{O}(\epsilon), \end{aligned} \tag{III.10}$$

so that $I_5^{D=6}$ drops out as claimed. To arrive at the last line, we used the finiteness of $I_5^{D=6}$ and that $\delta^{\mu\nu}_{[-2\epsilon]}$ can yield only $\mathcal{O}(\epsilon)$ contributions. This shows that there are no 'left-over' pieces of $I_5^{D=6}$ remaining when all the pieces are combined to form an amplitude derived from a loop momentum integral with up to two powers of momenta in the numerator.

The three Feynman parameter case is similar. Again applying the identity (III.6) we have

$$\begin{aligned} I_5(p^\mu p^\nu p^\rho) &= -\frac{1}{2}\Big(\delta^{\mu\nu}_{[4-2\epsilon]}\sum_i p_i^\rho I_5^{D=6}[a_{i+1}] + \text{cyclic}\Big) + \sum_{ijk} p_i^\mu p_j^\nu p_k^\rho c_{ijk} I_5^{D=6}[1] + \text{boxes} \\ &= -\frac{1}{2}\Big(\delta^{\mu\nu}_{[-2\epsilon]}\sum_i p_i^\rho \frac{\alpha_{i+1}\gamma_{i+1}}{\hat\Delta_5} I_5^{D=6}[1] + \text{cyclic}\Big) + \text{boxes} + \mathcal{O}(\epsilon) \\ &= \text{boxes} + \mathcal{O}(\epsilon), \end{aligned} \qquad \text{(III.11)}$$

where we used

$$c_{ijk} = \frac{\alpha_k\gamma_k}{\hat\Delta_5}\, c_{ij} + \text{cyclic} \qquad \text{(III.12)}$$

and

$$I_5^{D=6}[a_i] = \frac{\alpha_i\gamma_i}{\hat\Delta_5} I_5^{D=6}[1] + \text{boxes}, \qquad \text{(III.13)}$$

from equation (6.11).

It is straightforward to continue in this way, demonstrating that $I_5^{D=6}$ drops out from the loop momentum integrals encountered in relativistic field theories. For gauge theories, up to five factors of the loop momentum in the numerator can appear.

Appendix IV. Constants of Integration for Box Integrals

In this appendix we evaluate the constant of integration for the box with two adjacent massive legs, or with three massive legs, by performing the integral at the point where all the α_i are equal. The constant of integration for the adjacent two-mass box is a special case of that for the three-mass box, with $\lambda = 0$. We have

$$\hat I_0 \equiv \hat I_4^{3\text{m}}(\alpha_i = 1, \lambda) = \Gamma(2+\epsilon)\int d^4u_i\ \frac{\delta\big(1-\sum u_i\big)}{\big[(u_1+\lambda u_2)u_3 + u_4(1-u_4)\big]^{2+\epsilon}}\,. \qquad \text{(IV.1)}$$

We let

$$u_1 = z(1-y), \quad u_2 = (1-z)(1-y), \quad u_3 = y(1-x), \quad u_4 = xy. \qquad \text{(IV.2)}$$

The z integral is elementary and leads to

$$\hat I_0 = -\frac{\Gamma(1+\epsilon)}{1-\lambda}\int_0^1 dx\int_0^1 dy \frac{y^{-1-\epsilon}}{1-x}\Big\{\big[(1-x)(1-y)+x(1-xy)\big]^{-1-\epsilon} - \big[\lambda(1-x)(1-y)+x(1-xy)\big]^{-1-\epsilon}\Big\}\,. \qquad \text{(IV.3)}$$

The y integral can be done in terms of hypergeometric functions,

$$
\begin{aligned}
\hat{I}_0 \;=\; & \frac{\Gamma(1+\epsilon)}{\epsilon\,(1-\lambda)} \int_0^1 \frac{dx}{1-x} \Big\{ {}_2F_1(1+\epsilon,-\epsilon;1-\epsilon;1-x+x^2) \\
& - \big(x+\lambda(1-x)\big)^{-1-\epsilon} {}_2F_1\left(1+\epsilon,-\epsilon;1-\epsilon;\frac{\lambda(1-x)+x^2}{\lambda(1-x)+x}\right)\Big\}\;.
\end{aligned}
\tag{IV.4}
$$

For $\lambda \neq 0$, the integrand has no singularities as $\epsilon \to 0$, so we may expand it in ϵ; the hypergeometric functions have the following expansion for small ϵ,

$$
{}_2F_1(1+\epsilon,-\epsilon;1-\epsilon;v) \;=\; 1 \;+\; \epsilon\,\ln(1-v) \;+\; \epsilon^2\left[-2\;\mathrm{Li}_2(v) - \tfrac{1}{2}\ln^2(1-v)\right] \;+\; \mathcal{O}(\epsilon^3) \tag{IV.5}
$$

(we only need the first two terms here), which leads to

$$
\begin{aligned}
\hat{I}_0 \;=\; & r_\Gamma \int_0^1 dx\,\left[\frac{-\frac{1}{\epsilon} - \ln x - \ln(1-x) + 2\ln(\lambda+(1-\lambda)x)}{\lambda+(1-\lambda)x} + \frac{2\ln(\lambda+(1-\lambda)x)}{(1-\lambda)(1-x)}\right] \\
=\; & \frac{r_\Gamma}{1-\lambda}\left\{\frac{\ln\lambda}{\epsilon} \;+\; \int_\lambda^1 \frac{du}{u}\big(-\ln(u-\lambda) - \ln(1-u) + 2\ln u + 2\ln(1-\lambda)\big) \;+\; 2\int_0^{1-\lambda}\frac{dv}{v}\ln(1-v)\right\} \\
=\; & \frac{r_\Gamma}{1-\lambda}\left[\frac{\ln\lambda}{\epsilon} \;-\; \frac{1}{2}\ln^2\lambda\right]\;.
\end{aligned}
\tag{IV.6}
$$

In the case $\lambda = 0$, we add and subtract terms in (IV.4) to obtain

$$
\begin{aligned}
\hat{I}_0(\lambda=0) \;=\; & \frac{\Gamma(1+\epsilon)}{\epsilon}\int_0^1 \frac{dx}{1-x}\Big\{ {}_2F_1(1+\epsilon,-\epsilon;1-\epsilon;1-x+x^2) - x^{-\epsilon}\,{}_2F_1\left(1+\epsilon,-\epsilon;1-\epsilon;x\right)\Big\} \\
& - \frac{\Gamma(1+\epsilon)}{\epsilon}\int_0^1 dx\;x^{-1-\epsilon}\left({}_2F_1(1+\epsilon,-\epsilon;1-\epsilon;x) - 1\right) - \frac{\Gamma(1+\epsilon)}{\epsilon}\int_0^1 dx\;x^{-1-\epsilon}\;.
\end{aligned}
\tag{IV.7}
$$

In the first and second integrals, the integrand is again nonsingular everywhere, and we can expand in ϵ; the third is elementary:

$$
\begin{aligned}
\hat{I}_0(\lambda=0) \;=\; & 2\,\Gamma(1+\epsilon)\int_0^1 dx\frac{\ln x}{1-x} - \Gamma(1+\epsilon)\int_0^1 dx\frac{\ln(1-x)}{x} + \frac{\Gamma(1+\epsilon)}{\epsilon^2} \\
=\; & \frac{\Gamma(1+\epsilon)}{\epsilon^2}\left(1 - \frac{\pi^2}{6}\epsilon^2\right) \\
=\; & \frac{r_\Gamma}{\epsilon^2}\;,
\end{aligned}
\tag{IV.8}
$$

so that $c_0(0) = 0$.

Appendix V. The Triangle with Three External Masses

The differential equations approach also provides an easy way to derive a compact expression for the three-mass triangle integral to all orders in ϵ. (The integral is in fact finite, so only the leading order is needed in practical calculations; but in order to examine explicitly the limit in which one of the external masses vanishes, it is convenient to have the forms derived here, or ones equivalent to them [24].)

The three-mass triangle with massless internal lines satisfies the following system of equations (using $N_3 = 1$):

$$\frac{\partial\left(\hat{\Delta}_3^{1/2-\epsilon}\hat{I}_3\right)}{\partial\alpha_i} = -\tfrac{1}{2}(1-2\epsilon)\hat{\Delta}_3^{1/2-\epsilon}\sum_{j=1}^{3}\left(\eta_{ij}-\frac{\gamma_i\gamma_j}{\hat{\Delta}_3}\right)\hat{I}_2^{(j)}\ , \tag{V.1}$$

where

$$\hat{\Delta}_3 = -\alpha_1^2-\alpha_2^2-\alpha_3^2+2\alpha_1\alpha_2+2\alpha_2\alpha_3+2\alpha_3\alpha_1, \tag{V.2}$$

so that

$$\begin{aligned}\gamma_i &= \sum_{j=1}^{3}\alpha_j-2\alpha_i,\\ \hat{\Delta}_3 &= \gamma_1\gamma_2+\gamma_2\gamma_3+\gamma_3\gamma_1.\end{aligned} \tag{V.3}$$

Also, the two-point integrals $\hat{I}_2^{(i)}$ are very simple,

$$\begin{aligned}\hat{I}_2^{(i)} &= \Gamma(\epsilon)\ (\alpha_{i+1}\alpha_{i-1})^{\epsilon-1}\int_0^1 dx\ x^{-\epsilon}(1-x)^{-\epsilon}\\ &= \frac{r_\Gamma}{\epsilon(1-2\epsilon)}(\alpha_{i+1}\alpha_{i-1})^{\epsilon-1}\ .\end{aligned} \tag{V.4}$$

Notice that a function of

$$\delta_j \equiv \frac{\gamma_j}{\sqrt{\hat{\Delta}_3}}$$

obeys

$$\frac{\partial F(\delta_j)}{\partial\alpha_i} = \hat{\Delta}_3^{-1/2}\left(\eta_{ij}-\frac{\gamma_i\gamma_j}{\hat{\Delta}_3}\right)F'(\delta_j), \tag{V.5}$$

and that

$$\alpha_{i+1}\alpha_{i-1} = \tfrac{1}{4}(\gamma_i+\gamma_{i-1})(\gamma_i+\gamma_{i+1}) = \tfrac{1}{4}\,\hat{\Delta}_3\,(1+\delta_i^2). \tag{V.6}$$

Therefore we may solve the differential equations (V.1) by

$$\hat{I}_3 = \hat{\Delta}_3^{-1/2+\epsilon}\left[F(\delta_1)+F(\delta_2)+F(\delta_3)+C\right]\ , \tag{V.7}$$

where $F(\delta)$ satisfies

$$F'(\delta) = -\tfrac{1}{2}(1-2\epsilon)\,\hat{\Delta}_3^{1-\epsilon}\ \left[\frac{r_\Gamma}{\epsilon(1-2\epsilon)}\left(\tfrac{1}{4}\,\hat{\Delta}_3\,(1+\delta^2)\right)^{\epsilon-1}\right] = -\frac{2^{1-2\epsilon}r_\Gamma}{\epsilon}\ (1+\delta^2)^{\epsilon-1}, \tag{V.8}$$

and C is a constant of integration.

We need the integral

$$\begin{aligned}
\int_0^\delta dz\ (1+z^2)^{\epsilon-1} &= \int_0^\delta dz\ (1+iz)^{\epsilon-1}(1-iz)^{\epsilon-1} \\
&= -i\int_1^{1+i\delta} dw\ w^{\epsilon-1}2^{\epsilon-1}(1-w/2)^{\epsilon-1} \\
&= -\frac{2^{\epsilon-1}i}{\epsilon}\left[(1+i\delta)^\epsilon\, {}_2F_1\left(1-\epsilon,\epsilon;1+\epsilon;\frac{1+i\delta}{2}\right) - {}_2F_1\left(1-\epsilon,\epsilon;1+\epsilon;\frac{1}{2}\right)\right] \\
&= -\frac{2^{2\epsilon-1}i}{\epsilon}\left[\left(\frac{1+i\delta}{1-i\delta}\right)^\epsilon {}_2F_1\left(2\epsilon,\epsilon;1+\epsilon;-\frac{1+i\delta}{1-i\delta}\right) - {}_2F_1\left(2\epsilon,\epsilon;1+\epsilon;-1\right)\right] \\
&= \frac{4^{\epsilon-1}}{\epsilon}\frac{1}{i}\left[\left(\frac{1+i\delta}{1-i\delta}\right)^\epsilon {}_2F_1\left(2\epsilon,\epsilon;1+\epsilon;-\frac{1+i\delta}{1-i\delta}\right)\right. \\
&\qquad \left. - \left(\frac{1-i\delta}{1+i\delta}\right)^\epsilon {}_2F_1\left(2\epsilon,\epsilon;1+\epsilon;-\frac{1-i\delta}{1+i\delta}\right)\right],
\end{aligned} \tag{V.9}$$

where we have symmetrized the result in the last line. Alternatively, we may do the integral as

$$\begin{aligned}
\int_0^\delta dz\ (1+z^2)^{\epsilon-1} &= \int_0^\delta dz \sum_{m=0}^\infty (\epsilon-1)(\epsilon-2)\cdots(\epsilon-m)\frac{z^{2m}}{m!} \\
&= \sum_{m=0}^\infty (\epsilon-1)(\epsilon-2)\cdots(\epsilon-m)\frac{\delta^{2m+1}}{m!(2m+1)} \\
&= \delta\sum_{m=0}^\infty \frac{(1-\epsilon)(2-\epsilon)\cdots(m-\epsilon)(\frac{1}{2})(\frac{3}{2})\cdots(m-\frac{1}{2})\ (-\delta^2)^m}{(\frac{3}{2})\cdots(m-\frac{1}{2})(m+\frac{1}{2})\ m!} \\
&= \delta\ {}_2F_1(1-\epsilon,\tfrac{1}{2};\tfrac{3}{2};-\delta^2).
\end{aligned} \tag{V.10}$$

The two expressions for the integral can be related using a variety of hypergeometric identities.

Thus we have

$$\hat{I}_3(\alpha_i) = -\frac{1}{2}\frac{r_\Gamma}{\epsilon^2}\hat{\Delta}_3^{-1/2+\epsilon}\big[f(\delta_1)+f(\delta_2)+f(\delta_3)+c\big], \tag{V.11}$$

where

$$\begin{aligned}
f(\delta) &= \epsilon\, 4^{1-\epsilon}\,\delta\ {}_2F_1(1-\epsilon,\tfrac{1}{2};\tfrac{3}{2};-\delta^2) \\
&= \frac{1}{i}\left[\left(\frac{1+i\delta}{1-i\delta}\right)^\epsilon {}_2F_1\left(2\epsilon,\epsilon;1+\epsilon;-\frac{1+i\delta}{1-i\delta}\right) - \left(\frac{1-i\delta}{1+i\delta}\right)^\epsilon {}_2F_1\left(2\epsilon,\epsilon;1+\epsilon;-\frac{1-i\delta}{1+i\delta}\right)\right].
\end{aligned} \tag{V.12}$$

To fix the constant c, it is easiest to consider the integral at the following, somewhat asymmetric, kinematic point:

$$s_{12} = \frac{-1}{\alpha_3\alpha_1} = -\frac{1}{2}, \quad s_{23} = \frac{-1}{\alpha_1\alpha_2} = -\frac{1}{2}, \quad s_{31} = \frac{-1}{\alpha_2\alpha_3} = -1, \tag{V.13}$$

or

$$\alpha_1 \;=\; 2, \quad \alpha_2 \;=\; \alpha_3 \;=1; \qquad \hat{\Delta}_3 \;=\; 4; \qquad \delta_1 \;=\; 0, \quad \delta_2 \;=\; \delta_3 \;=\; 1. \tag{V.14}$$

At this kinematic point, we make the change of variables

$$a_1 \;=\; 1-y, \qquad a_2 \;=\; xy, \qquad a_3 \;=\; (1-x)y, \tag{V.15}$$

with Jacobian equal to y, and obtain

$$\begin{aligned}\hat{I}_3(2,1,1) \;&=\; \frac{\Gamma(1+\epsilon)}{2}\int_0^1 d^3a_i\;\frac{\delta(1-\sum a_i)}{(\frac{1}{2}a_3a_1+\frac{1}{2}a_1a_2+a_2a_3)^{1+\epsilon}}\\ &=\; 2^\epsilon\Gamma(1+\epsilon)\int_0^1 dx\int_0^1 dy\; y^{-\epsilon}\big[1-(1-2x(1-x))y\big]^{-1-\epsilon}\\ &=\; 2^\epsilon\frac{\Gamma(1+\epsilon)}{1-\epsilon}\int_0^1 dx\;\;{}_2F_1(1+\epsilon,1-\epsilon;2-\epsilon;1-2x(1-x))\;.\end{aligned} \tag{V.16}$$

Next we use the change-of-variables,

$$x \;=\; \frac{1-\sqrt{1-z}}{2}\;, \qquad z \;=\; 4x(1-x), \tag{V.17}$$

and a hypergeometric identity, to get

$$\begin{aligned}\hat{I}_3(2,1,1) \;&=\; 2^{\epsilon-1}\frac{\Gamma(1+\epsilon)}{1-\epsilon}\int_0^1 dz\;(1-z)^{-1/2}\Bigg[\frac{\Gamma(2-\epsilon)\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)}\;{}_1F_0(1-\epsilon;z/2)\\ &\quad+\;\frac{\Gamma(2-\epsilon)\Gamma(\epsilon)}{\Gamma(1+\epsilon)\Gamma(1-\epsilon)}2^\epsilon\,z^{-\epsilon}\;{}_2F_1(1,1-2\epsilon;1-\epsilon;z/2)\Bigg]\\ &=\; 2^{\epsilon-1}\frac{\Gamma(1+\epsilon)}{1-\epsilon}\Bigg[2\frac{\Gamma(2-\epsilon)\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)}\;{}_2F_1(1-\epsilon,1;\tfrac{3}{2};\tfrac{1}{2})\\ &\quad+\;2^\epsilon\frac{\Gamma(2-\epsilon)\Gamma(\epsilon)\Gamma(\frac{1}{2})}{\Gamma(1+\epsilon)\Gamma(\frac{3}{2}-\epsilon)}\;{}_2F_1(1,1-2\epsilon;\tfrac{3}{2}-\epsilon;\tfrac{1}{2})\Bigg]\;,\end{aligned} \tag{V.18}$$

We then use the following hypergeometric identities,

$$\begin{aligned}{}_2F_1(1-\epsilon,1;\tfrac{3}{2};\tfrac{1}{2}) \;&=\; 2^{1-\epsilon}\;{}_2F_1(1-\epsilon,\tfrac{1}{2};\tfrac{3}{2};-1)\;,\\ {}_2F_1(1,1-2\epsilon;\tfrac{3}{2}-\epsilon;\tfrac{1}{2}) \;&=\; {}_2F_1(\tfrac{1}{2},\tfrac{1}{2}-\epsilon;\tfrac{3}{2}-\epsilon;1) \;=\; \frac{\Gamma(\frac{3}{2}-\epsilon)\Gamma(\frac{1}{2})}{\Gamma(1-\epsilon)}\;,\end{aligned} \tag{V.19}$$

to get

$$\hat{I}_3(2,1,1) \;=\; -\frac{2r_\Gamma}{\epsilon}\;{}_2F_1(1-\epsilon,\tfrac{1}{2};\tfrac{3}{2};-1) \;+\; \frac{4^\epsilon\pi\Gamma(1+\epsilon)}{2\epsilon}\;. \tag{V.20}$$

On the other hand, plugging the values of $\hat{\Delta}_3$ and δ_i from (V.14) into equation (V.11), we have

$$\hat{I}_3(2,1,1) \;=\; -\frac{4^{-1/2+\epsilon}r_\Gamma}{2\epsilon^2}\Big[2\,\epsilon\,4^{1-\epsilon}\;{}_2F_1(1-\epsilon,\tfrac{1}{2};\tfrac{3}{2};-1)+c\Big]\;. \tag{V.21}$$

Comparing equations (V.20) and (V.21), we find

$$c \;=\; -\frac{2\pi\epsilon\Gamma(1+\epsilon)}{r_\Gamma} \;=\; -2\pi\,\epsilon\,\frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)}. \tag{V.22}$$

Despite its appearance, equation (V.11) does have a finite limit as $\epsilon \to 0$,

$$\hat{I}_3(\alpha_i) \;=\; \frac{i}{\sqrt{\hat{\Delta}_3}} \sum_{j=1}^{3} \left[\mathrm{Li}_2\left(-\left(\frac{1+i\delta_j}{1-i\delta_j}\right)\right) - \mathrm{Li}_2\left(-\left(\frac{1-i\delta_j}{1+i\delta_j}\right)\right)\right] \;+\; \mathcal{O}(\epsilon), \tag{V.23}$$

which is the form given in ref. [17].

Appendix VI. Higher-Point Scalar Integrals

In this appendix, we discuss formulæ allowing the evaluation of higher-point scalar integrals ($n > 5$), in part to correct some statements we made in a previous paper [10]. The corrected results will be similar to results obtained previously by Melrose, and by van Neerven and Vermaseren [5,7]. The main difference is that the present results allow for external kinematics in the full $4-2\epsilon$ dimensions, which is useful for obtaining tensor integrals by the differentiation method discussed in sections 2, 6, and appendix VII.

We begin by recalling equation (3.23), which we rewrite here in a slightly different form,

$$\hat{I}_n \;=\; \sum_{i=1}^{n} \frac{\gamma_i}{2N_n}\,\hat{I}_{n-1}^{(i)} \;+\; (n-5+2\epsilon)\,\frac{\hat{\Delta}_n}{2N_n}\,\hat{I}_n^{D=6-2\epsilon}\;. \tag{VI.1}$$

For $n \geq 6$, in order to use equation (VI.1) to evaluate scalar integrals, it is desirable to take the external momenta $k_1, k_2, \ldots, k_n$ to be restricted to $D=4$. The loop momenta have to remain in $D = 4-2\epsilon$ in order to regulate infrared divergences. In the 't Hooft-Veltman variant of dimensional regularization, the external momenta appearing in the one-loop integral in a next-to-leading-order calculation are indeed taken to be four-dimensional. In the conventional dimensional regularization scheme, the external momenta are taken to be $4-2\epsilon$-dimensional, but this will generally lead to only $\mathcal{O}(\epsilon)$ corrections, since the integrals $I_n^{D=6-2\epsilon}$ are finite as $\epsilon \to 0$ for $n \geq 4$.

In reference [10], we argued that the term containing $\hat{I}_n^{D=6-2\epsilon}$ in equation (VI.1) could be dropped for $n \geq 6$, when the external momenta are restricted to $D=4$. The argument was based on the fact that for $n \geq 6$ the Gram determinant $\hat{\Delta}_n$ appearing in equation (VI.1) vanishes for $D=4$ kinematics, due to the linear dependence of the $(n-1)$ vectors $k_1, k_2, \ldots, k_{n-1}$ [18,25]. If the $\hat{I}_n^{D=6-2\epsilon}$ term could be dropped, then equation (VI.1) would reduce to a simple recursion relation expressing the scalar integrals $\hat{I}_n$ as a linear combination of the n $(n-1)$-point integrals $\hat{I}_{n-1}$. For $n=6$, the argument does indeed hold, and the scalar hexagon integral is given by

$$\hat{I}_6 \;=\; \sum_{i=1}^{6} \frac{\gamma_i}{2N_6}\,\hat{I}_5^{(i)} \qquad\qquad (D=4 \text{ kinematics}). \tag{VI.2}$$

Unfortunately, for $n \geq 7$ the situation is more complicated. It is true that for $n \geq 6$ the Gram determinant $\hat{\Delta}_n$ vanishes for $D = 4$ external kinematics. However, for $n \geq 7$, the factor of N_n in the denominator also vanishes. Indeed, N_n is given by $N_n = 2^{n-1} \det \rho = 2^{n-1} \left(\prod_{i=1}^n \alpha_i\right)^2 \det S$, and the dimension of the null space of the $n \times n$ matrix S_{ij} is $n - 6$ for $D = 4$ kinematics [5]. Therefore, for $n \geq 7$, the coefficients appearing in equation (VI.1) are not well-defined for $D = 4$ external kinematics (which is where we would like to use the equation).

Notice that both numerator and denominator of the coefficient ratios $\gamma_i/2N_n$ and $\hat{\Delta}_n/2N_n$ vanish for $D = 4$ kinematics: The matrices that give rise to $\hat{\Delta}_n$ and to $\gamma_i = \frac{1}{2}(\partial\hat{\Delta}_n/\partial\alpha_i)$ have null spaces of dimension $n - 5$ and $n - 6$ respectively. Based on the dimensions of the corresponding null spaces, we can argue that $\hat{\Delta}_n$ vanishes "faster" than N_n, and γ_i vanishes "equally fast", as $D = 4$ kinematics are approached. Thus we might expect that a modification of equation (VI.1) should exist, which is well-defined for $D = 4$ kinematics, and for which the coefficient of $\hat{I}_n^{D=6-2\epsilon}$ vanishes in this limit. In fact, van Neerven and Vermaseren [7] have shown how to obtain such an equation, which expresses an n-point scalar integral in terms of six $(n-1)$-point integrals. (Their derivation was carried out for $D = 4$ loop momenta; however it is easy to see that it is equally valid for $D = 4 - 2\epsilon$ loop momenta as well, as long as the external momenta are restricted to $D = 4$.)

Here we will obtain an equation similar to (VI.1), but where the coefficients have $N_{n-1}^{(k)}$ in the denominator instead of N_n. Since N_6 is nonzero for generic $D = 4$ kinematics, this equation will be well-defined for the heptagon integral ($n = 7$) in $D = 4$. It reduces to the above-mentioned equation of van Neerven and Vermaseren in $D = 4$, but it is also well-defined away from $D = 4$, which makes it a useful starting point if one wishes to apply the differentiation approach of sections 2 and 6 to compute tensor integrals. The reason why restricting external kinematics to $D = 4$ complicates the differentiation approach is that the α_i variables are then subject to various Gram-determinental constraints [18,25], which would have to be respected in performing the differentiations. After carrying out the differentiations it is permissible, and usually desirable, to restrict the external kinematics to $D = 4$, in order to take advantages of certain simplifications. An example of this procedure, for the one-parameter heptagon integrals, is provided in the next appendix.

To derive the new scalar equation, we first need some general relations between the quantities $\hat{\Delta}_n$, γ_i and N_n, which are associated with the integral $\hat{I}_n$, and the corresponding quantities $\hat{\Delta}_{n-1}^{(k)}$, $\gamma_i^{(k)}$ and $N_{n-1}^{(k)}$ associated with the $(n-1)$-point "daughter" integral $\hat{I}_{n-1}^{(k)}$. As in section 3, we choose the α_i variables for the daughter and parent integrals to be the same. We also take the kinematics to be general for now, i.e. not restricted to $D = 4$, so that all quantities are well-defined. The necessary relations follow from the observation: If A is a symmetric $n \times n$ matrix, and $B_{(k)}$ is the $(n-1) \times (n-1)$ matrix formed by crossing out the k^{th} row and k^{th} column of A, then the inverse of $B_{(k)}$ can be computed as

$$\left(B^{-1}_{(k)}\right)_{ij} \;=\; A^{-1}_{ij} - \frac{A^{-1}_{ik}A^{-1}_{kj}}{A^{-1}_{kk}}\;, \qquad\qquad i\neq k,\; j\neq k. \tag{VI.3}$$

The proof is simply to multiply equation (VI.3) on the left by $(B_{(k)})_{\ell i} = A_{\ell i}$, and simplify. Note also that $\left(B^{-1}_{(k)}\right)_{ij}$ vanishes for either $i=k$ or $j=k$.

Starting with the expression (3.8) for $\hat{\Delta}^{(k)}_{n-1}$ in terms of α_i and the matrix $\eta^{(k)}$, and using equations (3.9) and (3.10), we can rewrite $\hat{\Delta}^{(k)}_{n-1}$ as

$$\begin{aligned}
\hat{\Delta}^{(k)}_{n-1} \;&=\; \sum_{i,j\neq k}\alpha_i\eta^{(k)}_{ij}\alpha_j \;=\; \alpha^{\mathrm{T}}\eta^{(k)}\alpha \\
&=\; \gamma^{\mathrm{T}}\eta^{-1}\eta^{(k)}\eta^{-1}\gamma \;=\; \frac{N^{(k)}_{n-1}}{N_n^2}\gamma^{\mathrm{T}}\rho\left(\rho^{(k)}_{n-1}\right)^{-1}\rho\gamma \\
&=\; \frac{N^{(k)}_{n-1}}{N_n^2}\sum_{i,j=1}^{n}(\gamma^{\mathrm{T}}\rho)_i\left(\rho^{-1}_{ij} - \frac{\rho^{-1}_{ik}\rho^{-1}_{kj}}{\rho^{-1}_{kk}}\right)(\rho\gamma)_j \\
&=\; \frac{N^{(k)}_{n-1}}{N_n^2}\left[\gamma^{\mathrm{T}}\rho\gamma - \frac{\gamma_k^2}{\rho^{-1}_{kk}}\right]\;.
\end{aligned} \tag{VI.4}$$

Using the definitions (3.9) $N_n = 2^{n-1}\det\rho$, $N^{(k)}_{n-1} = 2^{n-2}\det\rho^{(k)}$, and the fact that $\det\rho^{(k)} = \rho^{-1}_{kk}\det\rho$ is the cofactor of the kk element of ρ, we have

$$\frac{\eta_{kk}}{N_n} \;=\; \rho^{-1}_{kk} \;=\; \frac{2N^{(k)}_{n-1}}{N_n}\;. \tag{VI.5}$$

Using equations (VI.4), (VI.5) and the relation $\gamma^{\mathrm{T}}\rho\gamma = N_n\hat{\Delta}_n$ which follows from equation (3.11), we obtain expressions for $\hat{\Delta}^{(k)}_{n-1}$ and its derivatives with respect to α_i:

$$\begin{aligned}
\hat{\Delta}^{(k)}_{n-1} \;&=\; \frac{\eta_{kk}\hat{\Delta}_n - \gamma_k^2}{2N_n}\;, \\
\gamma^{(k)}_i \;&=\; \frac{\eta_{kk}\gamma_i - \eta_{ik}\gamma_k}{2N_n}\;, \\
\eta^{(k)}_{ij} \;&=\; \frac{\eta_{kk}\eta_{ij} - \eta_{ik}\eta_{kj}}{2N_n}\;.
\end{aligned} \tag{VI.6}$$

One can iterate this procedure to get expressions for $\hat{\Delta}^{(k,p)}_{n-2}$, etc., if necessary.

Now we proceed to derive the new scalar equation which is of use for $n=7$. To do this, we consider equation (VI.1), and also the one-parameter equation

$$\hat{I}_n[a_k] \;=\; \sum_{i=1}^{n}\frac{\eta_{ki}}{2N_n}\;\hat{I}^{(i)}_{n-1} \;+\; (n-5+2\epsilon)\,\frac{\gamma_k}{2N_n}\;\hat{I}^{D=6-2\epsilon}_n\;. \tag{VI.7}$$

Multiply equation (VI.7) by γ_k/η_{kk} and subtract it from equation (VI.1), to get

$$\hat{I}_n = \sum_{i=1}^{n}\left[\frac{\gamma_i}{2N_n} - \frac{\eta_{ik}\gamma_k}{\eta_{kk}\cdot 2N_n}\right]\hat{I}_{n-1}^{(i)} + \frac{\gamma_k}{\eta_{kk}}\hat{I}_n[a_k] + (n-5+2\epsilon)\left[\frac{\hat{\Delta}_n}{2N_n} - \frac{\gamma_k^2}{\eta_{kk}\cdot 2N_n}\right]\hat{I}_n^{D=6-2\epsilon}\,. \quad \text{(VI.8)}$$

The coefficients in brackets in equation (VI.8) can now be rewritten in terms of $(n-1)$-point quantities using equations (VI.6). We get

$$\hat{I}_n = \sum_{i=1}^{n}\frac{\gamma_i^{(k)}}{2N_{n-1}^{(k)}}\,\hat{I}_{n-1}^{(i)} + \frac{\gamma_k}{2N_{n-1}^{(k)}}\,\hat{I}_n[a_k] + (n-5+2\epsilon)\,\frac{\hat{\Delta}_{n-1}^{(k)}}{2N_{n-1}^{(k)}}\,\hat{I}_n^{D=6-2\epsilon}\,. \quad \text{(VI.9)}$$

Any value of $k = 1, 2, \ldots, n$ may be used in this formula. Note that $\gamma_k^{(k)} = 0$, so there are only $n-1$ terms in equation (VI.9).

For $n = 7$ and (generic) $D = 4$ kinematics, we have $N_6^{(k)} \neq 0$, while $\hat{\Delta}_6^{(k)} = 0$ and $\gamma_k = 0$. So equation (VI.9) reduces to

$$\hat{I}_7 = \sum_{i=1}^{7}\frac{\gamma_i^{(k)}}{2N_6^{(k)}}\,\hat{I}_6^{(i)} \qquad (D = 4 \text{ kinematics}), \quad \text{(VI.10)}$$

which contains only six hexagons due to the vanishing of $\gamma_k^{(k)}$. Indeed, the formula can be shown to be equivalent to the scalar integral formula of Melrose, and van Neerven and Vermaseren [5,7]. For $n > 7$, equation (VI.9) is still ill-defined. Presumably one could go on to construct equations in terms of $\gamma^{(k,l)}$, $\hat{\Delta}_{n-2}^{(k,l)}$, etc. that will be well-defined for $n = 8$, and so on. This would be useful for evaluating the corresponding tensor integrals via differentiation.

Appendix VII. Higher-Point Tensor Integrals

In this appendix, we derive formulæ allowing the evaluation of tensor integrals for the pentagon ($n = 5$) and hexagon ($n = 6$) integrals, for arbitrary internal and external masses. We also briefly discuss tensor heptagon ($n = 7$) integrals. The situation regarding tensor integrals is similar to the case of the pentagon discussed in section 6 and appendix III. In order to effectively use the differentiation approach, one must show two things: First, that the $1/\epsilon$ pole encountered in the basic formula (2.20), at the level of $n-3$ Feynman parameter insertions in the n-point integral, does not present any problems; and second, that the "hard" six-dimensional integrals $\hat{I}_n^{D=6-2\epsilon}$ (for $n \geq 5$) always drop out of any "physical" tensor integral, i.e. any integral which is the Feynman-parametrization of some loop-momentum integral. We discuss these issues here to some extent for $n = 6, 7$; presumably both points can be shown to hold for arbitrary n.

For $n = 5$ and $n = 6$, the insertion of a single Feynman parameter can be treated using equation (VI.7). For $n = 5$, the term containing the $D = 6 - 2\epsilon$ scalar integral $\hat{I}_n^{D=6-2\epsilon}$ is

$\mathcal{O}(\epsilon)$ and can be ignored. For $n = 6$, this term is $\mathcal{O}(1)$, and so we would like to show that for "physical" one-parameter integrals (linear combinations of the parameters corresponding to Feynman-parametrization of some loop-momentum integral), and for $D = 4$ external kinematics, the integral $\hat{I}_6^{D=6-2\epsilon}$ drops out. Feynman parametrization of the loop-momentum integral $I_n[p^\mu]$ leads to a linear combination of one-parameter integrals, similar to the first equation in (III.3),

$$\sum_{i=2}^{n} p_{i-1}^{\mu} I_n[a_i] \;\propto\; \sum_{i=2}^{n} p_{i-1}^{\mu} \alpha_i \hat{I}_n[a_i] \; . \tag{VII.1}$$

So we can show that $\hat{I}_6^{D=6-2\epsilon}$ drops out by showing that

$$\sum_{i=2}^{6} p_{i-1}^{\mu} \frac{\alpha_i \gamma_i}{N_6} \;=\; 0 \tag{VII.2}$$

for $D = 4$ external kinematics. To show that equation (VII.2) holds, it suffices to contract the equation with a set of vectors p_{j-1}^{μ} that span $D = 4$ (we can pick any four of $j = 2, \ldots, 6$ for nonexceptional momentum configurations). We then use equation (III.7) to write $p_{i-1} \cdot p_{j-1}$ in terms of the matrix $\rho = N_6 \eta^{-1}$, and equation (3.11) to simplify the sum:

$$\begin{aligned}
\sum_{i=2}^{6} \frac{\alpha_i \gamma_i}{N_6}\; p_{i-1} \cdot p_{j-1} \;&=\; \sum_{i=1}^{6} \frac{\alpha_i \gamma_i}{N_6} \left(\frac{\rho_{ij}}{\alpha_i \alpha_j} - \frac{\rho_{i1}}{\alpha_i \alpha_1} - \frac{\rho_{1j}}{\alpha_1 \alpha_j} + \frac{\rho_{11}}{\alpha_1^2} \right) \\
&=\; \sum_{i=1}^{6} \frac{\gamma_i}{N_6} \left(\frac{\rho_{ij}}{\alpha_j} - \frac{\rho_{i1}}{\alpha_1} \right) \;+\; \frac{\hat{\Delta}_6}{N_6} \frac{1}{\alpha_1} \left(-\frac{\rho_{1j}}{\alpha_j} + \frac{\rho_{11}}{\alpha_1} \right) \\
&=\; \frac{\hat{\Delta}_6}{N_6} \frac{1}{\alpha_1} \left(-\frac{\rho_{1j}}{\alpha_j} + \frac{\rho_{11}}{\alpha_1} \right) \; .
\end{aligned} \tag{VII.3}$$

But $\hat{\Delta}_6 = 0$ while $N_6 \neq 0$ for $D = 4$ external kinematics, so $\hat{I}_6^{D=6-2\epsilon}$ does drop out as desired.

We turn next to the insertion of two Feynman parameters. The first part of the derivation parallels the derivation of the one-parameter equations (3.4) and (3.16) in section 3. We consider the integrals

$$\begin{aligned}
J_{n;i}[a_k] \;\equiv\; \Gamma(n-3+\epsilon) \int_0^1 da_{n-1} \int_0^{1-a_{n-1}} da_{n-2} \cdots \int_0^{1-a_1-a_2-\cdots-\widehat{a_i}-\cdots-a_{n-1}} da_i \\
\times \frac{d}{da_i} \frac{a_k}{\left[\sum_{i,j=1}^{n} S_{ij} a_i a_j\right]^{n-3+\epsilon}} \Bigg|_{a_n = 1-a_1-a_2-\cdots-a_{n-1}} ,
\end{aligned} \tag{VII.4}$$

evaluated two different ways, to obtain the set of equations

$$\sum_{j=1}^{n} \left(\frac{\rho_{ij}}{\alpha_i} - \frac{\rho_{nj}}{\alpha_n} \right) \hat{I}_n[a_j a_k] \;=\; \frac{1}{2} \left[\frac{\hat{I}_{n-1}^{(i)}[a_k]}{\alpha_i} - \frac{\hat{I}_{n-1}^{(n)}[a_k]}{\alpha_n} \right] + \frac{1}{2} \left(\frac{\delta_{ik}}{\alpha_i} - \frac{\delta_{nk}}{\alpha_n} \right) \hat{I}_n^{D=6-2\epsilon}[1] \; . \tag{VII.5}$$

Solving for $\hat{I}_n[a_i a_j]$, we get

$$\hat{I}_n[a_i a_j] \;=\; \frac{1}{2N_n}\sum_{\ell=1}^{n}\left(\eta_{j\ell}-\frac{\gamma_j\gamma_\ell}{\hat{\Delta}_n}\right)\hat{I}^{(\ell)}_{n-1}[a_i]+\frac{\gamma_j}{\hat{\Delta}_n}\hat{I}_n[a_i]+\frac{1}{2N_n}\left(\eta_{ij}-\frac{\gamma_i\gamma_j}{\hat{\Delta}_n}\right)\hat{I}_n^{D=6-2\epsilon}\;. \quad \text{(VII.6)}$$

We can rewrite the right-hand-side of equation (VII.6) in terms of scalar integrals only, with the help of equation (3.25):

$$\begin{aligned}\hat{I}_n[a_i a_j] \;=\; & \frac{1}{2N_n}\sum_{\ell=1}^{n}\left(\eta_{j\ell}-\frac{\gamma_j\gamma_\ell}{\hat{\Delta}_n}\right)\frac{1}{2N^{(\ell)}_{n-1}}\left[\sum_{\substack{p=1\\p\neq\ell}}^{n}\eta^{(\ell)}_{ip}\;\hat{I}^{(\ell,p)}_{n-2}\;+\;(n-6+2\epsilon)\,\gamma^{(\ell)}_i\;\hat{I}^{D=6-2\epsilon\;(\ell)}_{n-1}\right]\\ &+\frac{\gamma_j}{\hat{\Delta}_n}\frac{1}{2N_n}\left[\sum_{\ell=1}^{n}\eta_{i\ell}\;\hat{I}^{(\ell)}_{n-1}\;+\;(n-5+2\epsilon)\,\gamma_i\;\hat{I}_n^{D=6-2\epsilon}\right]\\ &+\frac{1}{2N_n}\left(\eta_{ij}-\frac{\gamma_i\gamma_j}{\hat{\Delta}_n}\right)\hat{I}_n^{D=6-2\epsilon}\;.\end{aligned} \quad \text{(VII.7)}$$

In this equation, $\hat{I}^{(\ell,p)}_{n-2}$ is the $(n-2)$-point scalar integral obtained from $\hat{I}^{(\ell)}_{n-1}$ by eliminating the p-th propagator. We keep the original kinematic α_i-variables defined for $\hat{I}_n$; $\hat{I}^{(\ell,p)}_{n-2}$ will be independent of α_ℓ and α_p. The other quantities — $N^{(\ell)}_{n-1}$, $\hat{\Delta}^{(\ell)}_{n-1}$ and its derivatives — refer to the normalization, rescaled Gram determinant, and so on, associated with $\hat{I}^{(\ell)}_{n-1}$. We can eliminate $\hat{I}^{(\ell)}_{n-1}$ from equation (VII.7) in favor of $\hat{I}^{(\ell,p)}_{n-2}$ and $\hat{I}^{(\ell)\;D=6-2\epsilon}_{n-1}$, and use equations (VI.6) to simplify things. We get finally

$$\begin{aligned}\hat{I}_n[a_i a_j] \;=\; & \frac{\eta_{ij}\hat{\Delta}_n+(n-6+2\epsilon)\gamma_i\gamma_j}{2N_n\hat{\Delta}_n}\;\hat{I}_n^{D=6-2\epsilon}\\ &+\frac{n-6+2\epsilon}{4N_n^2}\sum_{\ell=1}^{n}\left[\eta_{i\ell}\gamma_j+\eta_{j\ell}\gamma_i-\frac{\eta_{i\ell}\eta_{j\ell}\gamma_\ell}{\eta_{\ell\ell}}-\frac{\gamma_i\gamma_j\gamma_\ell}{\hat{\Delta}_n}\right]\hat{I}^{D=6-2\epsilon\;(\ell)}_{n-1}\\ &+\frac{1}{4N_n^2}\sum_{\ell,p=1}^{n}\left[\frac{\eta_{ip}\eta_{j\ell}\eta_{\ell\ell}-\eta_{i\ell}\eta_{j\ell}\eta_{\ell p}}{\eta_{\ell\ell}}\right]\hat{I}^{(\ell,p)}_{n-2}\;.\end{aligned} \quad \text{(VII.8)}$$

This formula merits several comments:

1) The expression $\hat{I}^{(\ell,p)}_{n-2}$ has no meaning for $\ell=p$; however, $\ell\neq p$ is enforced automatically by the prefactor.

2) For $n=5$, and all-massless kinematics, this equation reduces to equation (6.16); notice that $\eta_{\ell\ell}=1$ in this case, and that we wrote out the $\hat{I}_{n-2}$ terms — in this case triangles — more explicitly there.

3) For $n=5$ and general kinematics, we now have $\hat{I}_5[a_i a_j]$ to $\mathcal{O}(1)$, which means that we have surmounted the "$1/\epsilon$ barrier" for the pentagon. That is, insertions of more than two Feynman

parameters can be obtained using just the differentiation formula (2.20), and equation (VII.8) evaluated to $\mathcal{O}(1)$. The argument in appendix (III) for the cancellation of $\hat{I}_5^{D=6-2\epsilon}$ from physical quantities works for general kinematics too.

4) For $n = 6$, we will have surmounted the "$1/\epsilon$ barrier" if we can produce $\hat{I}_6[a_i a_j a_k]$ to $\mathcal{O}(1)$, or alternatively the derivative $\partial/\partial\alpha_k$ of equation (VII.8) to $\mathcal{O}(\epsilon)$. The $\hat{I}_{n-2}^{(\ell,p)}$ term presents no problem, because we can easily compute the first derivatives of box integrals to $\mathcal{O}(\epsilon)$, using equation (3.17) with $n = 4$. The $\hat{I}_{n-1}^{D=6-2\epsilon\ (\ell)}$ term also presents no problem, due to the manifest ϵ prefactor for $n = 6$. Finally, the $\hat{I}_n^{D=6-2\epsilon}$ term works out as well: the $\gamma_i\gamma_j$ term has a manifest ϵ, and the η_{ij} term requires us to know the first derivatives of $\hat{I}_6^{D=6-2\epsilon}$ to $\mathcal{O}(\epsilon)$; which we can again compute using equation (3.17), this time with $n = 6$ and $\epsilon \to \epsilon - 1$, in terms of the integrals $\hat{I}_5^{D=6-2\epsilon\ (\ell)}$ and $\hat{I}_6^{D=6-2\epsilon}$ through $\mathcal{O}(1)$.

There is one last step to surmounting the "$1/\epsilon$ barrier" for the hexagon, which is showing that $\hat{I}_6^{D=6-2\epsilon}$ and $\hat{I}_5^{D=6-2\epsilon\ (\ell)}$ drop out of "physical" quantities. Before looking at the three-parameter expression, let's look at the two-parameter expression (VII.8) again and see how how $\hat{I}_6^{D=6-2\epsilon}$ drops out of Feynman-parametrized loop integrals. Feynman parametrization of the integral $I_n[p^\mu p^\nu]$ leads to

$$-\frac{1}{2}\delta^{\mu\nu}_{[4-2\epsilon]} I_n^{D=6-2\epsilon}[1] \ + \ \sum_{i,j=2}^{n} p^\mu_{i-1} p^\nu_{j-1} I_n[a_i a_j] \ , \tag{VII.9}$$

which means that we want to show that

$$\sum_{i,j=2}^{6} p^\mu_{i-1} p^\nu_{j-1} \left[\frac{\eta_{ij}}{2N_6} + \frac{\epsilon\gamma_i\gamma_j}{N_6\hat{\Delta}_6} \right] \alpha_i\alpha_j \ = \ \frac{1}{2}\delta^{\mu\nu}_{[4]} + \mathcal{O}(\epsilon). \tag{VII.10}$$

Because of the factor of $\hat{\Delta}_6$ in the denominator, we should really be slightly more careful about how we go to "$D = 4$ kinematics", than in the one-parameter discussion above. We choose four of the vectors p^μ_{i-1} to lie in $D = 4$ and therefore to span $D = 4$; we will permit the remaining two vectors to have components in the $[-2\epsilon]$ directions, and we will only take $\epsilon \to 0$ at the end. In order to prove that equation (VII.10) holds, it suffices to contract it with $p^\mu_{i'-1} p^\nu_{j'-1}$, where i' and j' each run over the set of four vectors spanning $D = 4$. (In the expression (VII.9) we can consider μ and ν to belong to $D = 4$, not $[-2\epsilon]$, since we intend to contract the result with $D = 4$ vectors.) The derivation of equation (VII.3) continues to be valid, since we are taking p_{j-1} to be one of the momenta that lie in $D = 4$. Thus each factor of γ_i in (VII.10) will end up with a factor of $\hat{\Delta}_6$, and the $\gamma_i\gamma_j$ term in the equation drops out in the limit $\epsilon \to 0$. The η_{ij} term has a smooth limit; using equation (III.7) it is easy to show that it gives the desired result, $\frac{1}{2} p_{i'-1} \cdot p_{j'-1}$.

We now sketch how $\hat{I}_6^{D=6-2\epsilon}$ drops out of the integral $I_6[p^\mu p^\nu p^\lambda]$, which after Feynman-parametrization becomes

$$\sum_{i,j,k=2}^{6} p^\mu_{i-1} p^\nu_{j-1} p^\lambda_{k-1} I_6[a_i a_j a_k] - \frac{1}{2}\Big(\delta^{\mu\nu}_{[4-2\epsilon]} \sum_{k=2}^{6} p^\lambda_{k-1} I_6^{D=6-2\epsilon}[a_k] \ + \ \text{cyclic}\Big) \,. \qquad \text{(VII.11)}$$

In order that the coefficient of $\hat{I}_6^{D=6-2\epsilon}$ vanishes from the combination (VII.11) we find, after differentiating (VII.8), that we must have

$$\sum_{i,j,k=2}^{6} p^\mu_{i-1} p^\nu_{j-1} p^\lambda_{k-1} \left(\frac{\eta_{ij}\gamma_k + \eta_{ik}\gamma_j + \eta_{jk}\gamma_i}{2N_6\hat{\Delta}_6}\right) \alpha_i\alpha_j\alpha_k \ = \ \frac{1}{2}\delta^{\mu\nu}_{[4]} \sum_{k=2}^{6} p^\lambda_{k-1} \left(\frac{\gamma_k}{\hat{\Delta}_6}\right) \alpha_k \ + \ \text{cyclic}. \qquad \text{(VII.12)}$$

We have already dropped terms with more γ_i's in their numerators than $\hat{\Delta}_6$'s in their denominators, which will vanish in the $D = 4$ limit, following the same logic used earlier. If we now use equation (VII.10), and again drop terms vanishing in the $D = 4$ limit, then we can see that equation (VII.12) is indeed true, and so $\hat{I}_6^{D=6-2\epsilon}$ drops out of a momentum-space integral with three loop-momentum insertions. Similar considerations apply to $\hat{I}_5^{D=6-2\epsilon\ (\ell)}$.

For the case of heptagon ($n = 7$) tensor integrals, here we will be content to obtain a well-defined one-parameter equation. By similar manipulations to those giving equation (VI.9), we can get the one-parameter equation,

$$\hat{I}_n[a_i] - \frac{\eta_{ik}}{\eta_{kk}}\hat{I}_n[a_k] \ = \ \sum_{j=1}^{n} \frac{\eta^{(k)}_{ij}}{2N^{(k)}_{n-1}} \ \hat{I}^{(j)}_{n-1} \ + \ (n-5+2\epsilon)\,\frac{\gamma^{(k)}_i}{2N^{(k)}_{n-1}} \ \hat{I}_n^{D=6-2\epsilon} \,. \qquad \text{(VII.13)}$$

This equation is not adequate as it stands, since $\hat{I}_n[a_i]$ appears twice; however, by differentiating equation (VI.9) with respect to α_i, we get a second one-parameter equation,

$$\begin{aligned}\hat{I}_n[a_i] \ &= \ \frac{1}{n-4+2\epsilon}\frac{\partial \hat{I}_n}{\partial\alpha_i} \\ &= \ \frac{1}{n-4+2\epsilon}\Bigg[\sum_{j=1}^{n}\left(\frac{\eta^{(k)}_{ij}}{2N^{(k)}_{n-1}} \ \hat{I}^{(j)}_{n-1} + \frac{\gamma^{(k)}_j}{2N^{(k)}_{n-1}} \ \frac{\partial \hat{I}^{(j)}_{n-1}}{\partial\alpha_i}\right) + \frac{\eta_{ik}}{2N^{(k)}_{n-1}} \ \hat{I}_n[a_k] \\ &\quad + \ (n-5+2\epsilon)\left(\frac{2\gamma^{(k)}_i}{2N^{(k)}_{n-1}} \ \hat{I}_n^{D=6-2\epsilon} + \frac{\hat{\Delta}^{(k)}_{n-1}}{2N^{(k)}_{n-1}} \ \frac{\partial \hat{I}_n^{D=6-2\epsilon}}{\partial\alpha_i}\right)\Bigg] \,.\end{aligned} \qquad \text{(VII.14)}$$

Solving the two equations (VII.13) and (VII.14) for $\hat{I}_n[a_i]$, we get

$$\hat{I}_n[a_i] \ = \ \sum_{j=1}^{n} \frac{\gamma^{(k)}_j}{2N^{(k)}_{n-1}} \ \hat{I}^{(j)}_{n-1}[a_i] \ + \ \frac{\gamma^{(k)}_i}{2N^{(k)}_{n-1}} \ \hat{I}_n^{D=6-2\epsilon} \ + \ \frac{\hat{\Delta}^{(k)}_{n-1}}{2N^{(k)}_{n-1}} \ \frac{\partial \hat{I}_n^{D=6-2\epsilon}}{\partial\alpha_i} \,. \qquad \text{(VII.15)}$$

For $n = 7$ and $D = 4$ kinematics, we have $\hat{\Delta}^{(k)}_6 = 0$ while $N^{(k)}_6 \neq 0$, so we can drop the last term, to get:

$$\hat{I}_7[a_i] \ = \ \sum_{j=1}^{7} \frac{\gamma^{(k)}_j}{2N^{(k)}_6} \ \hat{I}^{(j)}_6[a_i] \ + \ \frac{\gamma^{(k)}_i}{2N^{(k)}_6} \ \hat{I}_7^{D=6-2\epsilon} \qquad (D=4 \text{ kinematics}). \qquad \text{(VII.16)}$$

As with the scalar heptagon equation (VI.10), in this equation any value of k, $k = 1, \ldots, 7$, may be chosen. In using equation (VII.16), we would like to know that $\hat{I}_7^{D=6-2\epsilon}$ drops out of "physical" quantities. This amounts to showing that

$$\sum_{i=2}^{7} p_{i-1}^{\mu} \frac{\alpha_i \gamma_i^{(k)}}{2N_6^{(k)}} \ = \ 0, \tag{VII.17}$$

for $D = 4$ external kinematics. To show that equation (VII.17) holds, we contract it with four independent vectors spanning $D = 4$ Minkowski space, namely p_{j-1}^{μ} for $j \in \{2, \ldots, n\}$, $j \neq k$. We thus have to show

$$\sum_{\substack{i=2 \\ i \neq k}}^{7} \frac{\alpha_i \gamma_i^{(k)}}{2N_6^{(k)}} p_{i-1} \cdot p_{j-1} \ = \ 0. \tag{VII.18}$$

But this is the same sum encountered in showing that $\hat{I}_6^{(k)\ D=6-2\epsilon}$ drops out of "physical" linear combinations of $\hat{I}_6^{(k)}[a_i]$, which we have already shown above.

Finally, the linear combinations of $\hat{I}_6^{(j)}[a_i]$ that appear explicitly in equation (VII.16), namely $\sum_{i=2}^{7} p_{i-1}^{\mu}\, \alpha_i\, a_i$, are also the same as those occurring in "physical" one parameter hexagon integrals (using $\hat{I}_6^{(j)}[a_j] = 0$). So $\hat{I}_6^{(j)\ D=6-2\epsilon}$ drops out there too. Therefore "physical" combinations of $\hat{I}_7[a_i]$ in equation (VII.16) are given in terms of well-defined, $D = 4$ quantities, as desired.

To get heptagon integrals with two or more Feynman parameters inserted, one can differentiate equation (VII.15) with respect to the α_i, and then take the limit of $D = 4$ kinematics; it remains to show that the unwanted six-dimensional integrals drop out for "physical" quantities.

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Figure Captions

Fig. 1: A schematic depiction of equation (3.24), with the coefficients suppressed.

Fig. 2: A diagram containing a triangle loop with one massive (or off-shell) leg.

Fig. 3: A diagram containing a box loop with one massive leg.

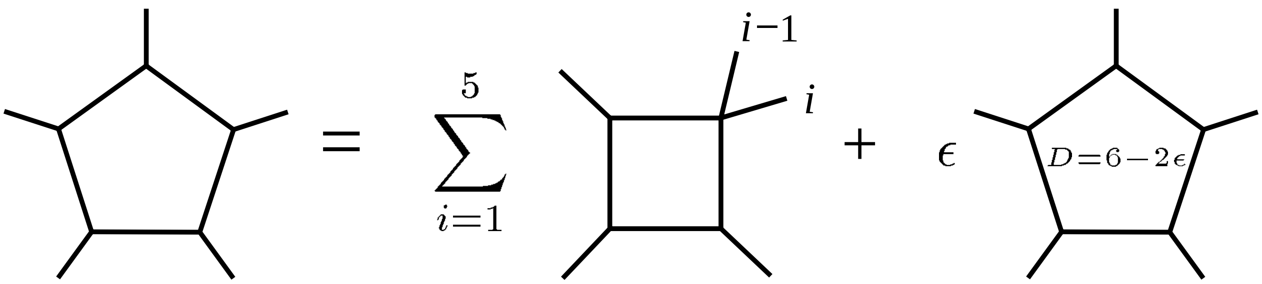

Fig. 1

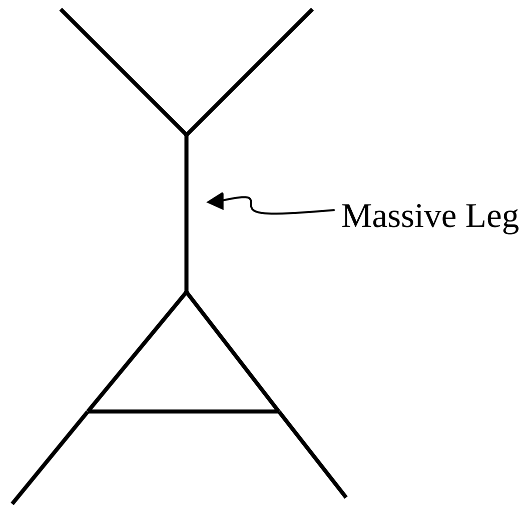

Fig. 2

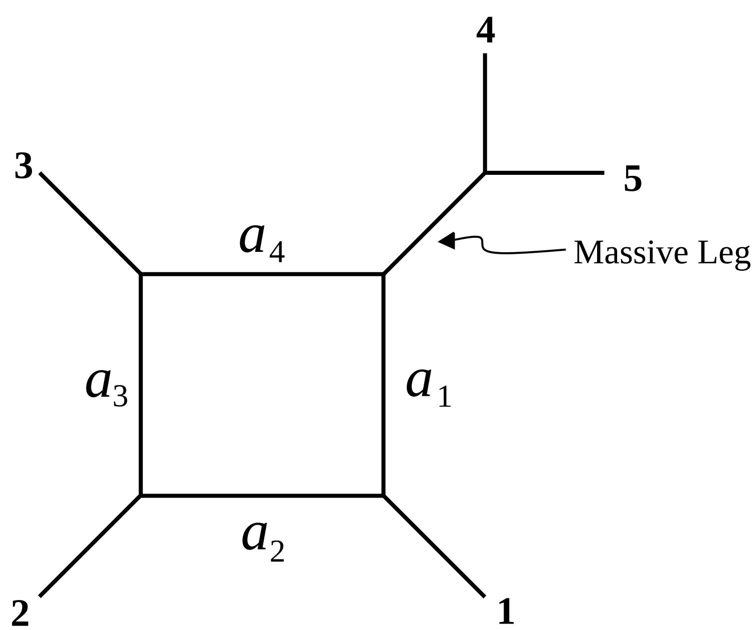

Fig. 3