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Using Lie Algebra Methods to Analytically Study Non-Perturbative Effects in Beam Lines*

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ABSTRACT

Non-perturbative effects can arise in beam lines from strong chromatic and nonlinear elements or from large lattice errors. We present a general approach using Lie algebra methods which provides analytic treatment of beam lines with strong elements or large errors. In addition to affording insight into lattice design and performance, these techniques can answer a broad spectrum of tolerance questions without use of numerical simulations. We give several detailed examples.

INTRODUCTION

Lie algebra calculations are usually regarded as perturbative, since they rely strongly on the Campbell-Baker-Hausdorff (CBH) theorem, which is a perturbative composition rule.[1] However the similarity transformation,[2] which is exact, is another very powerful algebraic tool that can be used in composing elements and finding the generator of a beam line. The requirement for the applicability of the similarity transformation, that a particular transformation be followed in the beam line with the exact inverse of the same transformation, seems unlikely. However, as we will show, this situation occurs far more often than one may imagine. In a sensitive beam line like the Final Focus Test Beam (FFTB),[3] where fifth-order terms play a major role in optimizing the design, it is possible to identify and compute all important terms using only first order CBH.

The principal reason that strong transformations are invariably followed by their inverse is that, whether the beam line be circular or linear, one wishes to find a configuration, or correct to a situation where the sum of all aberrations in the line is small. In the Lie algebra language, each distinct aberration corresponds to a unique monomial in the beam line generator, so within this framework one requires that all monomials have small coefficients. If a large aberration occurs somewhere in the line, by design or as a result of an error, then it must be canceled at another place in the line, either by a specifically designed element or corrector, or in the most complex situation, a series of elements or correctors. This element, or combination of elements thus contains the inverse of the aberration source. These conditions are exactly the conditions that establish the applicability of the similarity transformation rule.

There are a set of important tolerance questions faced by designers, which are of the form, "How far from design can parameters vary before the line no longer functions adequately?" Using similarity-transformation techniques these questions can usually be answered analytically, alleviating the need for numerical simulations.

The power and generality of this method is perhaps best communicated through examples. These examples fall into two types: non-interleaved and interleaved. In the interleaved case two or more large non-commutative aberrations must be present.

NON-INTERLEAVED EXAMPLES

Steering Correction

The well-known steering-correction procedures can be viewed from this perspective. Suppose that a linear lattice contains a badly-displaced quadrupole, and suppose there are two steering-correction dipoles in the line. How far can this quadrupole be displaced and steering corrected before complications arise?

The generator for a steering dipole is $\theta_i x_i$ for a kick of angle θ_i at a longitudinal location designated by i=1 or i=2. The correctors must be non-degenerate, which in Lie algebra language means $[x_1,x_2]\neq 0$. The kick due to a displaced quadrupole may be written, $k_q \Delta x_q\, x_q$. Because of the non-degeneracy condition, x_q may be written as a linear combination of the two correctors, $x_q=a_1x_1+a_2x_2$. Since the Poisson bracket for two linear operators is a constant, the Lie operator

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$\exp[:a_1x_1+a_2x_2:]=\exp[:a_1x_1:]\ \exp[:a_2x_2:]$. If we choose $\theta_1= -a_1\ k_q\Delta x_q$ and $\theta_2= - a_2\ k_q\Delta x_q$, the mis-steering of the quadrupole will be corrected.

Assuming for the moment that the intermediate chromatic corrections are small, the generator for the linear lattice between corrector and displaced quadrupole can be found using the first order CBH theorem, and will be of the form

$$H = 1/2\ c_{11}\ \delta\ x_1^2 + c_{12}\ \delta\ x_1\ x_1' + 1/2\ c_{22}\ \delta\ x_1'^2$$

The beam line between the displaced quadrupole and the steering corrector, including the corrector is thus given by

$$\exp[:-\theta_1 x_1:]\ \exp[:H(x_1,x_1'):]\ \exp[:\theta_1 x_1:]$$

which, using the similarity transformation, may be evaluated *exactly* as $\exp[:H(x_1,x_1'-\theta_1):]$.[2] This generator now has an additional term of the form $(c_{12}+ c_{22})\ \theta_1\ \delta\ x_1$, which is a dispersion term. As is well known, the limit on the strength of steering errors is set by the limit on the tolerable dispersion, given explicitly by this term.

Dispersion Correction

Dispersion correction follows almost exactly the discussion above. If the beam line has provisions for dispersion correction one could proceed now, in identical manner, to set the dispersion corrector to cancel this dispersion. Again a similarity transformation arises, now of the form

$$\exp[:-\xi_1\delta\ x_1:]\ \exp[:H(x_1,x_1'):]\ \exp[:\xi_1\delta\ x_1:],$$

which may be evaluated exactly as $\exp[:H(x_1,x_1'-\xi_1\delta):]$. This generator has an additional term proportional to $\xi_1\ \delta^2\ x_1$, which is a second-order dispersion term. In a case where a beam line has provisions for first-order dispersion correction, but not second-order, the magnitude of this term sets the limit on dispersion that can be corrected, which in turn sets the limit on the amount of steering error which can be corrected.

Chromatic Correction

To correct chromaticity it is usual to introduce a pair of sextupoles at -I. In the FFTB there is a pair of sextupoles for horizontal chromatic correction and a pair for vertical chromatic correction. Since the two pairs are not interleaved the similarity transformation may be applied. The full generator from the sextupole, with horizontal dispersion present is

$$H_S= 1/3!\ k_S\ \{x^3-3xy^2 +3\ \eta\delta\ (x^2-y^2) + 3\ \eta^2\delta^2 x\ \}$$

At -I from this point x -> -x, y -> -y, and typically $\eta \to \eta$, so the second generator is not exactly $-H_S$. To treat such situations it is necessary to introduce a unit operator at the second sextupole in the form

$$\exp(:-\eta\delta\ k_S(x^2-y^2):)\ \exp(:+\eta\delta\ k_S(x^2-y^2):).$$

Combining the first factor with H_S, we now obtain a generator which is the exact inverse of the first sextupole generator. This has a physical interpretation. The chromaticity generated at the first sextupole is large, as well as the geometric sextupole terms, and both must be propagated across the -I. After this has been done we still have $\exp(:+\eta\delta\ k_S(x^2-y^2):)$ following

$$\exp[:-H_S:]\ \exp[:H_I:]\ \exp[:H_S:].$$

In other words the chromaticity of the two sextupoles has been added together, and remains at the position of the second sextupole. This large term is intended to compensate the chromaticity of the final doublet. This chromatic correction is now amenable to calculation with a similarity transformation. The intermediate generator of the final transformer looks the same as the intermediate generator of our first examples, and for example, a new term coming from $\exp[:H(x_1,x_1'+2\eta\delta\ k_S x,y,y'-2\eta\delta\ k_S y):]$ is proportional to $\eta^2\ k_S^2\ \delta^3\ x^2$.

INTERLEAVED EXAMPLES

Large-Small Factorization

Interleaved examples proceed as above with the additional need to compute products of the form

$$\exp:\mathbf{A}:\ \exp:\mathbf{B}:\ \exp:-\mathbf{A}:\ \exp:-\mathbf{B}:,$$

where **A** and **B** are large generators. The product of the first three factors is a similarity transformation and can be calculated exactly. The result will be of the form

$$\exp(:\mathbf{B}+\Delta_{\mathbf{B}}:)\ \exp:-\mathbf{B}:.$$

To proceed further it is necessary to extract Δ_B from the first generator. This is possible, to first order in Δ_B,[4] which is what we desire. The answer may be written in the factored form exp:Δ′: exp:**B**: with

$$\Delta' = \int_0^1 dt e^{:t\mathbf{B}:} \Delta_B$$

Typically **B** is a kick, and Δ_B a monomial or a sum of monomials, hence the integral above may be easily evaluated.

Also this large-small factorization is required in the setup of a similarity transformation structure when a large monomial occurs in a generator with other small but non-negligible terms. For example the large chromaticity of the final doublet in the FFTB occurs in a generator with another much smaller first-order chromatic term and a small second-order chromatic term.

CONCLUSION

Large nonlinear elements and large errors in beam lines can be grouped so that Lie operators for these large aberrations are paired with their inverses. Unit operators of the form exp:-A: exp:A: may be needed and inserted as appropriate. The large-small factorization described above may be necessary to extract small residuals occurring in generators with large terms. Between operators with large generators, generators for beam-line segments can be found using first-order, or if necessary, second-order CBH. Large pairs together with intermediate beam-line generators can then be collapsed using the similarity transformation. This method of composing beam-line generators offers insight into the optical system and is useful in tolerance calculations.

REFERENCES

1. John Irwin, *The Application of Lie Algebra Techniques to Beam Transport Design*, Nuclear Instruments and Methods in Physics Research, A298 (1990) 440–472, for an introduction to notation and methods. The CBH theorem is given in section 10.5, page 471.

2. The general similarity transformation is

exp:**A**: exp:**B**: exp:-**A**: = exp:(exp:**A**: **B**):, with

exp:**A**: **B**(x,x′,y,y′) = **B**(exp:**A**: x, exp:**A**: x′, exp:**A**: y, exp:**A**: y′).

If **A** is a kick, then exp:**A**: z = z + [**A**,z].

3. Ghislain Roy, *Analysis of the Optics of the Final Focus Test Beam Using Lie Algebra-Based Techniques*, Ph.D. thesis, SLAC Report 397, May 1992, contains an introduction to Lie algebra methods, and applies them to the FFTB. Roy discusses use of the similarity transformation for the sextupoles placed at -I.

4. Alex Dragt, *Lectures on Nonlinear Orbit Dynamics*, American Institute of Physics Proceedings No. 87 (1982), p. 147. The large-small factorization may be found in equation 5.88, page 221. The integral can be understood as the first-order CBH sum of similarity transformed slices, with each slice being transformed through a different fraction of the large operator. Slicing techniques are described in reference 1.