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SOLUTION OF A THREE-THIN-LENS SYSTEM WITH ARBITRARY TRANSFER PROPERTIES*

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ABSTRACT

An iterative method is developed for solving a three-thin-lens system whose combined effect equals that of an arbitrary linear block-diagonal 4×4 transfer matrix in both planes. The explicit algebraic solution is given and application discussed.

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1 INTRODUCTION

In optical designs involving multiple lens systems, one often finds it advantageous to be able to achieve the desired optical properties by starting with thin lenses wherever allowed. The conditions set by tolerances and higher order aberrations are then used to optimize this linear design; therefore, an explicit solution for thin-lens systems satisfying the most general transform properties (i.e., highest degrees of freedom) is much desired. With such a solution, one can decouple the linear optical constraints from higher order optimization. Even in the presence of additional thick quadrupoles or other linear elements, this solution can be applied by absorbing all the other elements into the generic transfer matrix.

From simple degree-of-freedom count, one can see that in order to construct a thin-lens system whose net effect equals an arbitrary linear block-diagonal 4×4 transfer map (M_x, M_y) satisfying only the linear optical constraint

$$\det M_x = 1 , \qquad \det M_y = 1 , \tag{1.1}$$

one would need at least three thin lenses:

$$T_{ix} = \begin{pmatrix} 1 & 0 \\ K_i & 1 \end{pmatrix} , \qquad T_{iy} = \begin{pmatrix} 1 & 0 \\ -K_i & 1 \end{pmatrix} , \qquad i = 1, 2, 3$$

with corresponding drifts

$$D_i = \begin{pmatrix} 1 & L_i \\ 0 & 1 \end{pmatrix} , \quad i = 1,2,3 .$$

Thus the task appears to be solving the following set of matrix equations for K_i's and L_i's:

$$\begin{aligned} T_{3x} \cdot D_3 \cdot T_{2x} \cdot D_2 \cdot T_{1x} \cdot D_1 &= M_x , \\ T_{3y} \cdot D_3 \cdot T_{2y} \cdot D_2 \cdot T_{1y} \cdot D_1 &= M_y . \end{aligned} \tag{1.2}$$

However, if one blindly goes about solving Eq. (1.2) by eliminating all but one unknown variable, the resulting equation would take on a high order with many spurious or unphysical roots. Apart from the intrinsic difficulty in solving such equations, one would be burdened with the task of sorting through all the roots in order to locate the desired one. In carrying out an optical design where Eq. (1.2) needs be solved repeatedly for many different conditions, this can become a discouraging task, and insight into the problem compromised.

In this note we report an iterative algorithm aimed at solving Eq. (1.2) which, when followed single-mindedly, yields a second order equation in one of the parameters K_i or L_i. The intrinsic order of this problem is exactly second order simply because both roots to the above mentioned second order equation are seen to satisfy Eq. (1.2), thus no spurious roots have been introduced in the process. The explicit solution of the K_i's and L_i's in terms of the M_{xij}'s and M_{yij}'s allows insight into questions such as constraints on the transfer map (M_x, M_y) for physical solutions, and makes easy the extension into family of solutions when additional lenses are included.

2 THE ITERATIVE SOLUTION

To solve Eq. (1.2), we adopt an iterative approach where a number (one or two) of the elements on the left-hand side of Eq. (1.2) is added to the whole system at each step, starting from one or two elements. At each step the ***newly added unknowns alone*** are solved from a set of consistency conditions determined in the previous step, exclusively in terms of the M_{xij}'s and M_{yij}'s. The last fact guarantees that when we go to the next step, the only unknowns will again be the newly added ones. The rationale of this algorithm is that since we have the control of the unknowns at any given step, the proliferation of equations is minimized. In the extreme case, one can choose to add only one element at a time, in which case one has to solve an equation with only one unknown all the time, and no proliferation of roots due to elimination of unknowns will occur.

2.1 Degree-of-freedom count

It is important to make sure at every step that we are not over- or under-constraining the degrees of freedom allowed in the intermediate system; thus, a discussion on the degree-of-freedom count is given here.

The equations (1.2) are equivalent to the following equations:

$$\begin{aligned} A_x(K_1, L_1, K_2, L_2, \ldots, K_n, L_n) &= M_x(M_{x11}, M_{x12}, \ldots)\ , \\ A_y(K_1, L_1, K_2, L_2, \ldots, K_n, L_n) &= M_y(M_{y11}, M_{y12}, \ldots)\ , \end{aligned} \tag{2.3}$$

where both A_x and A_y are 2×2 matrices depending on variables $K_1, L_1, K_2, L_2, \ldots, K_n, L_n$. For the discussion so far let's make $n < 4$. Together, A_x and A_y can have only $2n$ degrees of freedom (DOF). Thus there must be $(8-2n)$ constraints among the matrix elements of A_x and A_y. These same constraints must apply exactly among the M_{xij}'s and M_{yij}'s, due to the equality. M_{xij}'s and M_{yij}'s are not further constrained, since the DOF counts must be equal on both sides.

We can also look at Eq. (2.3) as being a system of $2n+8$ variables (that is, $2n$ of the K_i, L_i's and 8 of the M_{xij}, M_{yij}'s) satisfying eight equations. But since there are only $2n$ DOF, everything else must be expressible in terms of the $2n$ K_i's and L_i's. If we choose instead $2n$ out of the $8 M_{xij}$'s and M_{yij}'s as the independent variables, say, M_1, $M_2, \ldots, M_{2n}$, then everything else is expressible in terms of these M_i's. In particular, we have $2n$ equations stating that

$$\begin{aligned} K_i \quad (\text{or} \quad L_i) \quad &= \quad G_i(M_1, \ldots, M_{2n}) \qquad 2n \;\; \text{equations} \\ M_{xij} \quad (\text{or} \quad M_{yij}) \quad &= \quad F_{xij}(M_1, \ldots, M_{2n}) \qquad 8-2n \;\; \text{equations} \end{aligned} \tag{2.4}$$

where M_{xij}'s and M_{yij}'s are those matrix elements other than $M_1, \ldots, M_{2n}$. Together these make up the eight equations as required.

Notice that the second part of Eq. (2.4) can simply be written as

$$F_k(\text{all } M_{xpq}, M_{ypq}) = 0 \ , \quad k = 1, \ \ldots, \ 8-2n \ , \tag{2.5}$$

namely, a consistency condition among all the elements M_{xpq}'s and M_{ypq}'s of the transfer matrices.

In the above discussion, we should keep in mind that the constraints on the transfer matrix elements M_{xpq}'s and M_{ypq}'s always implicitly include the linear optical condition (1.1).

2.2 Step-by-step construction

As an example of the iterative method, let us start by considering the one-lens system

$$T_{1x} \cdot D_{1x} \;\; = M_x \ , \qquad\qquad T_{1y} \cdot D_{1y} \;\; = M_y \ . \tag{2.6}$$

From the earlier discussion, it is clear that if we manipulate the algebra correctly and forget about singular cases for the time being, we can always rewrite Eq. (2.6) as

$$K_1 = G_1(M_{xij} \ , M_{xkl}) \ , \qquad L_1 = G_2(M_{xij} \ , M_{xkl}) \ ,$$

$$F_k(M_{xpq}, M_{ypq}) = 0 \ , \quad k = 1, \ 2, \ \ldots, \ 6 \ , \tag{2.7}$$

where M_{xij} and M_{xkl} represent *two* smartly chosen elements from *either* M_x or M_y, and M_{xpq}'s M_{ypq}'s are all the elements in M_x and M_y.

Equipped with Eq. (2.7), we add another set of matrices to the left-hand side of Eq. (2.6),

$$T_{2x} \cdot D_{2x} \cdot T_{1x} \cdot D_{1x} = M_x \ , \qquad T_{2y} \cdot D_{2y} \cdot T_{1y} \cdot D_{1y} = M_y \ ,$$

or

$$T_{1x} \cdot D_{1x} = D_{2x}^{-1} \cdot T_{2x}^{-1} \cdot M_x \ , \qquad T_{1y} \cdot D_{1y} = D_{2y}^{-1} \cdot T_{2y}^{-1} \cdot M_y \ . \tag{2.8}$$

Notice Eq. (2.8) now has the same form as Eq. (2.6) if we make proper substitutions on the right-hand side. We can readily apply Eq. (2.7) and get

$$K_1 = G_1(M_{xij}, M_{xkl}, K_2, L_2) \ , \qquad L_1 = G_2(M_{xij}, M_{xkl}, K_2, L_2) \ ,$$

$$F_k(M_{xpq}, M_{ypq}, K_2, L_2) = 0 \ , \quad k = 1, \ 2, \ \ldots, \ 6 \ . \tag{2.9}$$

Of course the functional forms of G_1, G_2, F_k are not the same any more, neither are the M_{xij}'s and M_{yij}'s. They are a consequence of the substitution of Eq. (2.8) into Eq. (2.6). But notice that now K_1, L_1

are totally separated from K_2, L_2 by construction. Thus with the six equations for F_k's containing ten variables, we can choose *four* of the M_{xpq}'s as independent and re-express Eq. (2.9) as

$$
\begin{aligned}
K_1 &= G_1(M_{xij}, M_{xkl}) \qquad & L_1 &= G_2(M_{xij}, M_{xkl}) \\
K_2 &= H_1(M_{xij}, M_{xkl}) \qquad & L_2 &= H_2(M_{xij}, M_{xkl})
\end{aligned}
$$

$$
F_k(M_{xpq}, M_{ypq}) = 0 \ , \quad k = 1,\ 2,\ \ldots,\ 4 \ . \tag{2.10}
$$

where now the M_{xij}, M_{xkl}'s are the *four* chosen independent elements from *either* M_x or M_y. Notice there are four F_k's now, consistent with the DOF count earlier for M_x and M_y.

We can iterate this process one step further for three lenses. Again K_3, L_3 will be separated from the other K_i, L_i's, and we only need to worry about two unknowns and two chosen equations. If we go to the extreme and add only one optical element at a time, we would have only one unknown and one equation at every step, by construction.

3 The explicit solution

The order of adding elements as demonstrated in the previous section is of course not unique. In the following we present an explicit solution obtained by a different order of adding elements. It is observed that a "conjugate" system to equation (1.2), where the order of successive optical elements is reversed and the *formal* roles of kicks and drifts interchanged, can be obtained through the transformations described in the following.

First let us consider the 4-space spanned by (x, x', y, y'), amounting to filling the diagonal 2×2 blocks of any 4×4 transfer matrices with corresponding x and y 2×2 submatrices, such as those in Eq. (1.2), and leaving the off-diagonal blocks zero in the absence of coupling. Thus the 4×4 transfer matrix for the thin lens K_i becomes

$$T_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ K_i & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -K_i & 1 \end{pmatrix} ,$$

and that for the drift L_i becomes

$$D_i = \begin{pmatrix} 1 & L_i & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_i \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Now we introduce the operator R, which is equivalent to reversed propagation in y only,

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} ,$$

and apply the following transformations to the three-lens system, Eq. (1.2):

$$R^{-1} \cdot (T_3 \cdot D_3 \cdot T_2 \cdot D_2 \cdot T_1 \cdot D_1)^T \cdot R = R^{-1} \cdot M^T \cdot R \ ,$$

or

$$D_1^{TR} \cdot T_1^{TR} \cdot D_2^{TR} \cdot T_2^{TR} \cdot D_3^{TR} \cdot T_3^{TR} = M^{TR} \ . \tag{3.11}$$

where the superscript T stands for matrix transpose and the superscript R for multiplication by R^{-1} and R as indicated. The M is the 4×4 matrix containing M_x and M_y diagonally. We then have in Eq. (3.11):

$$D_i^{TR} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ L_i & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -L_i & 1 \end{pmatrix} , \quad \text{and} \quad T_i^{TR} = \begin{pmatrix} 1 & K_i & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & K_i \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Thus the explicit form of the left-hand side of Eq. (3.11) takes on a *formal* resemblance to Eq. (1.2) with the roles of kicks and drifts interchanged. This is the "conjugate" system to Eq. (1.2). In particular if one makes the following substitutions in the left hand side of Eq. (3.11) :

$$K_1 \leftrightarrow L_3 \ , \qquad K_3 \leftrightarrow L_1 \ , \qquad K_2 \leftrightarrow L_2 \ ,$$

then it is seen to be exactly equal to the left-hand side of Eq. (1.2). The fact that there is a symmetry between the pairs (K_1, L_3), (K_3, L_1), and (K_2, L_2), as is also evident in the explicit form of Eq. (1.2) if all products are expanded, suggests the following order of adding elements in the iterative program that affords much simplified final expressions:

$$T_2 \cdot D_2 \ \rightarrow \ D_3 \cdot T_2 \cdot D_2 \cdot T_1 \ \rightarrow \ T_3 \cdot D_3 \cdot T_2 \cdot D_2 \cdot T_1 \cdot D_1 \ . \qquad (3.12)$$

As mentioned earlier, this is a straightforward algorithm, and not too many choices or too much thinking is needed in the process. The only exception is possibly in choosing which of the M_{xpq}'s to use as independent variables in say, equation (2.9) to get equation (2.10), or, equivalently, how to manipulate the equations in (2.9) to solve for K_2 and L_2. The optimal order of adding elements mentioned above does provide a natural option. In the following, we outline the intermediate steps that lead to the end solution. The algebraic calculation is done with Mathematica.

3.1 One-lens solution

Starting with the first step in Eq. (3.12), namely,

$$T_{2x} \cdot D_{2x} = M_x \ , \qquad T_{2y} \cdot D_{2y} = M_y \ ,$$

we obtain the trivial but potentially useful expressions for K_2, L_2 in the spirit of the first line of Eq. (2.4):

$$K_2 = M_{x21} \ , \qquad L_2 = M_{x12} \ , \tag{3.13}$$

and the consistency conditions among the matrix elements in the spirit of the second line of Eq. (2.4):

$$\begin{pmatrix} 1 & M_{x12} \\ M_{x21} & 1 + M_{x12}\ M_{x21} \end{pmatrix} = \begin{pmatrix} M_{x11} & M_{x12} \\ M_{x21} & M_{x22} \end{pmatrix} ,$$

$$\begin{pmatrix} 1 & M_{x12} \\ -M_{x21} & 1 - M_{x12}\ M_{x21} \end{pmatrix} = \begin{pmatrix} M_{y11} & M_{y12} \\ M_{y21} & M_{y22} \end{pmatrix} . \tag{3.14}$$

Of the above, of course, only six equations are independent.

3.2 Two-lens solution

We proceed to add the next set of matrices as indicated in the second step in Eq. (3.12), namely,

$$T_{2x} \cdot D_{2x} = D_{3x}^{-1} \cdot M_x \cdot T_{1x}^{-1} \ , \qquad T_{2y} \cdot D_{2y} = D_{3y}^{-1} \cdot M_y \cdot T_{1y}^{-1} \ , \tag{3.15}$$

As discussed before, we can now impose the consistency conditions (3.14) on the right-hand side of Eq. (3.15) and obtain equations in K_1 and L_3 *only*. These are linear equations. Having solved for K_1 and L_3, we also obtain expressions for K_2 and L_2 in the new system, which is obtained by using the expressions Eq. (3.13) with the new M_{xij}, M_{yij}'s from Eq. (3.15) and the solutions for K_1 and L_3 just obtained. The complete solution is:

$$K_1 = \frac{M_{x21} + M_{y21}}{M_{x22} - M_{y22}}, \qquad K_2 = \frac{M_{x22}M_{y21} + M_{x21}M_{y22}}{M_{y22} - M_{x22}},$$

$$L_3 = \frac{M_{x12} - M_{y12}}{M_{x22} - M_{y22}}, \qquad L_2 = \frac{M_{x12}M_{y22} - M_{x22}M_{y12}}{M_{y22} - M_{x22}}. \qquad (3.16)$$

From the degree-of-freedom count one easily sees that this system is subject to extra constraints among the M_{xij}, M_{yij}'s. They are implied in Eq. (3.14) with the new matrix elements in Eq. (3.15) substituted. These remaining consistency conditions in principle allow us to solve for the three-lens system in the next step.

3.3 Three lens solution

We proceed to add the final set of matrices as indicated in the last step in Eq. (3.12), namely,

$$\begin{aligned} D_{3x} \cdot T_{2x} \cdot D_{2x} \cdot T_{1x} &= T_{3x}^{-1} \cdot M_x \cdot D_{1x}^{-1}, \\ D_{3y} \cdot T_{2y} \cdot D_{2y} \cdot T_{1y} &= T_{3y}^{-1} \cdot M_y \cdot D_{1y}^{-1}, \end{aligned} \qquad (3.17)$$

where the left-hand side has been completely solved in the previous step and thus the right-hand side must satisfy the same consistency conditions as before, with proper factors of K_3 and L_1 substituted. Thus we obtain a set of equations with K_3 and L_1 being the *only* unknowns. The algebra involved in this step is too massive to be reproduced here. It is however useful to mention that instead of looking at the conditions given in Eq. (3.17), one obtains far simpler expressions by looking at the sum and difference of the x- and y-parts of Eq. (3.17). The final equation for either K_3 or L_1 is a quadratic one. Solving for L_1 in terms of the M_{xij}'s and M_{yij}'s we obtain:

$$\tilde{L_1} = \frac{Y \pm \sqrt{Z}}{X} \, , \tag{3.18}$$

where X, Y, and Z are polynomials of the M_{xij}'s and M_{yij}'s, and $\tilde{L_1}$ is a simple function of L_1 and some of the matrix elements. A compact parametrization of these is given in the Appendix. Both roots in Eq. (3.18) satisfy Eq. (1.2). At this point It is not clear if a simple criterion exists for ensuring physical solutions (i.e., all L_i's > 0). Neither is it clear if Z in Eq. (3.18) is positive definite, although all numerical tests so far yield positive Z's. However, since we have achieved an explicit solution that can be numerically calculated in real time, one can readily verify if there is a physical solution for any given set of matrix elements. The remaining K_i's and L_i's are obtained by substituting

Eq. (3.18) into equations obtained at earlier stages of the iteration with fewer elements:

$$K_3 = -\frac{-2 + M_{x22} + M_{y22} - M_{x21}L_1 - M_{y21}L_1}{-M_{x12} + M_{y12} + M_{x11}L_1 - M_{y11}L_1} ,$$

$$K_1 = \frac{K_{1D}}{D} , \qquad K_2 = \frac{K_{2D}}{D} ,$$

$$L_2 = \frac{L_{2D}}{D} , \qquad L_3 = \frac{L_{3D}}{D} , \tag{3.19}$$

where

$$K_{1D} = -M_{x21} - M_{y21} + M_{x11}K_3 - M_{y11}K_3 ,$$

$$\begin{aligned} L_{2D} &= -(M_{y12} - M_{y11}L_1)(M_{x22} - M_{x12}K_3 - (M_{x21} - M_{x11}K_3)L_1) + \\ &\quad (M_{x12} - M_{x11}L_1)(M_{y22} + M_{y12}K_3 - (M_{y21} + M_{y11}K_3)L_1) , \end{aligned}$$

$$\begin{aligned} K_{2D} &= (M_{y21} + M_{y11}K_3)(M_{x22} - M_{x12}K_3 - (M_{x21} - M_{x11}K_3)L_1) \\ &\quad + (M_{x21} - M_{x11}K_3)(M_{y22} + M_{y12}K_3 - (M_{y21} + M_{y11}K_3)L_1) , \end{aligned}$$

$$L_{3D} = -M_{x12} + M_{y12} + M_{x11}L_1 - M_{y11}L_1 ,$$

$$\begin{aligned} D &= -M_{x22} + M_{y22} + M_{x12}K_3 + M_{y12}K_3 + M_{x21}L_1 - \\ &\quad M_{y21}L_1 - M_{x11}K_3L_1 - M_{y11}K_3L_1 . \end{aligned}$$

Equations (3.18) and (3.19) sum up the solution obtained by our iterative method. Since the explicit form is readily available, many optical design questions can be directly answered without resorting to mathematical tricks. For example, one can find out if a particular set of transfer matrices would lead to physical thin-lens solutions by direct substitution. It should also be easy to see what changes in the transfer matrices one needs to make in order to achieve a certain property for the L_i's and K_i's. With Eq. (3.18) and Eq. (3.19), it is also easy to obtain parametrized family of solutions if one wants to use more than three thin lenses.

This method is applicable, of course, to a different order of added elements. Depending on the emphasis, one may find an alternative intermediate solution that is advantageous for a specific problem. However, the final complete three-thin-lens solution should always be the same as those given in Eqs. (3.18) and (3.19).

ACKNOWLEDGMENTS

During the development of this work, we were informed of related work by O. Napoly on telescopic systems [1]. However, we believe our work differs both in scope and in technique.

REFERENCE

1. O. Napoly, Thin lens telescopes for final focus systems, CERN/LEP–TH/89–69, CLIC Note 102.

APPENDIX A EXPLICIT FORMS OF X, Y, AND Z IN EQ. (3.18)

By careful manipulation of Eq. (1.2), taking advantage of the simplified forms for the sum and difference of its x-part and y-part, one arrives at two independent and relatively simple equations for L_1 and K_3 as follow:

$$\begin{aligned}
[\underline{a}^2 - 2(\overline{a}-1)\overline{a}]\tilde{K_3}\tilde{L_1} + (\overline{a}-1)\hat{B}\tilde{K_3} + (\overline{a}-1)\hat{C}\tilde{L_1} + \underline{a}(\overline{a}-1)^2 &= 0, \\
[(\overline{a}-1)\,\overline{a}^2 - \underline{a}^2]\ \tilde{K_3}\tilde{L_1} - (\overline{a}-1)\overline{a}\hat{B}\tilde{K_3} & \qquad (A.1) \\
-(\overline{a}-1)\overline{a}\hat{C}\tilde{L_1} + [(\overline{a}-1)\hat{C}\hat{B} - \underline{a}^3(\overline{a}-1)] &= 0
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{K_3} &= \underline{a}K_3 - \overline{c}\ , & \widetilde{L_1} &= \underline{a}L_1 - \underline{b} \\
\hat{B} &= (\underline{a}\overline{b} - \overline{a}\underline{b})\ , & \hat{C} &= (\underline{a}\underline{c} - \overline{a}\overline{c})\ , \\
\overline{a} &= \frac{M_{x11} + M_{y11}}{2}\ , & \underline{a} &= \frac{M_{x11} - M_{y11}}{2}\ , \\
\overline{b} &= \frac{M_{x12} + M_{y12}}{2}\ , & \underline{b} &= \frac{M_{x12} - M_{y12}}{2}\ , \\
\overline{c} &= \frac{M_{x21} + M_{y21}}{2}\ , & \underline{c} &= \frac{M_{x21} - M_{y21}}{2}\ .
\end{aligned}$$

Elimination of say, $\tilde{K_3}$ in Eq. (A.1) thus leads to the following equation:

$$A_l\ \tilde{L_1}^2\ +\ B_l\ \tilde{L_1}\ +\ C_l\ =\ 0\ , \qquad (A.2)$$

where

$$
\begin{aligned}
A_l &= (\overline{a}-1)\,(\overline{a}-\underline{a})\,(\overline{a}+\underline{a})\,\hat{C}\ , \\
B_l &= -(2\overline{a}^2-2\overline{a}-\underline{a}^2)\,\hat{B}\,\hat{C}-\underline{a}(\overline{a}^4-2\overline{a}^3-2\underline{a}^2\overline{a}^2+\overline{a}^2+\underline{a}^2\overline{a}+\underline{a}^4+\underline{a}^2)\ , \\
C_l &= (\overline{a}-1)\,\hat{B}(\hat{B}\hat{C}+\underline{a}\overline{a}^2-\underline{a}\overline{a}-\underline{a}^3)\ .
\end{aligned}
$$

The above expression is really more symmetric in the matrix elements if one uses the explicit forms of $\underline{a}$ and $\overline{a}$:

$$
A_l = \frac{\hat{C}M_{x11}M_{y11}(M_{y11}+M_{x11}-2)}{2}\ ,
$$

$$
\begin{aligned}
B_l = {} & \frac{-\hat{B}\hat{C}(M_{y11}^2+6M_{x11}M_{y11}-4M_{y11}+M_{x11}^2-4M_{x11})}{4} \\
& -\frac{(M_{y11}-M_{x11})}{16}\quad(M_{y11}^3-8M_{x11}^2M_{y11}^2+7M_{x11}M_{y11}^2 \\
& \qquad -4M_{y11}^2+7M_{x11}^2M_{y11}+M_{x11}^3-4M_{x11}^2)\ ,
\end{aligned}
$$

$$
C_l = \frac{-\hat{B}(M_{y11}+M_{x11}-2)\ (2M_{x11}M_{y11}^2-M_{y11}^2-2M_{x11}^2M_{y11}+M_{x11}^2-4\hat{B}\hat{C})}{8}\ .
$$

Finally X, Y, and Z in equation Eq. (3.18) are given by

$$
X = 2A_l\ , \qquad Y = -B_l\ , \qquad \text{and} \qquad Z = B_l^2 - 4A_lC_l\ . \tag{A.3}
$$

The complexity of Eqs. (A.2) and (3.19) does not pose any problem in terms of coding into a FORTRAN program and maintaining high degrees of accuracy and efficiency.