SLAC–PUB–5811
OHSTPY–HEP–T–92–005
June 1992
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THE CHALLENGE OF LIGHT-CONE QUANTIZATION OF GAUGE FIELD THEORY *

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Submitted to *Particle World*

* Work supported in part by the Department of Energy, contract DE–AC03–75SF00515.

ABSTRACT

In recent years light-cone quantization of quantum field theory has emerged as a promising method for solving problems in the strong coupling regime. This approach has a number of unique features that make it particularly appealing, most notably, the ground state of the free theory is also a ground state of the full theory. The method, therefore, seems to be well suited to solving QCD, and contrary to other approaches, the relativistic wavefunctions transform trivially to a boosted frame. These features make light-cone quantization of quantum field theory sufficiently different from the standard approaches to field theory that new technologies need to be developed. At this point, the two most popular approaches are the discrete light-cone quantization and the light-cone Tamm-Dancoff methods. They are designed to overcome the problems that have prevented other methods for the last twenty years from accurately calculating anything in the strong coupling regime of QCD. Moreover, their language is appealingly close to experiment and phenomenology. Both methods require computing resources that are within reach of present day computers, and the general structure of solution seems to have simple physical interpretations within the confines of the constituent quark model and the Feynman-Bjorken parton model. Both methods, however, face a host of new challenges not seen in the usual approaches to field theory that must be overcome before tackling QCD. This paper is devoted to a discussion of these new problems and to the various proposed solutions that are currently being investigated.

1 Introduction

Over the past twenty years two fundamentally different pictures of hadronic matter have developed. One is closely related to experimental observation and is known as the Feynman-Bjorken Quark Parton Model, or as the Constituent Quark Model. The other, Quantum Chromodynamics (QCD), is based on an elegant non-abelian quantum field theory. In this paper we will discuss some of the contradictions between these two pictures of hadronic matter and we will argue that the light-cone formulation of QCD provides a method for reconciling QCD with the Constituent Quark Model. This elegant approach to QCD avoids many of the most difficult problems as they appear in the equal time formulation of the theory; however, we will see that the light-cone formulation has its own unique complexities. The main thrust of this paper will be to discuss these complexities. We will attempt to present sufficient background material to allow the reader to see some of the advantages of the light-cone formulation, but we shall not undertake to give a complete review.

In the Constituent Quark Model, hadrons are relativistic bound states of a few confined quark and gluon quanta. The momentum distributions of quarks making up the nucleons in the Constituent Quark Model, the structure functions, are well-determined experimentally from deep inelastic lepton scattering, but there has been relatively little progress in computing the wavefunctions of hadrons from first principles in QCD. The most interesting progress has come from lattice gauge theory [1] and QCD sum rule calculations [2], both of which have given predictions for the proton's distribution amplitude. The distribution amplitude $\phi_p(x_1, x_2, x_3, Q)$, with $\sum_i x_i = 1$, is the fundamental gauge invariant wavefunction which describes the distribution of the fractional longitudinal momenta x_i of the valence quarks in a hadron integrated over transverse momentum up to the scale Q. However, the results from the two analyses are in strong disagreement: The QCD sum rule analysis predicts a strongly asymmetric three-quark distribution, whereas the lattice results, obtained in the quenched approximation, favor a symmetric distribution in the x_i. Even less is known from first principles in non-perturbative QCD about the gluon and non-valence quark contributions to the proton wavefunction, although data from a number of experiments now suggest non-trivial spin correlations, a significant strangeness content, and a large x component to the charm quark distribution in the proton.

There are many reasons why knowledge of hadron wavefunctions, particularly at the amplitude level, will be necessary for future progress in particle physics. For example, in electroweak theory, the central unknown required for reliable calculations of weak decay amplitudes are the hadronic matrix elements. The coefficient functions in the operator product expansion needed to compute many types of experimental quantities are essentially unknown and can only be estimated at this point. The calculation of form factors and exclusive scattering processes, in general, depend in detail on the basic amplitude structure of the scattering hadrons in a general Lorentz frame. Even the calculation of the magnetic moment of a proton requires wavefunctions in a boosted frame. We thus need a practical computational method for QCD which not only determines its spectrum, but also the wavefunction in a general Lorentz frame.

It is clearly a formidable task to calculate the structure of hadrons in terms of their fundamental degrees of freedom. Even in the case of abelian quantum electrodynamics, very little is known about the nature of the bound state solutions in the large α, strong-coupling, domain. A calculation of bound state structure in QCD has to deal with many complicated aspects of the theory simultaneously: confinement, vacuum structure, spontaneous breaking of chiral symmetry (for massless quarks), while describing a relativistic many-body system which apparently has unbounded particle number.

The first step is to find a language in which one can represent the hadron in terms of a few relativistic confined quarks and gluons. The Bethe-Salpeter formalism has been the central method for analyzing hydrogenic atoms in QED, providing a completely covariant procedure for obtaining bound state solutions. However, calculations using this method are extremely complex and appear to be intractable much beyond the ladder approximation. It also appears impractical to extend this method to systems with more than a few constituent particles.

An intuitive approach for solving relativistic bound-state problems would be to solve the Hamiltonian eigenvalue problem

$$H|\Psi\rangle = \sqrt{\vec{P}^2 + M^2}|\Psi\rangle \tag{1}$$

for the particle's mass, M, and wavefunction, $|\Psi\rangle$. Here, one imagines that $|\Psi\rangle$ is an expansion in free multi-particle occupation number Fock states, and that the operators H and $\vec{P}$ are second-quantized Heisenberg picture operators. Unfortunately, this method, as described by Tamm and Dancoff [3], is severely complicated by its non-covariant structure and the necessity to first understand its complicated vacuum eigen-solution over all space and time. The presence of the square root operator also presents severe mathematical difficulties. Even if these problems could be solved, the eigen-solution is only determined in its rest system; determining the boosted wavefunction is as complicated as diagonalizing H itself.

Fortunately, "light-cone" quantization, which can be formulated independent of the Lorentz frame, offers an elegant avenue of escape [4]. The square root operator does not appear in the light-cone formalism, and the vacuum structure is relatively simple; for example, there is no spontaneous creation of massive fermions in the light-cone quantized vacuum. There are, in fact, many reasons to quantize relativistic field theories at fixed light-cone time $\tau = t + z/c$. Dirac [5], in 1949, showed that a maximum number of Poincaré generators become independent of the dynamics in the "front form" formulation, including certain Lorentz boosts. In fact, unlike the traditional equal-time Hamiltonian formalism, quantization on a plane tangential to the light-cone can be formulated without reference to the choice of a specific Lorentz frame. The eigensolutions of the light-cone Hamiltonian have Lorentz scalars M^2 as eigenvalues, *i.e.*

$$H_{\mathrm{LC}}|\Psi\rangle = M^2|\Psi\rangle\ , \tag{2}$$

and describe bound states of arbitrary four-momentum and invariant mass M, allowing the computation of scattering amplitudes and other dynamical quantities.

However, the most remarkable feature of this formalism is the apparent simplicity of the vacuum. In many field theories the vacuum state of the free Hamiltonian is an eigenstate of the total light-cone Hamiltonian. The Fock expansion constructed on this vacuum state provides a complete relativistic many-particle basis for diagonalizing the full theory, as given in a recent review [6].

For the past several years, an increasingly large and diverse group of physicists have been studying the possibility of combining Tamm-Dancoff procedures with the procedure of light-cone quantization to develop a practical method of performing non-perturbative calculations in quantum field theory. In the Tamm-Dancoff method one approximates the field theory by truncating the Fock space. The assumption, based on the Constituent Quark Model picture, is that a few excitations describe the essential physics and that adding more excitations only refines this initial approximation. This is in stark contradiction to the instant formulation of QCD where an infinite number of gluons are essential to formulating even the vacuum. If the efforts are successful, they could lead to procedures for calculating not only the hadron mass spectrum but all the quantities which depend on the hadron wavefunction such as structure functions, fragmentation functions, *etc.* Furthermore, the form of the answer, an expansion in a Fock basis, is one that appeals to many physicists in that it matches the intuitive picture of hadrons as composed of partons. Indeed, Wilson [7] has stressed the point that the success of the Constituent Quark Model model provides a reason to be hopeful for the eventual success of the light-cone methods.

The striking advantages of quantizing gauge theories on the light-cone have been realized by a number of authors, including Klauder, Leutwyler, and Streit [8], Kogut and Soper [9], Rohrlich [10], Leutwyler [11], Casher [12], Chang, Root, and Yan [13], and Lepage, Brodsky and others [14, 15, 16]. Leutwyler recognized the utility of defining quark wavefunctions on the light-cone to give an unambiguous meaning to concepts used in the parton model. Casher gave the first construction of the light-cone Hamiltonian for non-Abelian gauge theory and gave an overview of important considerations in light-cone quantization. Chang, Root, and Yan demonstrated the equivalence of light-cone quantization with standard covariant Feynman analysis. Franke [17, 18, 19], Karmanov [20, 21], and Pervushin [22] have also done important work on light-cone quantization.

A mathematically similar but conceptually different approach to light-cone quantization is the "infinite momentum frame" formalism. This method involves observing the system in a frame moving past the laboratory close to the speed of light. The first developments were given by Weinberg [23]. Although light-cone quantization is similar to infinite momentum frame quantization, it differs since no reference frame is chosen for calculations, and it is thus manifestly Lorentz covariant. The only aspect that "moves at the speed of light" is the quantization surface. Other works in infinite momentum frame physics include Drell, Levy, and Yan [24, 25], Susskind and Frye [26], Bjorken, Kogut, and Soper [27], and Brodsky, Roskies, and Suaya [28].

Much of the recent work so far has been in theories in $1+1$ dimensions. For these theories there is much success to report. Numerical solutions and studies have been performed for a variety of theories including $U(1)$ and $SU(N)$ for $N=2,3$ and 4 [29, 30, 31]; Yukawa [32, 33]; ϕ^4 [33, 34, 35]; and the Schwinger model [36, 37, 38, 39] has been presented. A smaller amount of work in $3+1$ dimensions has also been done [40, 41, 42, 43, 44, 45]. Numerical studies on positronium have provided the Bohr series, and the fine structure with good accuracy[43]. Formal work on renormalization in $3+1$ dimensions [41] has yielded some positive results but many questions remain. Attempts to more directly combine light-cone and lattice gauge calculations are also under study [46].

In the procedure of light-cone quantization it is essential that one specifies quantization conditions on a light-like surface, rather than the usual space-like surface. That has several effects: There is a change of independent variables and a reduction in the number of degrees of freedom for the system. The missing degrees of freedom are replaced by constraint relations and the introduction of non-local operators. This has a profound effect on how we renormalize the theory.

There is a change of representation since negative frequency modes of the fields along the characteristic [time-like] surface create different states than negative frequency modes of the fields along a space-like surface. It is this last effect, the change of basis, that many workers see as possibly allowing an easier implementation of Hamiltonian methods. These hopes stem principally from one striking feature of the light-cone representation: the bare vacuum is an eigenstate of the dynamical operators of the full theory. From a mathematical point of view: there is an argument that in simple cases the physical vacuum will just be the bare vacuum on which the theory is quantized—in some cases that expectation is known to be realized [47]. In other cases the naive expectation is not realized , but the vacuum is still *much* simpler in the light-cone representation than in the equal-time representation [37]. The simplicity of the vacuum, in turn, removes some of the difficulties involved in developing renormalization procedures originally proposed by Tamm and Dancoff [3] for equal time quantization. The unique features of light-cone quantization open the exciting prospective of combining the elegance of field theory with the success of the Constituent Quark Model.

In this article we shall briefly provide some motivation for, and describe some successes of the efforts; but we shall concentrate on presenting outstanding problems in the field in the hope that other physicists may find them interesting to work on.

2 General Features of Light-Cone Quantization

In general, the Hamiltonian is the evolution operator $H = i\frac{\partial}{\partial\tau}$ which propagates fields in generalized "time" τ from one space-like surface to another. As emphasized by Dirac [5], there are several choices for the evolution parameter τ. In the "Instant Form" $\tau = t$ is the ordinary Cartesian time. In the "Front Form", or light-cone quantization, one chooses $\tau = t + z/c$ as the light-cone time, with boundary conditions specified as a function of x, y, and $z^- = ct - z$. Kinematic relations are given in Table 1.

Table 1. Kinematic relations for conventional and light-cone space-time parameterization ($c = 1$)

time	$t = x^0$	$\tau = x^+ = t + z$
space	(x, y, z)	$(x^- = t - z, x, y)$
metric tensor $g^{\mu\nu}$	$g^{00} = 1;\ g^{11} = g^{22} = g^{33} = -1$	$g^{+-} = g^{-+} = 2;\ g^{11} = g^{22} = -1$
scalar product $p \cdot x$	$p^0x^0 - p^1x^1 - p^2x^2 - p^3x^3$	$\frac{1}{2}(p^+x^- + p^-x^+) - p^1x^1 - p^2x^2$
time derivative	$\partial_0 = \partial^0 = \frac{\partial}{\partial x^0}$	$\partial_+ = \frac{1}{2}\partial^- = \frac{\partial}{\partial x^+}$
space derivative	$\partial_i = -\partial^i = \frac{\partial}{\partial x^i};\ i = 1,2,3$	$\partial_- = \frac{1}{2}\partial^+ = \frac{\partial}{\partial x^-};\ \partial_i = -\partial^i;\ i = 1,2$

Notice that these forms become equivalent in the non-relativistic limit where, effectively, $c \to \infty$. A comparison of light-cone quantization with equal-time quantization is shown in Table 2.

Table 2. A comparison of equal-time and light-cone quantization.

Hamiltonian	$H = \sqrt{M^2 + \vec{P}_\perp^2 + P_z^2} + V$	$P^- = \frac{P_\perp^2 + M^2}{P^+} + V$
Conserved quantities	$E, \quad P_z, \quad \vec{P}_\perp$	$P^-, \quad P^+, \quad \vec{P}_\perp$
Momenta	$-\infty < P_z < +\infty$	$0 < P^+ < +\infty$
Bound state equation	$H\Psi = E\Psi$	$H_{\rm LC}\Psi = M^2\Psi$

Although the instant form is the conventional choice for quantizing a field theory, it is not always the practical form. For example, given the wavefunction of an n-electron atom, $\psi_n(\vec{r}_i, t = 0)$, at initial time $t = 0$. Then, in principle, one can use the Hamiltonian H to evolve $\psi_n(\vec{r}_i, t)$ to later times t. However, an experiment which could specify the initial wavefunction would require the simultaneous measurement of the positions of all of the bound electrons, such as by the simultaneous Compton scattering of n independent laser beams on the atom. In contrast, determining the initial wavefunction at fixed light-cone time $\tau = 0$ only requires an experiment which scatters one plane-wave laser beam, since the signal reaching each of the n electrons is received at the same light-cone time $\tau = t_i + z_i/c$.

As we shall discuss in this article, light-cone quantization allows a precise definition of the notion that a hadron consists of a few confined quarks and gluons consistent with the success of the Constituent Quark Model. In light-cone quantization, a free particle is specified by its four momentum $k^\mu = (k^+, k^-, \vec{k}_\perp)$ where $k^\pm = k^0 \pm k^3$. If the particle is on its mass shell and has positive energy, its light-cone energy is also positive, *i.e.* $k^- = (k_\perp^2 + m^2)/k^+ > 0$. In perturbation theory, the total transverse momentum of a system of particle, $\vec{P}_\perp = \sum \vec{k}_\perp$ and the total plus momentum $P^+ = \sum k^+$ are conserved at each vertex. The light-cone bound-state wavefunction thus describes constituents which are on their mass shell, but off the light-cone energy shell: $P^- < \sum k^-$.

The restriction $k^+ > 0$ for massive quanta is a key difference between light-cone quantization and ordinary equal-time quantization. In equal-time quantization, the state of a parton is specified by its ordinary three-momentum $\vec{k} = (k^1, k^2, k^3)$. Since each component of $\vec{k}$ can be either positive or negative, there exist zero total momentum Fock states of arbitrary particle number, and these will mix with the zero-particle state to build up the ground state. However, in light-cone quantization each of the particles forming a zero-momentum state must have vanishingly small k^+. Such a configuration represents a point of measure zero in the phase space, and therefore such states are usually neglected.

Actually some care must be taken here, since there are operators in the theory that are singular at $k^+ = 0$ — *e.g.* the kinetic energy $(\vec{k}_\perp^2 + M^2)/k^+$. In certain circumstances, states containing $k^+ \to 0$ quanta (zero modes) can significantly alter the ground state of the theory. One such circumstance is when there is spontaneous symmetry breaking. Another is the complication due to massless gluon quanta in a non-Abelian gauge theory. Nevertheless, the space of states that can play a role in the vacuum structure is much smaller for light-cone quantization than for equal-time quantization. This suggests that vacuum structure may be simpler to analyze using the light-cone formulation. The treatment of zero modes and massless gluons are among the most important open question we will be discussing.

Even in perturbation theory, light-cone quantization has overwhelming advantages over standard time-ordered perturbation theory. For example, in order to calculate a Feynman amplitude of order g^n one must suffer the calculation of the sum of n time-ordered graphs, each of which is a non-covariant function of energy denominators which, in turn, consist of sums of complicated square roots $p_i^0 = \sqrt{\vec{p}_i^{\,2} + m_i^2}$. On the other hand, in light-cone perturbation theory, only a few graphs give non-zero contributions, and those that are non-zero have light-cone energy denominators which are simple sums of rational forms $p_i^- = (\vec{p}_{\perp i}^{\,2} + m_i^2)/p_i^+$.

3 Representation of Hadrons on the Light-Cone Fock Basis

One can construct a complete basis of free Fock states $|n\rangle$ (eigenstates of the free light-cone Hamiltonian with $|n\rangle\langle n| = I$) in the usual way by applying products of free field creation operators to the vacuum state $|0\rangle$:

$$\begin{aligned}
&|0\rangle \\
&|q\bar{q} : \underline{k}_i \lambda_i\rangle = b^\dagger(\underline{k}_1 \lambda_1)\, d^\dagger(\underline{k}_2 \lambda_2)\, |0\rangle \\
&|q\bar{q}g : \underline{k}_i \lambda_i\rangle = b^\dagger(\underline{k}_1 \lambda_1)\, d^\dagger(\underline{k}_2 \lambda_2)\, a^\dagger(\underline{k}_3 \lambda_3)\, |0\rangle \\
&\vdots
\end{aligned} \tag{3}$$

where $b^\dagger$, $d^\dagger$ and $a^\dagger$ create bare quarks, antiquarks and gluons having three-momenta $\underline{k} = (k^+, k_\perp)$ and helicities λ. One of the most important advantages of light-cone quantization is that the light-cone Fock expansion can be used as the basis for representing the physical states of QCD. For example, a pion with momentum $\underline{P} = (P^+, \vec{P}_\perp)$ is described by the expansion,

$$|\pi : \underline{P}\rangle = \sum_{n,\lambda_i} \int \overline{\prod_i} \frac{dx_i d^2\vec{k}_{\perp i}}{\sqrt{x_i} 16\pi^3} \left| n : x_i P^+, x_i \vec{P}_\perp + \vec{k}_{\perp i}, \lambda_i \right\rangle \psi_{n/\pi}(x_i, \vec{k}_{\perp i}, \lambda_i) \tag{4}$$

where the sum is over all Fock states and helicities, and where

$$\overline{\prod_i} dx_i \equiv \prod_i dx_i \, \delta(1 - \sum_j x_j) \quad \text{and} \quad \overline{\prod_i} d^2\vec{k}_{\perp i} \equiv \prod_i d^2\vec{k}_{\perp i} \, 16\pi^3 \, \delta^{(2)}(\sum_j \vec{k}_{\perp j}) \, . \tag{5}$$

The wavefunction $\psi_{n/\pi}(x_i, \vec{k}_{\perp i}, \lambda_i)$ is thus the amplitude for finding partons in a specific light-cone Fock state n with momenta $(x_i P^+, x_i \vec{P}_\perp + \vec{k}_{\perp i})$ in the pion. The Fock state is off the light-cone energy shell: $\sum k_i^- > P^-$. The light-cone momentum coordinates x_i and $\vec{k}_{\perp i}$, with $\sum_{i=1}^n x_i = 1$ and with $\sum_{i=1}^n \vec{k}_{\perp i} = \vec{0}_\perp$, respectively, are actually relative coordinates; *i.e.* they are independent of the total momentum P^+ and $\vec{P}_\perp$ of the bound state. The special feature that light-cone wavefunctions do not depend on the total momentum is not surprising, since x_i is the longitudinal momentum fraction carried by the i^{th}-parton $(0 \le x_i \le 1)$, and $\vec{k}_{\perp i}$ is its momentum "transverse" to the direction of the meson. Both of these are frame independent quantities. The ability to specify wavefunctions simultaneously in any frame is a special feature of light-cone quantization.

In the light-cone Hamiltonian quantization of gauge theories, one chooses the light-cone gauge, $\eta \cdot A = A^+ = 0$, for the gluon field. The use of this gauge results in well-known simplifications in the perturbative analysis of light-cone dominated processes such as high-momentum hadronic form factors. It is indispensable if one desires a simple, intuitive Fock-state basis since there are neither negative-norm gauge boson states nor ghost states in the unitary $A^+ = 0$ gauge. Thus each term in the normalization condition

$$\sum_{n,\lambda_i} \int \overline{\prod_i} \frac{dx_i d^2\vec{k}_{\perp i}}{16\pi^3} |\psi_{n/\pi}(x_i, \vec{k}_{\perp i}, \lambda_i)|^2 = 1 \tag{6}$$

is positive.

The coefficients in the light-cone Fock state expansion are the parton wavefunctions $\psi_{n/H}(x_i, \vec{k}_{\perp i}, \lambda_i)$ which describe the decomposition of each hadron in terms of its fundamental quark and gluon degrees of freedom. The light-cone variables x_i are often identified with the constituent's longitudinal momentum fractions $x_i = k_i^z/P_z$, in a frame where the total momentum $P^z \to \infty$. However, in light-cone Hamiltonian formulation of QCD, x_i are the boost-invariant light-cone fractions,

$$x_i \equiv \frac{k_i^+}{P^+} = \frac{k_i^0 + k_i^z}{P^0 + P^z} , \tag{7}$$

independent of the choice of Lorentz frame.

Given the light-cone wavefunctions, $\psi_{n/H}(x_i, \vec{k}_{\perp i}, \lambda_i)$, one can compute virtually any hadronic quantity by convolution with the appropriate quark and gluon matrix elements. The proton form factors can be computed as a simple overlap of its initial and final state light-cone wavefunctions $\psi_{n/p}$. The behavior of large momentum transfer Q^2 exclusive processes is controlled by the hadron distribution amplitudes $\phi(x_i, Q)$, the valence Fock state wavefunction at impact separation $b_\perp = 1/Q$ [15, 48]. For example, the angular, helicity, and phase – dependence of large-momentum-transfer proton Compton scattering provides a sensitive measure of the shape of $\phi_p(x_i, Q)$ [49].

In the case of inclusive reactions, the leading-twist structure functions measured in deep inelastic lepton – proton scattering are immediately related to the light-cone probability distributions:

$$2M\, F_1(x, Q) = \frac{F_2(x, Q)}{x} \approx \sum_a e_a^2\, G_{a/p}(x, Q) , \tag{8}$$

where

$$G_{a/p}(x, Q) = \sum_{n, \lambda_i} \int \overline{\prod_i} \frac{dx_i d^2\vec{k}_{\perp i}}{16\pi^3} |\psi_n^{(Q)}(x_i, \vec{k}_{\perp i}, \lambda_i)|^2 \sum_{b=a} \delta(x_b - x) \tag{9}$$

is the number density of partons of type a with longitudinal momentum fraction x in the proton and total transverse momentum less than Q. This follows from the observation that deep inelastic lepton scattering in the Bjorken-scaling limit occurs if x_{Bj} matches the light-cone fraction of the struck quark. (The $\sum_b$ is over all partons of type a in state n.) However, the light-cone wavefunctions contain much more information for the final state of deep inelastic scattering, such as the multi-parton distributions, spin and flavor correlations, and the spectator jet composition.

4 The Light-Cone Hamiltonian Eigenvalue Problem

In principle, the problem of computing the spectrum in QCD and the corresponding light-cone wavefunctions for each hadron can be reduced to diagonalizing the QCD light-cone Hamiltonian in Heisenberg quantum mechanics: Any hadron state must be an eigenstate of the light-cone Hamiltonian. A pion state $|\pi\rangle$, for example, satisfies $(M_\pi^2 - H_{\rm LC})\,|\pi\rangle = 0$, and projecting this onto the various Fock states $\langle q\bar{q}|$, $\langle q\bar{q}g|\ldots$ results in an infinite number of coupled integral eigenvalue equations [14, 40],

$$\left(M_\pi^2 - \sum_i \frac{\vec{k}_{\perp i}^2 + m_i^2}{x_i}\right) \begin{bmatrix} \psi_{q\bar{q}/\pi} \\ \psi_{q\bar{q}g/\pi} \\ \vdots \end{bmatrix} = \begin{bmatrix} \langle q\bar{q}|V|q\bar{q}\rangle & \langle q\bar{q}|V|q\bar{q}g\rangle & \cdots \\ \langle q\bar{q}g|V|q\bar{q}\rangle & \langle q\bar{q}g|V|q\bar{q}g\rangle & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \psi_{q\bar{q}/\pi} \\ \psi_{q\bar{q}g/\pi} \\ \vdots \end{bmatrix} \quad (10)$$

where V is the interaction part of $H_{\rm LC}$. In principle, these equations determine the hadronic spectrum and wavefunctions.

At this point there are several approaches that are currently popular in dealing with these equations. Although these methods may look very different they have the same common elements. A central difficulty is that any truncation of the Fock space at a fixed number of states breaks gauge invariance. We can see this by examining the Ward-Takahashi identities of QED. These identities are a manifestations of gauge invariance, and at the one loop level they relate wavefunction and vertex renormalization constants. The wavefunction renormalization involves at lowest order a two particle intermediate state while the vertex renormalization involves a three particle intermediate state, thus if one were to truncate at the two particle intermediate state, the Ward-Takahashi identities are violated. The Ward-Takahashi identities in general implement gauge invariance, which then maintains unitarity and Lorentz covariance. In the light-cone gauge the theory has no ghost or negative norm state, therefore the theory will automatically be unitary. On the other hand it will break manifest Lorentz covariance, and this has been seen in Compton scattering [50], and in the Yukawa (3+1) model [44]. One finds in Yukawa (3+1) that the bound state spectrum in different helicity states does not have the degeneracy required by Lorentz covariance because the Tamm-Dancoff approximation also violates rotational invariance. Caveats on the violation of rotational invariance have also been raised by Burkardt and Langnau [51]. As one improves the Tamm-Dancoff approximation by increasing the number of basis states this violation of gauge invariance and rotational invariance will be pushed to higher Fock space sectors. To the extent that higher and higher Fock space sectors make smaller contribution, their violation of gauge and rotational invariance will be smaller and smaller. To date, however, this hypothesis has not been checked by explicit calculations.

In the complete formulation of the "discretized light-cone quantization" (DLCQ) method, one constructs a complete discretized light-cone Fock basis in momentum space. The size of the Fock state basis can be is limited by a cut-off in the total invariant mass $\mathcal{M}^2 = \sum \frac{(k_\perp^2 + m^2)}{x}$. The light-cone Hamiltonian can then be visualized as a matrix with a finite number of rows and columns assuming an invariant ultraviolet cut-off. Next, one formulates all necessary model assumptions, in accord with covariance and gauge-invariance, thus obtaining a discrete representation of the quantum field theory. At any stage, one can go to the continuum limit, convert the matrix equation to an integral equation, and solve it with suitably optimalized numerical methods. One should emphasize, that the regularization scheme of DLCQ [40] explicitly allows for such a procedure, since the regularization scales are equal both for discretization and the continuum, in contrast to lattice gauge theory.

In the DLCQ approach, the light-cone Fock state basis is rendered discrete by imposing periodic (or anti-periodic) boundary conditions [32], and the integral equation becomes a matrix eigenvalue equation. However, even though the QCD potential is essentially trivial on the light-cone momentum space basis, the many channels required to describe a hadronic state make these equations very difficult to solve, except by numerical methods. For example, Fock states with two or more gluons are required just to represent the effects of the running coupling constant of QCD.

A closer examination show that regulating and renormalizing light-cone field theory can be quite ambiguous. In general, one must introduce an infrared cutoff in k^+ and an ultraviolet cutoff in $\vec{k}_\perp$. A number of approaches to this problem can be found in the literature including DLCQ, dimensional regulation and invariant mass cutoffs, which preserves the light-cone Lorentz symmetries. It is useful to note that each light-cone field has two distinct length scales, x^- and $x_\perp$ (not to be confused with longitudinal momentum fraction). Table 3 gives the dimension of all the objects that appear in the canonical Hamiltonian in both the equal time and light-cone approaches.

Table 3

Object	Equal Time	Light-Cone
dimension of length	x	$x^-, x_\perp$
scalar field ϕ	$1/x$	$1/x_\perp$
fermion field ψ	$1/x^{3/2}$	$1/x_\perp\sqrt{x^-}$
mass	$1/x$	$1/x_\perp$
derivative	$\partial/\partial x \sim 1/x$	$\partial/\partial x^- \sim 1/x^-$
Hamiltonian	$1/x$	$x^-/x_\perp^2$
Hamiltonian density	$1/x^4$	$1/x_\perp^4$

In addition to the objects in Table 3 the canonical light-cone Hamiltonian contains the operator $[\partial/\partial x^-]^{-1}$. If one considers all the allowed terms that one might add to the canonical Hamiltonian as possible counter terms, one finds an explosion in the number of possibilities. The Ohio State approach attempts to turn this unseemly situation into a virtue [7], by arguing that additional counter terms provide the degrees of freedom necessary to repair the symmetries that were broken in formulating the theory, in particular gauge invariance, and therefore Lorentz covariance.

Even without solving the QCD light-cone equations of motion, one can anticipate some general features of the behavior of the light-cone wavefunctions[14, 15, 52]. Each Fock component describes a system of free particles with kinematic invariant mass squared:

$$M^2 = \sum_i^n \frac{\vec{k}_{\perp i}^2 + m_i^2}{x_i}. \tag{11}$$

On general dynamical grounds, one can expect that states with very high M^2 are suppressed in physical hadrons, with the highest mass configurations computable from perturbative considerations. One also notes that

$$\ln x_i = \ln \frac{(k^0 + k^z)_i}{(P^0 + P^z)} = y_i - y_P \tag{12}$$

is the rapidity difference between the constituent with light-cone fraction x_i and the rapidity of the hadron itself. Since correlations between particles rarely extend over two units of rapidity in hadron physics, this argues that constituents which are correlated with the hadron's quantum numbers are primarily found with $x > 0.2$.

The limit $x \to 0$ is normally an ultraviolet limit in a light-cone wavefunction. Recall, that in any Lorentz frame, the light-cone fraction is $x = k^+/P^+ = (k^0 + k^z)/(P^0 + P^z)$. Thus in a frame where the bound state is moving infinitely fast in the positive z direction ("the infinite momentum frame"), the light-cone fraction becomes the momentum fraction $x \to k^z/p^z$. However, in the rest frame $\vec{P} = \vec{0}$, $x = (k^0 + k^z)/M$. Thus $x \to 0$ generally implies very large constituent momentum $k^z \to -k^0 \to -\infty$ in the rest frame; it is excluded by the ultraviolet regulation of the theory – unless the particle has strictly zero mass and transverse momentum.

If a particle has non-relativistic momentum in the bound state, then we can identify $k^z \sim xM - m$. This correspondence is useful when one matches physics at the relativistic/non-relativistic interface. In fact, any non-relativistic solution to the Schrödinger equation can be immediately written in light-cone form by identifying the two forms of coordinates. For example, the Schrödinger solution for particles bound in a harmonic oscillator potential can be taken as a model for the light-cone wavefunction for quarks in a confining linear potential [16]:

$$\psi(x_i, \vec{k}_{\perp i}) = A \exp(-bM^2) = \exp - \left(b \sum_i^n \frac{k_{\perp i}^2 + m_i^2}{x_i} \right) . \tag{13}$$

This form exhibits the strong fall-off at large relative transverse momentum and at the $x \to 0$ and $x \to 1$ endpoints expected for soft non-perturbative solutions in QCD. The perturbative corrections due to hard gluon exchange give amplitudes suppressed only by power laws and thus will eventually dominate wavefunction behavior over the soft contributions in these regions. This *ansatz* is the central assumption required to derive dimensional counting perturbative QCD predictions for exclusive processes at large momentum transfer and the $x \to 1$ behavior of deep inelastic structure functions. A review is given in Ref. [16]. A model for the polarized and unpolarized gluon distributions in the proton which takes into account both perturbative QCD constraints at large x and coherent cancellations at low x and small transverse momentum is given in Ref. [52].

The fact that particle exchange only generates a power law fall-off wavefunction in the LCTD calculation is in many ways a virtue of this totally relativistic calculation. The Tamm-Dancoff equation will diverge at large $k_\perp$ as a result of this behavior and the wavefunctions and eigenfunctions will show strong cutoff dependence. Of course, this is a reflection of the fact that for a strongly coupled relativistic field theory one can not get away with just regulating the theory, one must renormalize it. Counter terms must be added to the canonical Hamiltonian to remove this dependence of the wavefunction and eigenvalues on the regulator [53]. In a standard field theory one adds or changes the operators appearing in the canonical approach, however, in the LCTD approach one must change the kernel in the Tamm-Dancoff integral equations. The changes must reflect the allowed

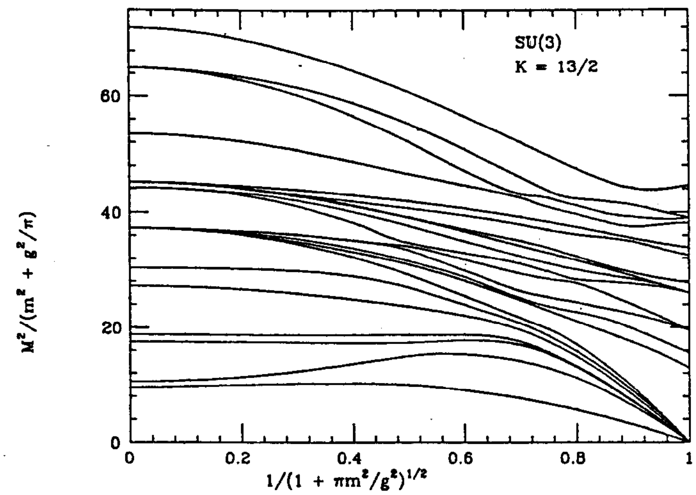

Figure 1: Representative baryon spectrum for QCD in one-space and one-time dimension [30].

operator structure in Table 2. This renormalization procedure has recently been applied to the Yukawa model in 3+1 dimension and can restore the degeneracies in the bound state spectrum expected from Lorentz invariance [44].

The DLCQ method was first used to obtain the mass spectrum and wavefunctions of Yukawa theory, $\overline{\psi}\psi\phi$, in one space and one time dimensions [32]. This success led to further applications including QED(1+1) for general mass fermions and the massless Schwinger model by Eller *et al.* [29], ϕ^4 theory in 1+1 dimensions by Harindranath and Vary [34], and QCD(1+1) for $N_C = 2,3,4$ by Hornbostel *et al.* [30]. Numerical solutions have been obtained for the meson and baryon spectra as well as their respective light-cone Fock state wavefunctions for general values of the coupling constant, quark masses, and color. A representative example of the invariant mass spectrum is shown in Fig. 1 for baryon states ($B = 1$) as a function of the dimensionless variable $\lambda = 1/(1 + \pi m^2/g^2)$. Notice that spectrum automatically includes continuum states with $B = 1$.

The structure functions for the lowest baryon states in SU(3) at two different coupling strengths $m/g = 1.6$ and $m/g = 0.1$ are shown in Fig. 2.

Higher Fock states have a very small probability; representative contributions to

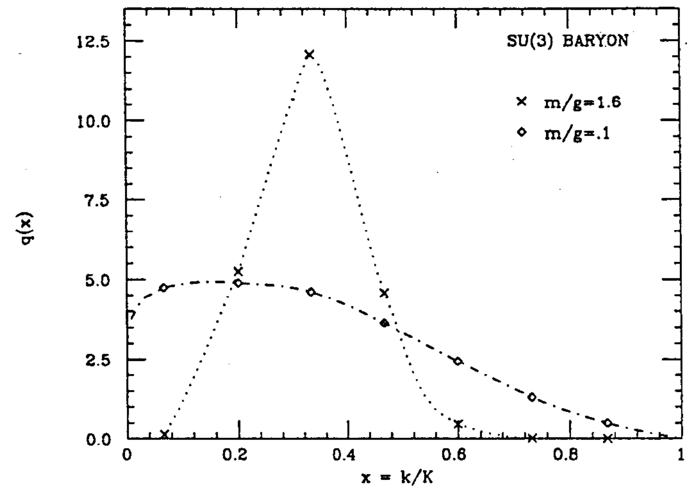

Figure 2: The baryon quark momentum distribution in QCD[1+1] computed using DLCQ [30].

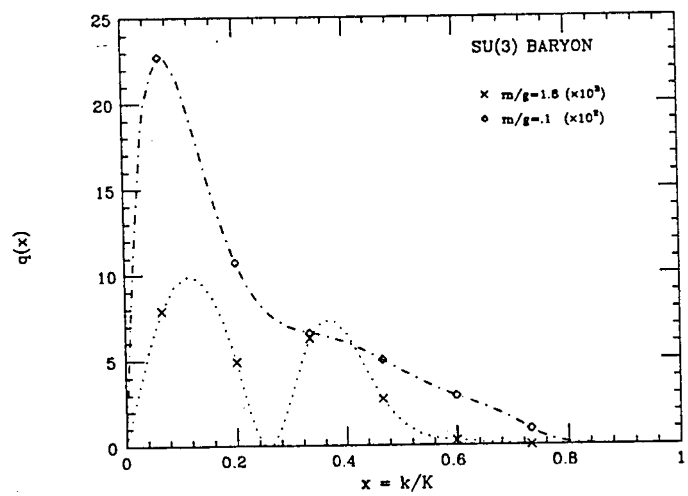

Figure 3: Contribution to the baryon quark momentum distribution from $qqq\bar{q}\bar{q}$ states for QCD[1+1] [30].

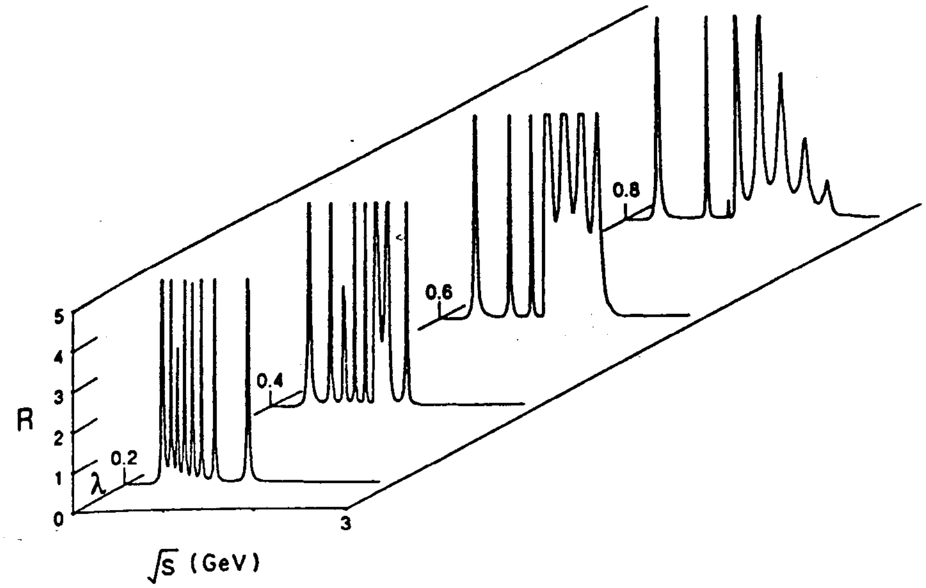

Figure 4: Calculation of $R_{e^+e^-}(s)$ in QED(1+1) using the DLCQ method. The results are shown for different coupling constants. For display purposes, the plot is clipped at $R = 5$. In addition, in order to give finite widths to what would have been δ–functions, the infinitesimal ϵ was set to 0.01 (from Ref. [54]).

the baryon structure functions are shown in Fig. 3. The interactions of the quarks in the pair state produce Fermi motion beyond $x = 0.5$. Although these results are for one-time one-space theory they do suggest that the sea quark distributions in physical hadrons may be highly structured.

Similar results for QCD(1+1) were also obtained by Burkardt [31] by solving the coupled light-cone integral equation in the low particle number sector. Burkardt was also able to study non-additive nuclear effects in the structure functions of nuclear states in QCD(1+1). More recently, Hiller [54] has used DLCQ and the Lanczos algorithm for matrix diagonalization method to compute the annihilation cross section, structure functions and form factors in 1+1 theories. A typical result is shown in Fig. 4. It would be interesting to repeat this non-perturbative calculation for a renormalizable theory like the Gross-Neveu model in (1+1) dimensions, and analyze how the channel-by-channel calculation merges into the asymptotic freedom result.

The Yukawa model (1+1) in LCTD was studied by Ref. [33]. Solution for the wavefunction and bound state mass are present including a careful analysis of self energy effects. This detailed numerical study used the sector dependent renormalization ideas proposed by Harindranath, Perry and Wilson [55]. Wilson has also emphasized the potential advantages of using a Gaussian basis similar to that used in many-electron molecular systems, and that method was used in Ref. [33].

The application to higher dimensions is much more involved due to the expansion of the degrees of freedom and the need to introduce ultraviolet and infrared regulators and truncation procedures which minimize violations of gauge invariance and Lorentz invariance. This is in addition to the work involved implementing two extra dimensions with their added degrees of freedom. There have been some initial attempts to apply DLCQ to gauge theories in 3+1 dimensions [40, 42, 45, 56].

Most important for the purpose of having an explicit test, positronium can serve as a crucial system to validate the DLCQ methods. In addition to the work by Krautgärtner, Pauli, and Wölz [43] to be discussed here in short, Kaluža [42] has recently used a DLCQ diagonalization approach to obtain the lepton structure function in positronium.

To simplify the model [43] one considers only the charge zero sector of QED(3+1) and includes only the $J_z = 0$ *electron-positron* ($e\bar{e}$) and the *electron-positron-photon* ($e\bar{e}\gamma$) Fock states, denoted collectively by $|e\bar{e}\rangle$ and $|e\bar{e}\gamma\rangle$, respectively. In effect one analyzes the muonium system μ^+e^- at equal lepton mass to avoid complications from the annihilation kernels. Even when one restricts the Fock states to one dynamical photon, one is considering a complex non-perturbative problem, similar to the ladder approximation in the Bethe-Salpeter formalism. The light-cone approach has the advantage that one obtains the Dirac-Coulomb equation in the heavy muon limit.

It is convenient to introduce the projectors $P = \sum_i |(e\bar{e})_i\rangle\langle(e\bar{e})_i|$ and $Q = \sum_i |(e\bar{e}\gamma)_i\rangle\langle(e\bar{e}\gamma)_i|$, with $P + Q = \mathbf{1}$. The index i runs over all discrete light-cone momenta and helicities of the partons (electron e, positron $\bar{e}$ and photon γ) subject to fixed total momenta and to covariant regularization by a sharp momentum cut-off [40]. Applying these projectors, the full Hamiltonian matrix equation can

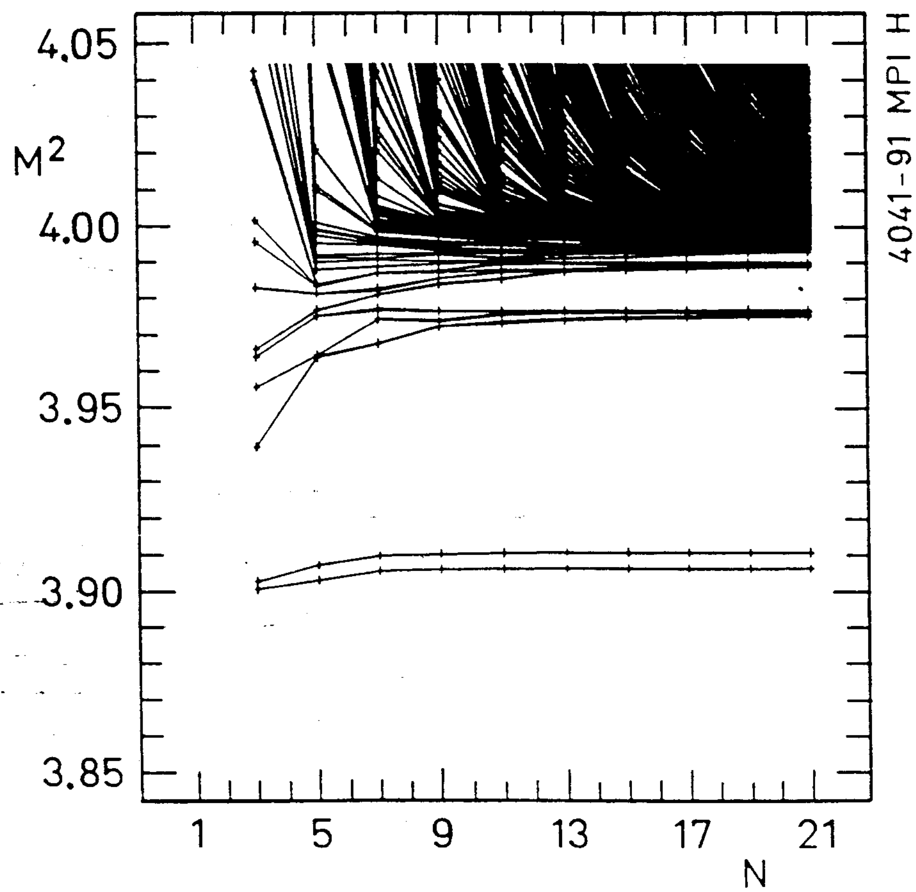

Figure 5: The invariant mass squared eigenvalues of the Tamm-Dancoff equation *versus* the number of integration points N. – Note the good convergence with N, and the appearance of the hyperfine splitting. Calculations are done for $J_z = 0$, $\Lambda = m_F$, and $\alpha = 0.3$.

be identically rewritten as

$$H_{eff}(\omega)\,|\psi_i(\omega)\rangle = M_i^2(\omega)|\psi_i(\omega)\rangle \ , \tag{14}$$

with the "effective Hamiltonian" acting only in P-space, *i.e.*

$$H_{eff}(\omega) \equiv PH_{LC}P + PH_{LC}\,\frac{1}{Q\,(\omega - H_{LC})\,Q}\,H_{LC}P \ . \tag{15}$$

Once $|\psi_i(\omega)\rangle = P|\psi_i\rangle$ is known, one can calculate the Q-space wave function by a quadrature.

In the continuum, the matrix above equation becomes an integral equation

$$\left\{\frac{m_{\rm F}^2 + \vec{k}_\perp^{\,2}}{x(1-x)} - M_i^2\right\}\psi_i(x,\vec{k}_\perp,s_1,s_2)$$
$$+\sum_{s_1',s_2'}\int_D dx'd^2\vec{k}_\perp{}'\left\langle x,\vec{k}_\perp;s_1,s_2\middle|V_{\rm eff}\middle|x',\vec{k}_\perp{}';s_1',s_2'\right\rangle\psi_i(x',\vec{k}_\perp{}',s_1',s_2') = 0 \ . \tag{16}$$

One has to keep track explicitly of s_2 and s_1, the helicities of the electron and the positron. The finite domain of integration D is set by covariant Fock space regularization [40],

$$\frac{m_{\rm F}^2 + \vec{k}_\perp^{\,2}}{x(1-x)} \le \Lambda^2 + 4m_{\rm F}^2 \ , \tag{17}$$

with given cut-off scale Λ. Note that to this degree of approximation Eqs. (14)-(16) are the same in both approaches, DLCQ and LCTD. Even at this level the integral equation develops a non-integrable colinear singularity. This colinear singularity [43] has been seen also in the Compton scattering [50]. Possible solutions have been proposed in Ref. [43].

The spectrum of the Tamm-Dancoff equation obtained using the above method is displayed in Fig. 5 as a function the number of integration points which play here the same role as the resolution in 1+1 dimensions. The two lowest states are identified as the singlet and the triplet state of positronium, since the wave functions have the corresponding symmetries. The agreement with former analytical solutions

[57] are excellent. The comparatively slow convergence of the higher excited states is not surprising.

A detailed study of Yukawa (3+1) by Glazek, Harindranath, Pinsky, Shigemitsu and Wilson [44] considers the implication of renormalization, self energy, triviality, and Lorentz covariance. They show that the fermion self energies give rise to a triviality bound which limits the ultraviolet cutoff for a given bare coupling. A new renormalization procedure for the LCTD equation is introduced which removes the cutoff dependence from the bound state masses and wavefunction even for large couplings [53]. This new renormalization procedure introduces additional degrees of freedom whic h are used to restore the Lorentz covariance of the spectrum. Numerical solutions to the LCTD equation for the spectrum are given in Fig. [6] [44]. It is amazing how similar the wavefunction as given in Figs. [7 and 8] [43, 44] are for different models and procedures.

5 The Light-Cone Vacuum

In the introduction we discussed the remarkable feature that the vacuum of the free light-cone theory can also be an eigenstate of the full Hamiltonian. Let us review the arguments: By definition, the perturbative vacuum is annihilated by the free Hamiltonian: ${H_{\rm LC}}^{(0)}|0\rangle = 0$. In gauge theory the interaction terms in $H_{\rm LC}$ are three- and four-point interactions; for example, in QED, the application on the vacuum of the interaction $H^I_{\rm LC} = \int d^3\underline{x}\bar{\psi}\gamma^\mu\psi A_\mu$ results in a sum of terms $b^\dagger(\underline{k}_1)a^\dagger(\underline{k}_2)d^\dagger(\underline{k}_3)|0\rangle$. As always, the conservation of P^+ requires $\sum_{i=1}^3 k_i^+ = 0$. However, $k_i^+ = 0$ is incompatible with finite energy for massive fermions. Thus the total light-cone Hamiltonian also annihilates the perturbative vacuum: $H_{\rm LC}|0\rangle = 0$. In contrast in equal-time quantization, the state $H|0\rangle$ is a highly complex composite of pair fluctuations.

The apparent simplicity of the vacuum in light-cone quantization is in contradiction to normal expectations for the structure of the lowest mass eigenstate of QCD. In the instant form, the QCD vacuum is believed to be a highly structured

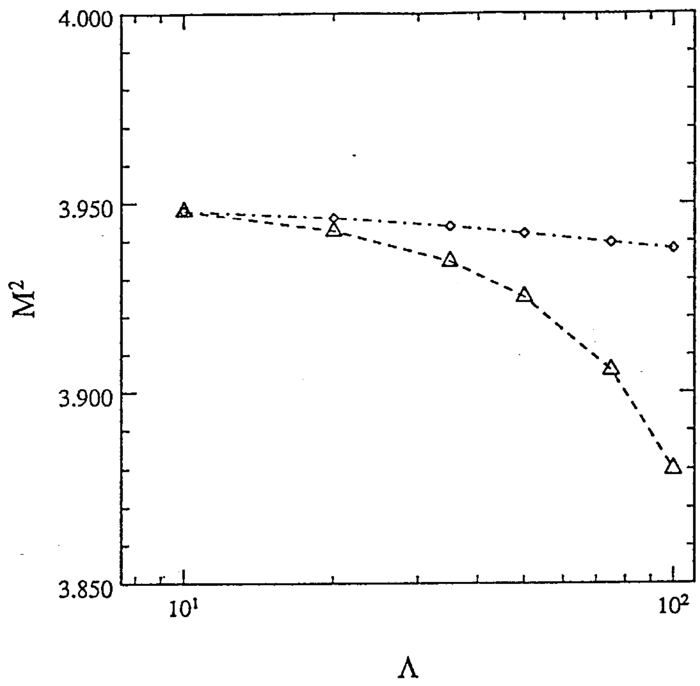

Fig. 6a

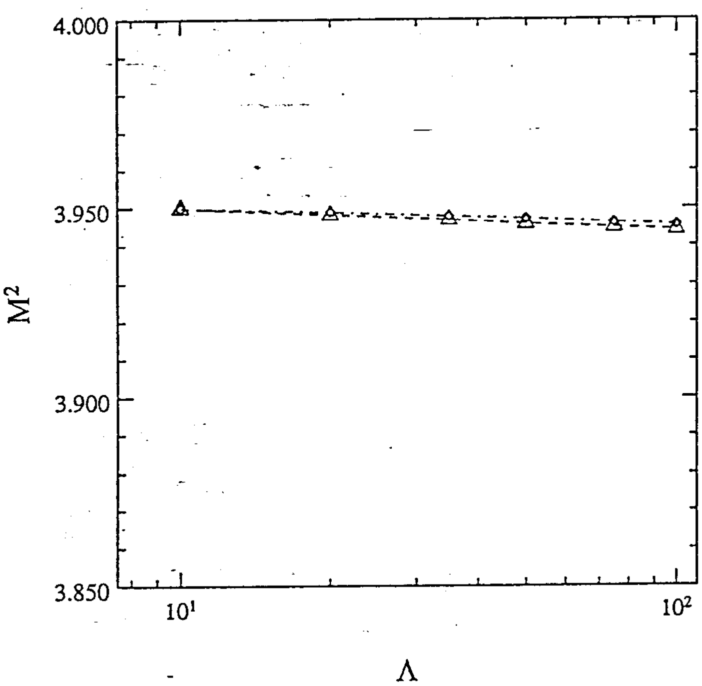

Fig. 6b

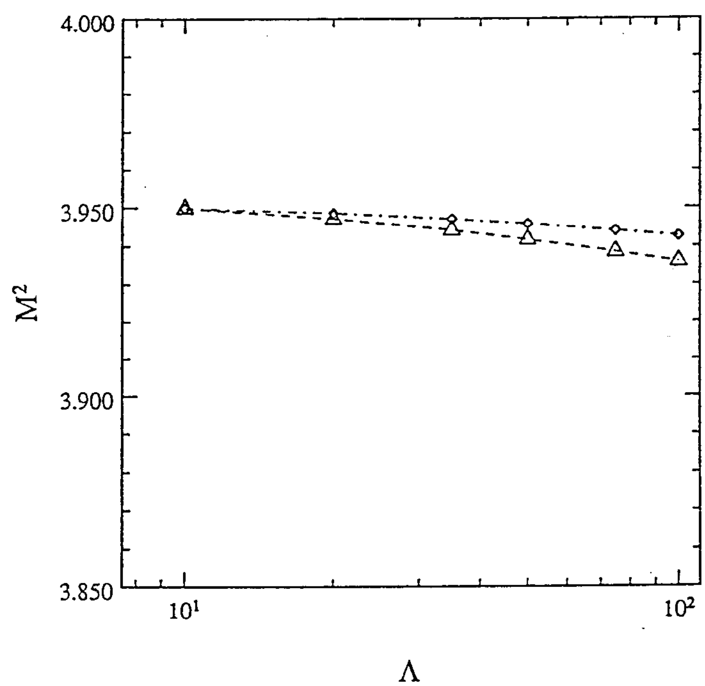

Fig. 6c

Figure 6: Bound state mass, M^2 versus cutoff Λ for $m_B = 0.25$ in units of the fermion mass with $\alpha = 1.184$. No counterterm is the dashed line with triangles, and with counterterm is the dot-dash line with diamonds. (a) $J_z = 0$ and mostly triplet, (b) $J_z = 0$ and mostly singlet, (c) $J_z = \pm 1$.

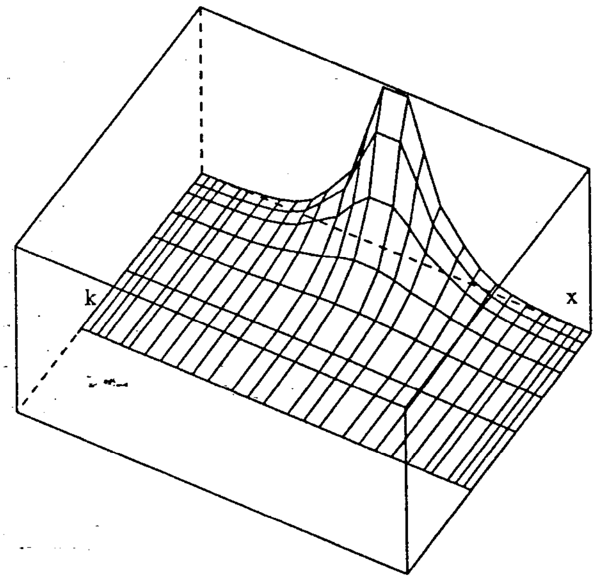

Fig. 7a

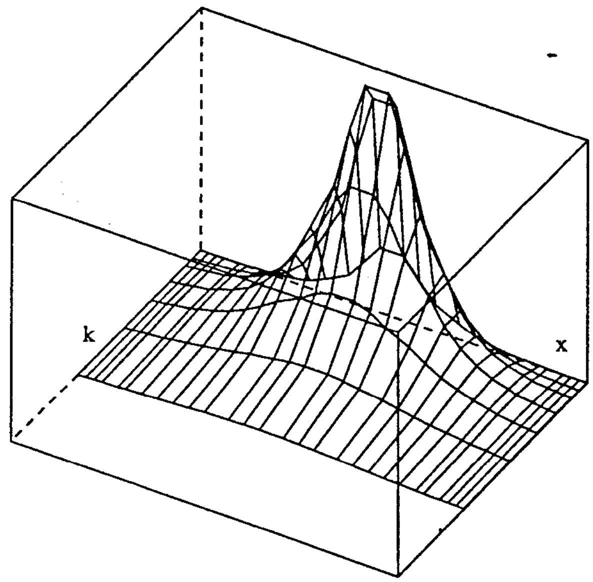

Fig. 7b

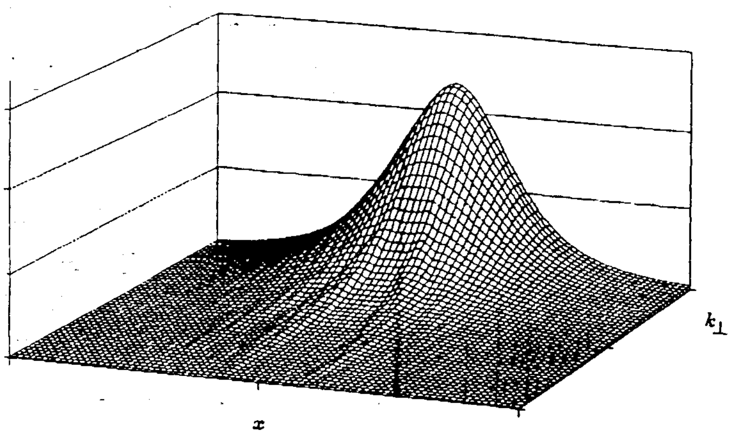

Fig. 7c

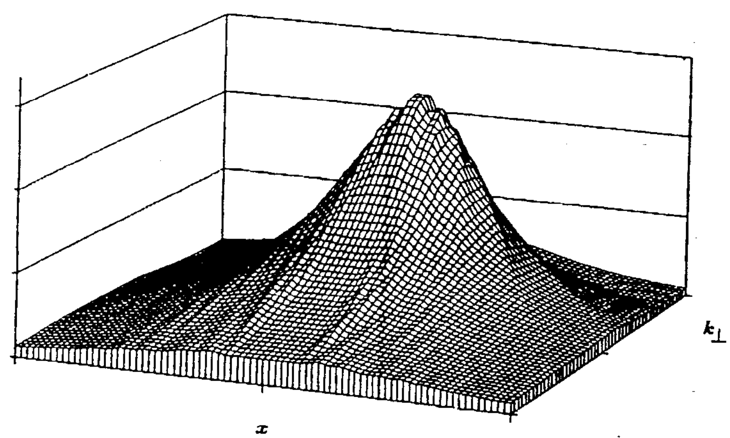

Fig. 7d

Figure 7: a) Singlet wavefunction for Yukawa ($\uparrow\downarrow$ component) $\alpha = 1.184 \quad m_B = 0.25$, b) Singlet wavefunction for Yukawa ($\uparrow\uparrow$ component) $\alpha = 1.184 \quad m_B = 0.25$, c) Singlet wavefunction for QED ($\uparrow\downarrow$ component) $\alpha = 0.03 \quad m_\gamma = 0$, d) Singlet wavefunction for QED ($\uparrow\uparrow$ component) $\alpha = 0.03 \quad m_\gamma = 0$.

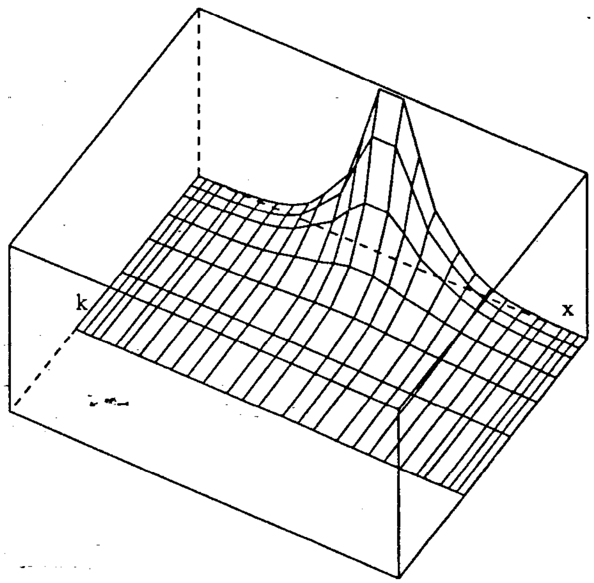

Fig. 8a

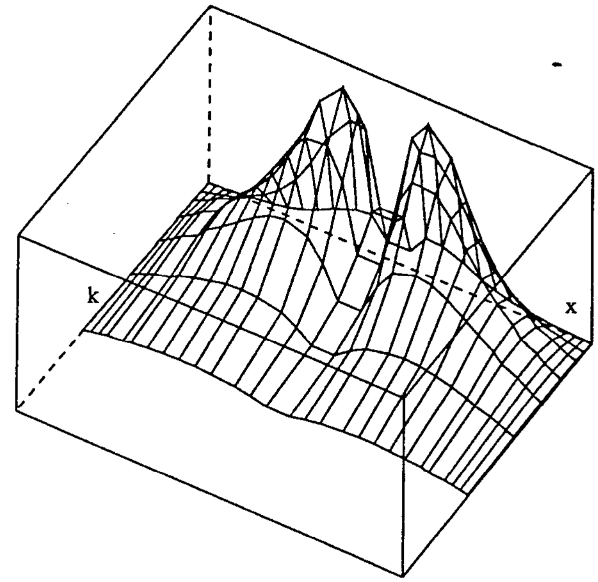

Fig. 8b

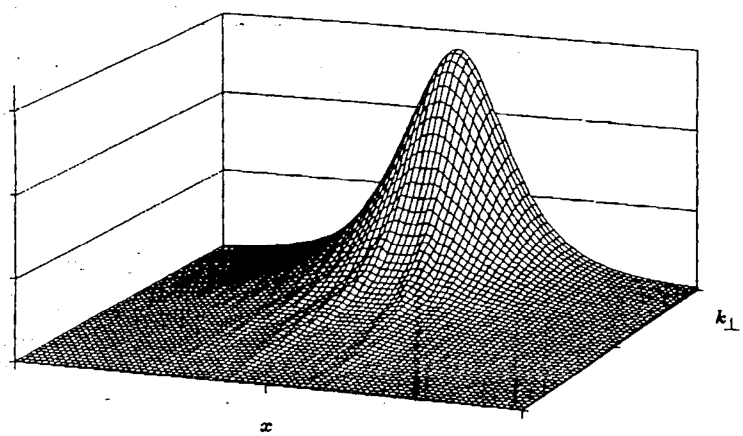

Fig. 8c

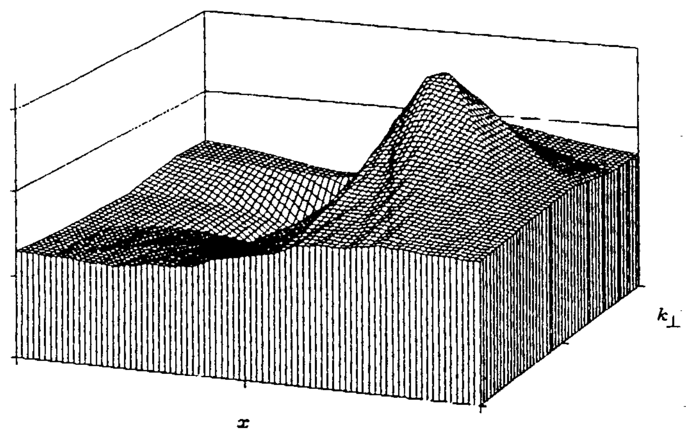

Fig. 8d

Figure 8: a) Triplet wavefunction for Yukawa ($\uparrow\uparrow$ component) $\alpha = 1.184 \quad m_B = 0.25$, b) Triplet wavefunction for Yukawa ($\uparrow\downarrow$ component) $\alpha = 1.184 \quad m_B = 0.25$ (absolute value), c) Triplet wavefunction for QED ($\uparrow\uparrow$ component) $\alpha = 0.3 \quad m_\gamma = 0$, d) Triplet wavefunction for QED ($\uparrow\downarrow$ component) $\alpha = 0.3 \quad m_\gamma = 0$.

condensate, which in turn is believed to be connected to color confinement, chiral symmetry breaking, the Goldstone pion, etc. [58]. In the standard model, the $W^{\pm}$ and Z bosons acquire their mass through the spontaneous symmetry breaking of the scalar Higgs potential. Thus an immediate question is how one can obtain non-trivial vacuum properties in a light-cone formulation of gauge field theory [59]. This problem has recently been attacked from several directions. The question of whether boundary conditions can be consistently set in light-cone quantization has been discussed by McCartor [37, 60] and Lenz [38, 61]. They have shown that for massive theories the energy and momentum derived using light-cone quantization are not only conserved, but also are equivalent to the energy and momentum one would normally write down in an equal-time theory. In the analyses of Lenz *et al.* [61] and Hornbostel [62], one traces the fate of the equal time vacuum in the limit $P_z \to \infty$ and equivalently in the limit $\theta \to \pi/2$ when rotating the evolution parameter $\tau = t\cos\theta + \frac{z}{c}\sin\theta$ into the light-cone time. Other authors [34, 59, 63] find that for theories allowing spontaneously symmetry breaking, there is a degeneracy of light-cone vacua, and the true vacuum state can differ from the perturbative vacuum through the addition of zero mode quanta with $k^+ = k^- = 0$.

Thus a number of the most important open questions are grouped under the label "zero modes". To illustrate some of the issues which arise let us consider the free Dirac theory in $1+1$ dimensions. The Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\left(i\overline{\psi}\gamma^\mu\partial_\mu\psi - i\partial_\mu\bar{\psi}\gamma^\mu\psi\right) - m\bar{\psi}\psi \ , \tag{18}$$

which gives the equations of motion for the two components of the Dirac spinor

$$i\partial_+\psi_+ = \frac{1}{2}m\psi_- \ , \tag{19}$$

$$i\partial_-\psi_- = \frac{1}{2}m\psi_+ \ . \tag{20}$$

If we quantize on the surface $x^+ = 0$ then (20) is a constraint equation for ψ_-. We thus choose ψ_+ as the field to provide the independent degrees of freedom and treat

ψ_- as a functional of ψ_+ :

$$\psi_- = -\frac{1}{2} im \int \psi_+ dx^- . \tag{21}$$

An immediate question is: what boundary conditions should we impose to specify the anti-derivative? Since that involves the possibility of adding an arbitrary function independent of x^- it is a "zero mode" question. To focus the discussion let us define a particular anti-derivative as

$$\phi\left(x^-\right) = \frac{1}{\sqrt{2\pi}} \int e^{ik_- x^-} \tilde{\phi}.\left(k_-\right) dk_- \tag{22}$$

Then we define $\underline{\int}$ by

$$\underline{\int} \phi\left(x^-\right) dx^- \equiv \frac{1}{\sqrt{2\pi}} \int e^{ik_- x^-} \tilde{\phi}\left(k_-\right) dk_- . \tag{23}$$

We write [21] as

$$\psi_- = -\frac{1}{2} im \underline{\int} \psi_+ dx^- + F(x^+) . \tag{24}$$

The problem may now be expressed as the problem of specifying F.

If the mass is not zero and we impose no periodicity conditions on ψ then we must choose F to be zero. It is easy to show that that choice leads to a light-cone theory isomorphic to the usual equal-time theory. That choice appears to most people to be the natural one. But now consider the case where m is zero. Now if we set $F = 0$ we will have $\psi_- = 0$ which is not isomorphic to the usual equal-time theory. The solution is to use the freedom in F to initialize ψ_-; that procedure is the same whether we work in the continuum or in a system made periodic in some way [60].

Finally, consider the case where the mass is nonzero but we make the system periodic by choosing anti-periodic boundary conditions for ψ_+. In that case we find that a consistent dynamical system can be obtained by setting F equal to zero. There is considerable question, however, as to whether that is the best choice for obtaining what we want: a system which, in some sense, best approximates the continuum limit [62]. Problems with the choice $F = 0$ include not only the obvious violation of parity, but the total absence of structure in the ψ_- anticommutator; it is easy to see that $\{\psi_-^*, \psi_-\}$ lacks the expected singular structure and such structure is not recovered for any value of the periodic length. Experience in models suggests that the absence of singular structure in ψ_- may be a bad defect in interacting theories [37]. A better choice may be to use the freedom in F to include the states parity-symmetric to those initialized on $x^+ = 0$. That possibility may involve starting in a basis in which neither P^+ or P^- are diagonal [62].

In interacting theories further zero mode questions arise. For instance, in gauge theories there is the problem of maintaining consistency between boundary conditions and gauge choice. That problem exists to some extent even in the equal-time formulation. For example, the gauge choice $A^1 = 0$ is inconsistent with anti-periodic boundary conditions on ψ in the case of the Schwinger model. The solution is to keep a zero mode in A^1 [64]. A similar effect is seen in light-cone gauge. In general this type of problem is more complex in light-cone quantization than in equal-time quantization due to the fact that in light-cone quantization points on the initialization surface ar e causally connected. Even for non-gauge theories similar issues arise. In Yukawa theory imposing periodicity conditions on the Fermi field requires the existence of a constrained zero mode in the Bose field. The mode is not a degree of freedom in the system but is a functional of the Fermi degrees of freedom which goes to zero as the coupling constant goes to zero.

When one considers theories in higher dimensions and considers only the continuum formulation, problems of the type we have been discussing are actually *reduced.* If one considers models made periodic by boundary conditions however, the problems persist much as in $1 + 1$ dimensions. Furthermore, the effects of an improper formulation are usually more severe in higher dimensions than in $1 + 1$. Often in $1 + 1$ the omission of a zero mode leads to a minor imperfection in the solution, but in higher dimensions such imperfections tend to pop up multiplied by divergent quantities. For example, omission of the constrained zero mode discussed above leads to the existence of a non-covariant, quadratically divergent self-mass term. Such problems can greatly complicate the proper formulation of a consistent renormalization scheme.

The Schwinger model provides an illustration of these questions which shows the potential for zero modes to resolve apparent paradoxes in the light-cone method. For example: the Schwinger model has a degenerate vacuum; one might wonder how this is possible in view of the argument outlined above that the light-cone "bare vacuum" is the only possible vacuum. The answer turns out to involve the necessity (in the simplest formulation) of keeping a zero mode in A^+. The existence of that mode leads to a gauge correction to the P^+ of the free theory. In that rather subtle way the interaction does "dress" the P^+ of the full theory and that modification brings certain states which in free theory have positive values of p^+ into degeneracy with the bare vacuum. These states become the other vacua of the theory; the bare vacuum is one of the vacua of the interacting theory. It is worth observing that even though the expectation that the bare vacuum will be the physical vacuum is not completely realized, all the vacua are much simpler in the light-cone representation than in the equal-time representation: in the light-cone representation they all involve only a finite number of bare quanta whereas in the equal-time representation they all involve an infinite number of bare quanta [36]. Some of these quanta also involve zero modes—the modes of ψ_- which must be specified along the line $x^- = 0$ [55]. This example illustrates the possibility that condensates may be more easily represented in the light-cone representation—whether this possibility will be realized in more complicated theories such as QCD we do not yet know. The Schwinger model also illustrates the possibility that $T^{\mu\nu}$ may involve integrals along surfaces other than $x^+ = 0$ [36,55]. In the case of the Schwinger model many of the expectations presented in the simplest discussions of the light-cone method are not fully realized, but the full solution exhibits enough of the expected properties that the solution is much simpler in the light-cone representation than in the equal-time representation [28,36,58].

An illuminating analysis of the influence of zero modes in QED(1+1) has been given by Werner, Heinzl and Krusche [36]. They show that although it is correct to impose the gauge condition $A^+ = 0$ on the particle sector of the Fock space, one must allow for $A^+ \neq 0$ if $k^+ = 0$. Allowing for this degree of freedom, one obtains a series of topological θ vacua on the light-cone which reproduce the known features of the massless Schwinger model including a non-zero chiral condensate. However, the effect of the infrared zero mode quanta decouples from the physics of zero charge bound states, so that the physical spectrum in one-space one-time gauge theories is independent of the choice of vacuum. The freedom in having a non-zero value for A^+ at $k^+ = 0$ can also be understood by using the gauge $\partial^+ A^+ \sim k^+ A^+ = 0$ [61, 65].

It is thus anticipated that zero mode quanta are important for understanding the light-cone vacuum for QCD in physical space-time. In particular, the non-Abelian four-point interaction term

$$H^I_{\rm LC} = -\frac{1}{2} g^2 \int d^3\underline{x} Tr\{[A^\mu, A^\nu][A^\mu, A^\nu]\} \tag{25}$$

plays a unique and an essential role, since $H^I_{\rm LC}|0\rangle \neq 0$ as long as one allows for zero mode gluon fields in the Fock space. Thus the true light-cone vacuum $|\Omega\rangle$ in QCD is not necessarily identical to the perturbative vacuum $|0\rangle$. In fact the zero mode excitations of $H^I_{\rm LC}$ produce a color-singlet gluon condensate $\langle\Omega|G_{\mu\nu}G^{\mu\nu}|\Omega\rangle \neq 0$ of the type postulated in the QCD sum rule analyses. The effect of such condensates will be to introduce "soft" insertions into the quark and gluon propagators and their effective masses $m(p^2)$, and to modify the perturbative interactions at large distances. This effectively introduces a small p^+ cutoff into the theory and to assure that the physical quantities are independent of the scale one must introduce counter terms. These counter terms should carry the essential physics associated with small p^+ that has been removed by the cutoff. Thus unlike the one-space one-time theory, the zero-mode gluon excitations can affect the color-singlet bound states. On the other hand, such zero mode corrections to vacuum cannot appear in Abelian QED(3+1) as long as a non-zero fermion mass appears in the free Hamiltonian.

6 The Prospects and Challenges

Light-cone quantization provides a relativistic and frame-independent representation of quantum field theory amenable to computer solution with present day computer technology. The method reduces the light-cone Hamiltonian to a coupled set of matrix or integral equations and has the remarkable feature of generating the complete spectrum of the theory: bound states and continuum states alike. light-cone fiel d theory is also useful for studying relativistic many-body problems in relativistic nuclear and atomic physics. In the nonrelativistic limit the theory is equivalent to the many-body Schrödinger theory. As we have reviewed here, light-cone field theory has been successfully applied to a number of field theories in one-space and one-time dimension, providing not only the bound-state spectrum of these theories, but also the wavefunctions and the first calculations for the real world of (3+1) dimension that have recently become available. A number of more limited studies have been conducted with significant success, and have uncovered interesting and challenging new problems.

Although the primary goal has been to apply light-cone methods to nonperturbative problems in QCD in physical space-time, it is important to first validate the techniques – particularly the renormalization program – in the much simpler case of QED and the Yukawa model. Quantization on the light-cone allows practical numerical solutions for obtaining its spectrum and wavefunctions at arbitrary coupling strength. In the DLCQ and LCTD methods ultraviolet and infrared regularizations have made considerable progress and are continuing to be developed.

The intrinsic advantages and outstanding problems of light-cone field theory are:

- The light-cone wavefunctions are independent of the momentum of the bound state – only relative momentum coordinates appear.

- Fermions and derivatives are treated exactly; there is no fermion doubling problem.

- Eliminating extra fermion or gluon degrees of freedom introduces non-local operators into the theory.

- The ultraviolet and infrared regulators introduced in light-cone field theory break Lorentz covariance.

- The field theoretic and renormalization properties of the light-cone theory are fundamentally different than the equal time problem because there are two independent scales present.

- One can use the exact global symmetries of the continuum Lagrangian to pre-diagonalize the Fock space sectors.

- The minimum number of physical degrees of freedom are used because of the light-cone gauge. No Gupta-Bleuler or Faddeev-Popov ghosts occur and unitarity is explicit.

- Gauge invariance is lost in a Hamiltonian theory which lead to the breaking of Lorentz covariance.

- The output is the full color-singlet spectrum of the theory, both bound states and continuum, together with their respective wavefunctions.

- The number of degrees of freedom in the representation of the light-cone Hamiltonian increases rapidly with the maximum number of particles in the Fock space.

- Many problems of ultraviolet and infrared regulation remain.

- A cutoff in the invariant mass of the Fock state introduces extra renormalization terms.

- The renormalization procedure is not completely understood in the context of non-perturbative problems.

- The vacuum in QCD is not likely to be trivial since zero-modes might mix with the free vacuum state.

- Virtually all aspects of chiral symmetry breaking, condensate and confinement are thus far not understood in light-cone quantized field theories in (3+1) dimensions.

While the challenges for light-cone field theory are substantial, the prospects for exciting new breakthroughs towards a solution of QCD, for the first time, appear within reach.

7 Acknowledgements

Much of the material presented in this article is based on collaborations with Kent Hornbostel, Andrew Tang, Peter Lepage, Thomas Eller, Dave Robertson, Matjaz Kaluža, Michael Krautgärtner, Frank Wölz, Stan Glazek, Robert Perry, Junko Shigemitsu, K.G. Wilson, and A. Harindranath. We particularly thank Alex Langnau and Matthias Burkardt for many valuable discussions and suggestions. This work was supported in part by the U.S. Department of Energy under Contract No. DE-AC03-76SF00515, and Grant No. DE-FG02-91ER40690.

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