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BROADBAND IMPEDANCES OF ACCELERATING STRUCTURES: PERTURBATION THEORY*

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A perturbation theory for broadband impedance calculations has been developed, allowing evaluation of impedances for an accelerating structure of a rather arbitrary shape. General formulas are given for the longitudinal and transverse impedances. The method is checked by calculating impedances and comparing results with those for structures previously studied. Several new results, including impedance of a taper, are presented.

KEY WORDS:

1 INTRODUCTION

The interaction of a beam with the beam environment in accelerators is usually described in terms of the coupling impedances. There is a vast literature dedicated to the impedance calculations and their properties, see for example [1]. Most quantative results have been obtained using numeric codes [2]. Analytical results in most cases are limited to derivation of integral equations which have to be solved numerically; Kirchhoff's equations [3] are an example of such equations. Although integral equations for an impedance may be useful for studing its general properties, the numeric solution of the integral equations is hardly justified, since numerical codes based on the direct solution of Maxwell equations with appropriate boundary conditions already exist and have proved to be very successful.

However, the estimates based on the integral equations may be extended to give explicit analytic results for structures of rather general form, especially in the high-frequency limit. This approach—being complimentary to numerical calculations—may be very fruitful for modern accelerators, where bunches are short compared to the beam pipe radius (or, in other words, the frequency content extends well above the beam pipe cut-off frequency). At high frequencies, it is possible to formulate a perturbation theory based on Kirchhoff's equations, analogous to the Born series in the scattering theory. Essentially, a perturbation theory of this kind was used

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in the time domain by Novokhatsky [4] in 1988 for derivation of the impedances of a step in a beam pipe, and of a pillbox cavity with attached pipes. The author [5] used a perturbation theory in 1989 for more general structures in the frequency domain. The present paper is a revised version of an unpublished talk given at a KEK, Japan workshop in September 1990. The perturbative method is described in a systematic way, and is applied to study more cases—particularly, to study the impedance of a taper. A axial symmetry is implied in most cases, unless it is stated otherwise, although the method also may be applied to study impedances of structures without axial symmetry.

The most difficult problem is formulation of a criterion of applicability of the method. The rough estimate of the parameter of expansion can be obtained by comparing sequential terms of the perturbation series, as it is usually done for Born's series. It is naturally to expect that for small loss parameters, the parameter of expansion is small. However, even a small parameter of expansion does not guarantee that Born's series are convergent (usually, they are not). We choose, therefore, another approach, comparing results of the method with numerical and analytical results known previously, and judging applicability by this comparison.

This paper is organized in the following way. In the beginning, we review the basic definitions. The idea of the method then is demonstrated, using a simple example from electrostatics for which the answer is well known. This method then reproduces Bethe's result [6] for a field distorted by a hole in a conductive plane. The method is extended to electrodynamics, using Kirchhoff's integral equations, and applied to get Kurennoy's results [7] for the longitudinal and transverse impedances of a hole in a straight pipe. In the next section the general expressions for the longitudinal and transverse impedances are derived, Eqs. (65, 72), in the lowest nontrivial iteration for a axially symmetric beam pipe with arbitrary variation of the radius. These formulas are used for particular geometries: a shallow cavity, a pill-box cavity with attached tubes, an array of such cavities, an abrupt variation of the pipe radius, a collimator, and a taper. This is followed by examples for the transverse impedance. The results are summarized in the conclusion.

2 BASIC DEFINITIONS

In the ultra-relativistic case, particles interact only through the EM fields excited in accelerating structures. The interaction may be described in terms of the wake fields in the time domain, or in terms of their Fourier components in the frequency domain. The longitudinal wake function $W_l(s)$ is related to the energy loss ΔE of a particle with the charge e, following a point-like bunch with the charge q at the distance $s > 0$

$$\Delta E = -eqW_l(s) \;=\; e\int dz\; E^z\;[z, r, t \;=\; (z+s)/v]\;, \tag{1}$$

where E^z is the longitudinal component of the electric field of the leading bunch.The longitudinal impedance then is defined as the synchronous harmonic of E_z:

$$Z_l(k) = -(1/q)\int dz E^z_\omega(z,r)\ \exp\{-ikz/\beta\}$$
$$W_l(s) = \int (d\omega/2\pi)\ Z_l(k)\ \exp\{-iks/\beta\}\ . \qquad (2)$$

The loss factor is given by the convolution of the Fourier component of the bunch density with the wake $W_l(s)$. For a Gaussian bunch, the loss factor is

$$k_l = 2\int_{-\infty}^{\infty} dk\ \exp\{-k^2\sigma^2\}\ \mathrm{Re}\,(Z_l(k)/Z_0)\ , \qquad (3)$$

where $Z_0 = 4\pi/c = 377\ \Omega$ and σ is the rms bunch length.

The transverse wake potential $W_\perp(s)$ is related to the transverse momentum $\Delta p_\perp = eqr_0 W_\perp(s)$ experienced by the trailing particle, due to the field excited by the leading bunch moving parallel to the z-axes with the offset r_0. The transverse impedance $Z_\perp(k)$ is proportional to the synchronous harmonic of the transverse force and is a frequency harmonic of the transverse wake $W_\perp(s)$:

$$Z_\perp(k) = -(i/qr_0)\int_{-\infty}^{\infty} dz\ \exp\{-ikz/\beta\}\ [\vec{E}_\omega + (\vec{v}/c)\times B_\omega]_\perp\ ,$$
$$W_\perp(s) = i\int (d\omega/2\pi)\ Z_\perp(k)\ \exp\{-iks/\beta\}\ . \qquad (4)$$

According to Panofsky-Wenzel theorem, $Z_\perp$ is related to the longitudinal impedance:

$$kZ_\perp = (1/r_0)\ (\partial Z_l/\partial r)\ . \qquad (5)$$

This follows from the Maxwell equation $\vec{\nabla}\times\vec{E} = ik\vec{H}$ and definitions (2,4).

Note also the following analytic properties of the impedance:

$$Z_l(-k^*) = Z_l^*(k)\ , \qquad Z_\perp(-k^*) = -Z_\perp^*(k)\ .$$

Therefore, the problem of impedance calculations is reduced to the problem of defining EM fields excited by a bunch in an accelerating structure.

The current density of a particle in free space with a charge e moving parallel to the z-axis, with offset r_0 in the direction of the x-axis, has Fourier components

$$\vec{j}^p_\omega = \int dt\ \exp\{i\omega t\}\ \vec{j}^p = e\hat{z}\ \delta(\phi)\ [\delta(r-r_0)/r]\ \exp\{ikz/\beta\}\ . \qquad (6)$$

Here $\hat{z}$ is a unit vector along the z-axis,and azimuthal angle $\phi = 0$ on the x-axis. This current produces EM fields given in terms of the potential ϕ_ω:

$$E_z = [k^2 + (\partial^2/\partial z^2)]\,\phi_\omega\,, \quad E_\phi = (1/r)\,(\partial^2\phi_\omega/\partial\phi\partial z)\,, \quad E_r = (\partial^2\phi_\omega/\partial r\partial z)\,,$$

$$H_z = 0\,, \quad H_\phi = ik\,(\partial\phi_\omega/\partial r)\,, \quad H_r = -(ik/r)\,(\partial\phi_\omega/\partial\phi)\,, \tag{7}$$

where $k = \omega/c$ and, for $r > r_0$, the potential is

$$\begin{aligned}\phi_\omega &= -(2ie/kc)\;\exp\{ikz/\beta\}\sum\exp\{im\phi\}\\ &\times\left\{\delta_{m0}\ln(kr/2\gamma) - [(1-\delta_{m0})/2|m|]\;(r_0/r)^{|m|}\right\}\,.\end{aligned}\tag{8}$$

The EM field of an ultrarelativistic particle moving inside a straight, circular, cylindrical beam pipe with an ideally conductive wall is given by Eq. (7), where

$$\begin{aligned}\phi_\omega &= (2ie/kc)\;\exp\{ikz/\beta\}\sum\exp\{im\phi\}\\ &\times\left\{\delta_{m0}\ln(a/r) + [(1-\delta_{m0})/2|m|]\;\left[(r_0/r)^{|m|} - (rr_0/a^2)^{|m|}\right]\right\}\,.\end{aligned}\tag{9}$$

The field is zero outside the pipe and in the walls.

The field in the pipe for the monopole mode $m = 0$,

$$E_r = (2e/\beta cr)\;\exp\{ikz/\beta\}\,, \qquad H_\phi = \beta E_r\,, \tag{10}$$

is, with accuracy $o(1\gamma^2)$, the same as the field of a particle in free space. Therefore, in the limit $\gamma \to \infty$ there is no interaction of a particle with the beam pipe and the impedances are zero. For the longitudinal impedance it follows also from the fact that the radial component of the Pointing vector in this case is zero. For the dipole mode $|m| = 1$ the field in the pipe is

$$\begin{aligned}E_z &= H_z\,,\\ E_\phi &= -H_r = E_1\sin(\phi)\,[1-(r^2/a^2)]\,,\\ E_r &= H_\phi = E_1\cos(\phi)\,[1+(r^2/a^2)]\,,\end{aligned}\tag{11}$$

where

$$E_1 = (2er_0/cr^2)\;\exp\{ikz/\beta\}\,, \tag{12}$$

giving zero transverse impedance, because the electric and magnetic components of the Lorenz force cancel each other with $(1/\gamma)^2$ accuracy.

This is not the case when the beam pipe geometry varies with z. In the following we discuss a method to calculate impedances for rather general variation of the beam pipe cross-section. For simplicity, however, we assume axial symmetry, but leave the beam pipe radius $a(z)$ an arbitrary function of z.

3 ILLUSTRATION: THE METHOD IN THE ELECTROSTATIC

The problem of calculating EM fields excited by a bunch in an accelerating structure of arbitrary shape is rather complicated. However, the fields are usually small compared to the field of the bunch. It is naturally to look for a perturbation method that will allow calculation of the induced field by iteration.

Let us start with an example that illustrates this approach. Consider a well known electrostatic problem: find the field of a point-like charge e placed at distance $z = a$ from an ideal conducting x, y plane. The field potential for $z > 0$ is a superposition of the potential $\phi_{\rm ext}$ of a charge in free space,

$$\phi_{\rm ext}(r,z) \;=\; e/[(z-a)^2+r^2]^{1/2}\;, \qquad r^2 \;=\; x^2+y^2\;, \tag{13}$$

and the potential of the image charge $-e$ at $z = -a$:

$$\phi_0(r,z) \;=\; \{e/[(z-a)^2+r^2]^{1/2}\} \;-\; \{e/[(z+a)^2+r^2]^{1/2}\}\;. \tag{14}$$

This result may be obtained using Green theorem [3], which defines the field within a volume in terms of the field on the surface encompassing the volume:

$$\phi(\vec{R}) \;=\; \phi_{\rm ext}(\vec{R}) + \int (d\vec{S}'/4\pi)\; [G(\vec{R},\vec{R}')\;\vec{\nabla}'\;\phi(\vec{R}') - \phi(\vec{R}')\;\nabla'\;G(\vec{R},\vec{R}')]\;. \tag{15}$$

The first term here, $\phi_{\rm ext}$, is the field of a charge within the volume if there is any. The Green function of the Laplace equation is simply $G(\vec{R},\vec{R}') = 1/|(\vec{R}-\vec{R}')|$. The surface element $d\vec{S} = \vec{n}dS$, where $\vec{n}$ is a unit normal vector pointed to the outside of the volume. The vector $\vec{R} = (z,\vec{r})$, where the 2-D tangential $\vec{r}$ is orthogonal to $\vec{n}$. Derivatives on the surface are understood as the limit value of the derivatives calculated at the inside the volume.

The boundary condition on the conductive wall is $\phi = 0$. Choose the surface of integration on the metallic boundaries, and the last term in (15) vanishes. The remaining integral may be interpreted as the field of the induced surface charge, with the surface density proportional to $\vec{n}\vec{\nabla}\phi$ taken on the surface $z = 0$.

Solve (15) by iterations: $\phi = \phi^{(0)} + \phi^{(1)} + \ldots$. In the zeros approximation, $\phi^{(0)} = \phi_{\rm ext}$. In the nth approximation

$$\phi^{(n)}(\vec{R}) \;=\; \int (d\vec{S}'/4\pi)\; G\vec{\nabla}\phi^{(n-1)}\,(\vec{r},0)\;, \qquad n = 1,2,\ldots\;, \tag{16}$$

starting with $\phi^{(0)} = \phi_{\rm ext}$. The integral for $n = 1$ can be calculated easily using the integral representation of the Green function:

$$G(\vec{R}) \;=\; [4\pi/(2\pi)^3)\int (d\vec{q}/q^2)\;\exp\{i\vec{q}\vec{R}\}\;. \tag{17}$$

This gives

$$\phi^{(1)}(\vec{R}) \;=\; -(1/2)\;e/[(z+a)^2+r^2]^{1/2}\;. \tag{18}$$

In the next iteration,

$$\phi^{(2)}(\vec{R}) = -(1/2)\,\phi^{(1)} ,$$

and so forth.

The series converges

$$\phi(\vec{R}) = \phi_{\rm ext}(\vec{R}) - e/[(z+a)^2 + r^2]^{1/2}\,[(1/2) + (1/2)^2 + (1/2)^3 \ldots] , \qquad (19)$$

giving the correct answer, (14).

Note that although the final result satisfies the boundary condition, the result of any finite number of iterations does not. Hence, the solution of the Laplace equation is exact for each iteration, but the boundary conditions are satisfied only approximately.

The same method applied to a magnetostatic problem displays similar properties: the series converge giving the right answer. Here, also, boundary conditions for any finite number of iterations are satisfied only approximately.

Let us make the problem a little more complex by adding a small round hole with radius b in the conducting plane. Both the the center of the hole and the charge are at $x = y = 0$. The field is described by the potential Φ,

$$\Phi = \phi_0 + \phi , \qquad \text{for} \quad z > 0 ,$$
$$\Phi = \psi , \qquad \text{for} \quad z < 0 ,$$

where ϕ_0 is the solution of the problem without the hole given by (14) for $z > 0$, and $\phi_0 \equiv 0$ for $z < 0$. The functions ϕ, ψ are to be found by perturbations.

With the surface of integration in the plane $z = 0$, Green theorem takes the form

$$\phi(\vec{R}) = -\int (d\vec{S}'/4\pi)\,[G(\vec{R},\vec{R}')\,(\partial\phi(\vec{R}')/\partial z')]_{z'=0}$$
$$- (\partial/\partial z)\int_h (d\vec{S}'/4\pi)\,[G(\vec{R},\vec{R}')\,\phi(\vec{r}',z')]_{z'=0} . \qquad (20)$$

$$\psi(\vec{R}) = \int (d\vec{S}'/4\pi)\,G(\vec{R},\vec{R}')\,(\partial\psi(\vec{R}')/dz') + (\partial/dz)\int_h (d\vec{S}'/4\pi)\,G(\vec{R},\vec{R}')\,\psi . \qquad (21)$$

The last integrals in (20,21) are over the opening of the hole where, generally, $\Phi \neq 0$. These integrals can be considered as perturbation. The equations are homogeneous and have the trivial solution $\phi = \psi = 0$. However, this solution is not acceptable, because it does not satisfy continuity of the derivative $\partial\Phi/\partial z$ at the opening.

The conditions of continuity on the hole are

$$\psi = \phi , \qquad (\partial\psi/\partial z) = (\partial\phi_0/\partial z) + (\partial\phi/\partial z) . \qquad (22)$$

The solution of Eqs. (20,21) is given by series

$$\phi = \sum_1^\infty \phi_n , \qquad \psi = \sum_1^\infty \psi_n . \qquad (23)$$

The series for ϕ and ψ are bootstraped by conditions (22) in such a way that ψ_n are driven by the derivatives $\partial\phi_n/\partial z$ and ϕ_n are driven by ψ_n at the hole. The first order $n = 1$ correction is

$$\begin{aligned} \psi_1(\vec{R}) &= \int_h (d\vec{S}'/4\pi)\, G(\vec{R},\vec{R}')\, [\partial\phi_0(\vec{R}')/\partial z'] \,, \\ \phi_1(\vec{R}) &= -(\partial/\partial z) \int_h (d\vec{S}'/4\pi)\, \psi_1(z')\, G(\vec{R},\vec{R}') \,, \end{aligned} \tag{24}$$

and for $n \geq 2$,

$$\begin{aligned} \psi_n(\vec{R}) &= \int_h (d\vec{S}'/4\pi)\, \{G(\vec{R},\vec{R}')\, [\partial\phi_{n-1}(\vec{R}')/\partial z'] + \psi_{n-1}\, (\partial/\partial z)\, G(\vec{R},\vec{R}')]\} \\ &+ \int_M G(\vec{R},\vec{R}')\, [\partial\psi_{n-1}(\vec{R}')/\partial z'] \,, \\ \phi_n(\vec{R}) &= -\int_h (d\vec{S}'/4\pi)\, \{G(\vec{R},\vec{R}')\, [\partial\phi_{n-1}(\vec{R}')/\partial z'] - \psi_n(z')\, [\partial G(\vec{R},\vec{R}')/\partial z']\} \\ &- \int_M (d\vec{S}'/4\pi)\, G(\vec{R},\vec{R}')\, [\partial\phi_{n-1}(\vec{R}')/\partial z'] \,. \end{aligned} \tag{25}$$

Here indexes (h) and (M) mean integration over the hole and the metallic surfaces, correspondingly.

In the first approximation, we consider Eq. (24), with ϕ_0 given by Eq. (14). For a small hole $b \ll a$,

$$E_0(0) = (\partial\phi_0/\partial z)\,|_{z=0} = (2e/a^2) \,, \tag{26}$$

and

$$\psi_1(r,z) = (eb/a^2) \int_0^\infty (dq/q)\, J_1(qb)\, J_1(qr)\, \exp\{-q|z|\} \,. \tag{27}$$

Consider the first order correction ϕ_1 to the potential due to the hole for $z > 0$ at large distances $R \gg b$. Expanding $G(\vec{R},\vec{R}')$, we obtain

$$\begin{aligned} \phi_1(\vec{R}) &= (\partial/\partial z)\, (eb/R)\, (\partial/\partial a) \\ &\times \int_h (dS'/4\pi a)\, [1 + (\vec{r}\vec{r}'/R^2) + \ldots] \int_0^\infty (dq/q)\, J_1(qb)\, J_0(qr') \,. \end{aligned}$$

The term $\vec{r}\vec{r}'/R^2$ does not contribute. The rest of the integral is easy to calculate. This gives the field of an electric dipole d:

$$\phi_1(\vec{R}) = -(zd/R^3) \,, \qquad d = (2/3\pi)\, (eb^3/a^2) \,, \tag{28}$$

which is exactly the Bethe [6] result $d = E_0 b^3/3\pi$, where the unperturbed field E_0 at the opening is given by Eq. (26).

Note that the boundary condition for $z < 0$ is not satisfied and has to be corrected in the next approximation.

4 KIRCHOFF'S EQUATION—IMPEDANCE OF A HOLE

We use the same method of iteration to solve the exact integral equation of the electrodynamics. Kirchhoff's integral equations [3] are the analog of the Green theorem considered above:

$$\begin{aligned}\vec{E}(\vec{R}) &= \vec{E}_b - \int dS' \left[(\vec{n}'\vec{E})(\vec{\nabla}'G_k) + ikG_k(\vec{n}' \times \vec{H})\right] \\ &\quad + \int dS' \left[\vec{n}'(\vec{E}\vec{\nabla}')G_k - \vec{E}(\vec{n}'\vec{\nabla}')G_k\right] , \\ \vec{H}(\vec{R}) &= \vec{H}_b + \int dS' \left[\vec{n}'(\vec{H}\vec{\nabla}')G_k - \vec{H}(\vec{n}'\vec{\nabla}')G_k\right] \\ &\quad - \int dS' \left[(\vec{n}'\vec{H})(\vec{\nabla}'G_k) + ik(\vec{n}' \times \vec{E})G_k\right] . \end{aligned} \tag{29}$$

Here $\vec{n}$ is a unit vector normal to the surface pointed outside of the volume, G_k is the Green function of the wave equation

$$(\Delta + k^2)G_k(\vec{R}, \vec{R}') = -\delta(\vec{R} - \vec{R}') ; \tag{30}$$

$\vec{E}_b, \vec{H}_b$ are the fields excited by the beam, with the charge density ρ and current density $\vec{j}$ within the volume in the consideration:

$$\begin{aligned}\vec{E}_b &= 4\pi \int dV' \left[(\vec{\nabla}'G_k)\rho + i(k/c)\, G_k\vec{j}(\vec{R}')\right] , \\ \vec{H}_b &= (4\pi/c) \int dV' \vec{j} \times (\vec{\nabla}'G_k) . \end{aligned} \tag{31}$$

The Green function $G_k(R) = \exp\{ikR\}/(4\pi R)$ has the integral representation

$$G_k(\vec{R}, \vec{R}') = \int \{d\vec{q}/[(2\pi)^3]\}\, (\exp\{i\vec{q}(\vec{R} - \vec{R}')\})/(q^2 + k^2) , \tag{32}$$

or, in the cylindrical coordinates (r, ϕ, z) [18],

$$G_k(\vec{R}, \vec{R}') = (i/8\pi) \sum_m \exp\{im(\phi - \phi')\} \int dp \exp\{ip(z - z')\}\, G_{m,p}(r, r') , \tag{33}$$

where

$$G_{m,p}(r, r') = J_m(\Omega r)\, H_m^{(1)}(\Omega r')\, \theta(r' - r) + J_m(\Omega r')\, H_m^{(1)}(\Omega r)\, \theta(r - r') . \tag{34}$$

Here $J_m, H_m^{(1)}$ are the Bessel and the Hankel's functions of the first kind, $\theta(r)$ is the step function, and

$$\Omega = \left(k^2 - p^2\right)^{1/2}, \qquad \Omega(-p) = \Omega(p), \qquad \Omega(-k^*) = -\Omega^*(k). \tag{35}$$

Note that $G_{m,p}(k) = G^*_{-m,-p}(-k^*)$.

To illustrate the perturbation method based on Kirchhoff's equation, consider the impedance of a slot in a straight beam pipe.

It is easy to see that the field that is given by Eq. (10) inside the pipe and is zero outside the pipe satisfies Kirchhoff's equations (25).

Consider now a straight pipe with a small slot in it. Similarly to what has been done in the electrostatic case, we first calculate the field outside the pipe, due to the opening. In the first approximation, Eqs. (29) and (10) give the field outside of the pipe, $r > a$

$$\begin{aligned} \vec{E}(\vec{R}) &= -(2e/ca)\, \vec{\nabla}_\perp \int_h dS' \left[G_k(\vec{R},\vec{R}')\exp\{ikz'\}\right], \\ \vec{H}(\vec{R}) &= -(2e/ca)\, (\hat{z}\times\vec{\nabla}) \int_h dS' \exp\{ikz'\}\, G_k(\vec{R},\vec{R}'). \end{aligned} \tag{36}$$

The integrals here are over the surface area of the opening. The perturbation of the field inside the beam pipe is then given by the last integrals in (29), which vanished in the straight pipe due to boundary conditions. Requiring the tangential component of the electric field and the normal component of the magnetic field at the opening to be continuous, we define them from Eqs. (36) and obtain, in particular, E_z inside the pipe:

$$\begin{aligned} E_z(\vec{R}) &= (2e/ca) \int_h dS' \left(\left[\partial G_k(\vec{R},\vec{R}')/(\partial r')\right] \right. \\ &\times \left. \int_h dS''(\partial/\partial z'')\, G_k(\vec{R}',\vec{R}'')\ \exp\{ikz''\}\right)_{r'=r''=a}. \end{aligned} \tag{37}$$

The longitudinal impedance is defined by Eqs. (2),(37):

$$\begin{aligned} Z_l(k) &= -\,(2/ca) \int dz \exp\{-ikz\} \int_h dS' \left(\left[\partial G_k(\vec{R},\vec{R}')/(\partial r')\right] \right. \\ &\times \left. \int_h dS''\ (\partial/\partial z'')\ G_k(\vec{R}',\vec{R}'')\ \exp\{ikz''\} \right)_{r'=r''=a}. \end{aligned} \tag{38}$$

The integral over dz can be calculated using (33):

$$\begin{aligned} &\int dz \exp\{-ikz\}\ G_k(\vec{R},\vec{R}') = \\ &\quad (i/4)\ \exp\{-ikz'/\beta\} \sum_m \exp\{im(\phi-\phi')\}\ G_{m,k/\beta}. \end{aligned} \tag{39}$$

For $r' > r$,

$$G_{m,k/\beta}(r,r') = (2i/\pi)\,\delta_{m,0}\,[\ln(kr'/2\gamma) + C] + [(1-\delta_{m,0})/(i\pi|m|)]\ (r/r')^{|m|}\ . \qquad (40)$$

where $\delta_{m,n} = 1$ for $m = n$ and zero otherwise, and $C = 0.5772\ldots$.

The longitudinal impedance is dominated by the contribution of the monopole $m = 0$ mode. For a slot with length L and width $w = a\Delta\phi$, Eq. (38) gives

$$Z_l(k) = [1/(2\pi)^2 c]\int_{-L/2}^{L/2} dz'd\phi d\phi'\ \exp\{-ikz'\} \times \left[\exp\{ikz + ik\,[(z-z')^2+\Delta^2]^{1/2}\}/\{[(z-z')^2+\Delta^2]^{1/2}\}\right]_{z=-L/2}^{L/2} ,$$

where $\Delta^2 = 4a^2\sin^2[(\phi-\phi')/2]$. The integral over dz' can be written in the form

$$\int_{-L/2}^{L/2} dz'\exp\{-ikz'\} \times\left[\exp\{ikz + ik\,[(z-z')^2+\Delta^2]^{1/2}\}\,/\,[(z-z')^2+\Delta^2]^{1/2}\right]_{z=-L/2}^{L/2}$$

$$= 2i\int_0^L [dx\sin(kx)/(x^2+\Delta^2)^{1/2}]\ \exp\{ik(x^2+\Delta^2)^{1/2}\}\ .$$

The integrand here is finite at $x \to 0$. Hence, for a narrow slot $w \ll L$, Δ can be omitted. This gives the imaginary part of the impedance

$$ImZ_l(k) = \{w^2/[(2\pi)^2a^2c]\}\int_0^{2kL}(dx/x)\ \sin(x)\ . \qquad (41)$$

If the slot is short $kL \ll 1$, then

$$ImZ_l(k) = Z_0\ \{kLw^2/[(2\pi)^3\,a^2]\} \qquad (42)$$

reproduces the Kurennoy's result [7].

The impedance increases with L for short slots $kL \ll 1$, and goes to a constant for $kL \gg 1$:

$$ImZ_l(k) = \{Z_0\pi w^2/[4(2\pi)^3\,a^2]\}\ . \qquad (43)$$

This result may be confusing because the zero impedance may be expected for the infinitely long slot. It should be remembered, however, that result (43) implies that the slot is long but finite, so that the field outside the beam pipe is zero at infinity.

Dependence of $Z_l(k)$ on the length of a slot is shown in Fig. 1.

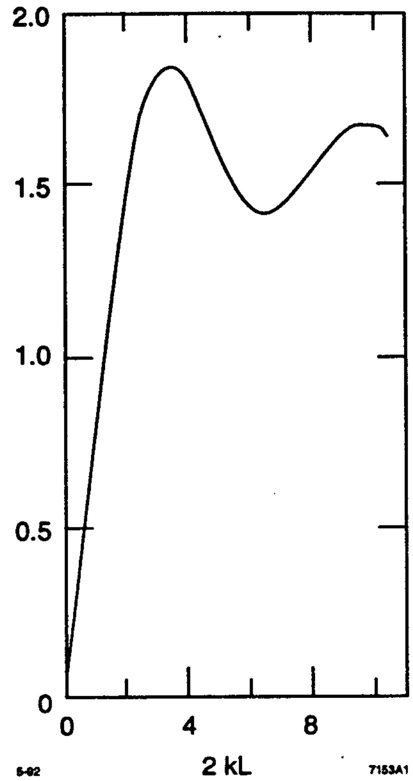

FIGURE 1: Dependence of the longitudinal impedance of a slot on the slot length, the integral in Eq. (41) as a function of $2kL$.

The real part of the impedance describes energy losses, and is very small [8]. Calculation of the real part of the impedance requires the next iteration, because the energy loss is related to the radiation through the hole, which cannot be described by the field (36). These calculations have to be corrected by the field of induced charges on the outer surface of the beam pipe, which can be obtained in the next approximation.

Similarly, consider the longitudinal dipole impedance of a narrow $w \ll a$ slot in a perfectly conductive pipe. Omitting intermediate calculations, we obtain from (5) the transverse impedance

$$Z_{\perp}(k) = \{iw^2/[(\pi)^2 kca^4]\} \int_0^{kL} (dx/x) \sin(x) \exp\{ix\} .$$

If $kL \ll 1$, the impedance does not depend on frequency [7]

$$Z_{\perp}(k) = i \{2Z_0 w^2 L/[(2\pi)^3 a^4]\} . \tag{44}$$

If the bunch is long $\sigma > L$, the impedance for all frequencies within the bunch spectrum is described by this formula. The wake potential (4) in this case is proportional to $\delta(s)$, and describes a transverse kick. The wake function depends on s in the same way as the bunch charge density.

5 LONGITUDINAL IMPEDANCE FOR A BEAM PIPE WITH ARBITRARY RADIUS VARIATION

If a volume is bounded by metallic walls, it is useful to perform the integration at these walls where the normal component of magnetic field and tangential components of the electric field are zero. In what follows we consider impedances due to variation in the beam pipe radius with z. In this case, the last integrals in Eqs. (29) vanish due to the boundary conditions on the metallic walls. Kirchhoff's equations take the form

$$\begin{aligned} \vec{E}(\vec{R}) &= \vec{E}_b - \int dS' \left[(\vec{n}'\vec{E})(\vec{\nabla}'G_k) + ikG_k(\vec{n}' \times \vec{H})\right] , \\ \vec{H}(\vec{R}) &= \vec{H}_b + \vec{\nabla} \times \int dS'(\vec{H} \times \vec{n}')\, G_k . \end{aligned} \tag{45}$$

Equation (45) can be solved by iteration. It is worth noting again that, as in the example from electrostatics, we iterate equations (45) where the exact boundary conditions are already imposed. If the equations were solved exactly, they would give the fields, satisfying the boundary conditions. However, the solution obtained with a finite number of iterations only approximately satisfies the boundary conditions.

Equation (45) can be written in terms of surface charge density σ and surface current density $\vec{I}$:

$$\begin{aligned} \vec{E} &= \vec{E}_b + 4\pi \int dS' \left[\sigma(z')(\vec{\nabla}'G_k)_{r'=a(z')} + i(k/c)\, G_k\vec{I}\right] , \\ \vec{H} &= \vec{H}_b + (4\pi/c)\, \vec{\nabla} \times \int dS'\vec{I}G , \end{aligned} \tag{46}$$

where $\sigma, \vec{I}$ are given by the fields at the boundary:

$$4\pi\sigma = -(\vec{E}\vec{n}) , \qquad (4\pi/c)\vec{I} = -(\vec{n} \times \vec{H}) . \tag{47}$$

The induced charge and current density $\rho, \vec{j}$ are related to σ, $\vec{I}$: $\rho = \sigma\delta(\eta)$, $\vec{j} = \vec{I}\delta(\eta)$ where η is the distance from the surface along the normal vector $\vec{n}(z)$,

$$\eta = [r - a(z)] \cos\alpha , \qquad \tan\alpha = a'(z) = [da(z)/dz] .$$

Note that $\vec{n}\vec{I} = 0$.

The continuity equation $ikc\rho = \vec{\nabla}\vec{j}$, which follows from Maxwell equations, relates σ and $\vec{I}$, giving

$$I_r = a'(z) I_z$$

and

$$ikc\sigma = [(\cos\alpha)/a]\,(\partial/\partial z)\,(aI_z/\cos\alpha) + (1/a)\,(\partial/\partial\phi)\,I_\phi\ . \tag{48}$$

Equations (46) and (47) can be solved by iteration.

We give explicit expression for two azimuthal modes of the longitudinal impedance:

1. the monopole mode $m = 0$, which dominates the longitudinal impedance, and
2. the dipole mode $m = 1$, which is related to the transverse impedance by the Panofsky-Wenzel theorem.

Only the induced part of the field E_z is relevant for the impedance calculations. For a circular cylindrical beam pipe with the radius $r = a(z)$, Eq. (46) gives

$$\begin{aligned} E_z - E_z^b &= Z_0 \int [dS'/a(z')] \\ &\times \{(i/k)\,[\cos\alpha(z')\,(\partial/\partial z')\,(aI_z/\cos\alpha) \\ &+ (\partial I_{\phi'}/\partial\phi')]\,(\partial G_k/\partial z) + ika(z')\,G_k I_z\}\ . \end{aligned} \tag{49}$$

Here

$$dS' = [a(z')dz'd\phi']/\cos\alpha\ .$$

Using the definition in Eq. (2) with Eq. (45) for E_z, we obtain

$$\begin{aligned} Z_l(k) &= -(iZ_0/ek)\int [dS'/a(z')] \\ &\times [\cos\alpha(z')\,(\partial/\partial z')\,(aI_z/\cos\alpha) + (\partial I_{\phi'}/\partial\phi')]\,[\exp\{-ikz\}\,G_k]_{z=-\infty}^{\infty} \\ &- (Z_0/e)\int [dS'/a(z')]\int dz\,\exp\{-ikz\}\,G_k\,\{ika(z')I_z(z') - (1/\beta) \\ &\times [\cos\alpha(z')\,(\partial/\partial z')\,(aI_z/\cos\alpha) + (\partial I_{\phi'}/\partial\phi')]\}\ . \end{aligned} \tag{50}$$

The first term would not vanish only if $z' \simeq z \to \infty$. However, at infinity the current is a surface current in a straight pipe

$$\vec{I} = -\hat{z}\,(e/2\pi a)\,\exp\{ikz/\beta\}\ , \qquad |z| \to \infty\ .$$

Hence, $\partial aI_z/\partial z = 0$ and the first term can be omitted.

The integration over dz can be performed using Eqs. (39) and (36); then integration by parts over dz' gives, for the $m=0$ and $m=1$ modes,

$$\begin{aligned} Z_l^{(0)}(k) &= (Z_0/2\pi)\ \ln(a_\infty/a_{-\infty}) \\ &- (2ik/ec\gamma^2)\int dS'[\ln\ (ka(z')/2\gamma)+C]\ I_z(z')\ \exp\{-ikz'/\beta\} \\ &+ (2/ec)\int dS'[a'(z')/a(z')]\ I_z(z')\ \exp\{-ikz'/\beta\}\ , \end{aligned} \qquad (51)$$

$$\begin{aligned} Z_l^{(1)}(k) &= -(Z_0 rr_0/2\pi)\ [(1/a_\infty^2)\ -\ (1/a_{-\infty}^2)]\ \cos\phi \\ &+ [(2ikr)(ec\gamma^2)]\int [dS'/a(z')]\ \exp\{-ikz'/\beta\}\ \cos(\phi-\phi')\ I_z \qquad (52) \\ &+ (2r/ec)\int [dS'/a^2(z')]\ \exp\{-ikz'/\beta\}\cos(\phi-\phi')\ [a'I_z+(\partial I_{\phi'}/\partial\phi')]\ . \end{aligned}$$

5.1 Zero approximation

In the zero approximation, the surface current is defined by Eq. (47) with the field (10,11) of a bunch. For calculation of the longitudinal impedance the offset, r_0 can be put to zero. Then $I_\phi^{(0)}=0$, the two other components of $I^{(0)}$ are independent of azimuth, and

$$Z_0\bar{I}_z^{(0)} = -(2e/ca(z))\ \exp\{ikz/\beta\}\ \cos\alpha\ . \qquad (53)$$

The impedance (51) in the zeroth approximation is

$$\begin{aligned} Z_l^{(0)}(k) &= (Z_0/2\pi)\ \ln(a_\infty/a_{-\infty} \\ &- (1/\pi c)\int dS[a'(z)/a^2(z)]\ \cos\alpha(z) \\ &- (2ik/ec\gamma^2)\int dS'\ \exp\{-ikz'\}[\ln(ka(z)/2\gamma)+C]\ . \end{aligned} \qquad (54)$$

The last term in Eq. (54) is interesting only because it gives the impedance of the straight pipe per unit length:

$$(d/dz)\ Z_l^{(0)}(k) = -[ikZ_0/(2\pi\gamma^2)]\ [\ln(ka/2\gamma)+C]\ . \qquad (55)$$

Usually, for $\gamma \gg 1$, it can be omitted.

The first two terms in (54) give the zero order result for the impedance of an abrupt change of a radius, usually called a step [12].

For the "step-out," i.e., the case of a particle entering a wider pipe, the impedance is

$$Z_l^{(0)}(k) = (Z_0/\pi) \ln[a(\infty)/a(-\infty)] . \tag{56}$$

It describes the change of the energy stored in the synchronous component of the field of a bunch due to difference of the beam pipe radii at $z = \pm\infty$. For a "step-in," i.e., for a particle entering a narrower pipe, the two terms in (52) cancel giving zero impedance. This is in good agreement with the numerical calculations [13], which show that there is a small energy gain in this case.

Note that (56) is actually valid for an arbitrary dependence $a(z)$, including an abrupt change of the radius.

Similarly, the dipole mode of the longitudinal impedance (52) in the zeroth approximation is given by the dipole component of the current $I^{(0)}$

$$Z_0 \vec{I}_z^{(0)} = -[4e/ca^2(z)] \exp\{ikz/\beta\} \cos\phi \cos\alpha , \qquad I^{(0)} = 0 . \tag{57}$$

The second term in Eq. (52) according to Eq. (5), gives the transverse impedance per unit length

$$(d/dz)\, Z_\perp = -(iZ_0/2\pi\gamma^2 a^2) . \tag{58}$$

The first and the last terms give for a step-in and a step-out, correspondingly

$$Z_\perp = -(Z_0/2\pi k)\, [(1/a_\infty^2) - (1/a_{-\infty}^2)]\, [1 \mp (1/2)] . \tag{59}$$

This impedance vanishes if the beam pipe has equal radii at infinity. Also, it does not contribute to the wake field of a symmetric (Gaussian) bunch.

5.2 First approximation

The nontrivial part of the longitudinal impedance is given by the last term in Eq. (51)

$$Z_l^0(k) = (Z_0/2\pi e) \int dS[a'(z)/a(z)]\, I_z^{(1)}(z) \exp\{-ikz/\beta\} . \tag{60}$$

The current $I_z^{(1)}$ of the first approximation is defined by Eq. (47)

$$Z_0 I_z^{(1)} = -H_\phi^{(1)} \cos\alpha$$

where

$$\vec{H}^{(1)} = \vec{\nabla} \times \vec{h} , \qquad \vec{h} = Z_0 \int dS'\, \vec{I}^{(0)}(z',\phi')\, G_k(\vec{R},\vec{R}') . \tag{61}$$

Substituting the zero-order current

$$Z_0 \vec{I}^{(0)} = -[2e/ca(z)] \exp\{ikz/\beta\}\, [\hat{r}a'(z) + \hat{z}] \cos\alpha , \tag{62}$$

we have $\vec{h} = -(\hat{r}h_1 + \hat{z}h_0)$ where

$$h_0 = (2e/c)\int [dS'/a(z')]\, G_k\, \exp\{ikz'\}\, \cos\alpha(z') , \quad (63)$$

$$h_1 = (2e/c)\int dS'\, [a'(z')/a(z')]\, G_k\, \exp\{ikz'\}\, \cos\alpha(z')\, \cos(\phi-\phi') .$$

It is convenient to write the component

$$Z_0 I_z^{(1)} = [(\partial h_1/\partial z) - (\partial h_0/\partial r)] \cos\alpha$$

using Green function, Eq. (33). The second term $(\partial h_0/\partial r)$ can be transformed, integrating h_0 by parts over dz'. After some algebra, we obtain

$$Z_l^0(k) = (Z_0 k/8\pi) \int dp \int dz a'(z) \int dz' a'(z')\, \exp\{i(p-k)(z-z')\}$$
$$\times\ \{\theta[a(z') - a(z)]\, J_1[\Omega a(z)]\, H_1^{(1)}[\Omega a(z')] + \theta[a(z)$$
$$-\ a(z')]\, J_1[\Omega a(z')]\ H_1^{(1)}[\Omega a(z)]\Big\} . \quad (64)$$

Equation (64) also can be written in the form

$$Z_l^0(k) = -(ikZ_0/2\pi) \int dz a'(z) \int dz' a'(z') \int d\phi'$$
$$\times\ [G_k(\vec{R}, \vec{R}')]_{r=a(z),\, r'=a(z')} \cos(\phi-\phi')\, \exp\{-ik(z-z')\} . \quad (65)$$

This expression is used below to study particular geometries $a(z)$.

The nontrivial part of the dipole longitudinal impedance is given by the two last terms in (52)

$$Z_l^{(1)}(k = (2r/ec) \int [dS'/a^2(z')]\, \exp\{-ikz'/\beta\}\, \cos(\phi-\phi')\, [a' I_z^{(1)} + (\partial I_{\phi'}^{(1)}/\partial\phi')] , \quad (66)$$

with the surface current determined in the first approximation

$$\vec{I}^{(1)} = -(1/Z_0)\, \vec{n} \times \vec{H}^{(1)} . \quad (67)$$

Here $\vec{H}^{(1)}$ is given by Eq. (61) with the zero-order current of a beam with offset r_0

$$Z_0 \vec{I}^{(0)}(z) = -[(\hat{r}\cos\phi + \hat{\phi}\sin\phi)\sin\alpha(z) + \hat{z}\cos\phi\cos\alpha(z)](2er_0/cr^2)\, \exp\{ikz/\beta\} . \quad (68)$$

After some calculations we have

$$\vec{h} = -(ier_0/2c) \int dp \int [dz'/a(z')]\, \exp\{ikz' + ip(z-z')\}$$
$$\times\ \{\hat{z}\, G_{1p}\, \cos\phi + a'(z')\, G_{2p}\, [\hat{r}\cos\phi + \hat{\phi}\sin\phi]\} . \quad (69)$$

Equations (66)-(69) give

$$\begin{aligned} Z_l^{(1)}(k) &= (irr_0 \cos\phi/4c) \int dp \int [dz/a(z)] \int [dz'/a(z')] \\ &\times \exp\{i(k-p)(z'-z)\} \{-a'(z') [(1/r)(\partial/\partial r) rG_{2p} + (1/r) G_{2p}] \\ &+ a'(z)[(G_{1p}/r) - (\partial G_{1p}/\partial r)] + 2ipa'(z)\, a'(z')\, G_{2p}\}_{r=a(z)} . \end{aligned} \quad (70)$$

This expression can be simplified by integration by parts. The final result for a pipe with equal radii at infinity is

$$\begin{aligned} Z_l^{(1)}(k) &= [(krr_0 \cos\phi)/2c] \int dp \int [dz/a(z)]\, [dz'/a(z')] \\ &\times a'(z)a'(z')\, G_{2p}(a(z), a(z'))\, \exp\{i(k-p)(z'-z)\} . \end{aligned} \quad (71)$$

This result can be rewritten also as

$$\begin{aligned} Z_l^{(1)}(k) &= -Z_0\, [(ikrr_0 \cos\phi)/2\pi] \int d\phi\, \cos 2(\phi-\phi') \int dz dz' \\ &\times \exp\{-ik(z-z')\}\, \{[a'(z)\, a'(z')]/[a(z)\, a(z')]\} \\ &\times [G_k(\vec{R}, \vec{R}')]_{r=a(z),\, r'=a(z')} . \end{aligned} \quad (72)$$

Transverse impedance then is given by Eq. (5).

Equations (65) and (72) give a close form of the longitudinal and transverse impedances for a axially symmetric beam pipe, with an arbitrary variation of the pipe radius $a(z)$. From these equations it is also easy to obtain the longitudinal and transverse wake fields.

In what follows, we apply these formulas to particular geometries.

6 EXAMPLES OF LONGITUDINAL IMPEDANCE

6.1 Impedance of a shallow cavity and a small collimator

We start with consideration of the simple case of a cavity with attached tubes:

$$r(z) = a\,,\ |z| > g/2\,, \qquad r(z) = b\,,\ |z| < g/2\,,\ b > a\,. \quad (73)$$

Let us consider a shallow cavity $(b-a) \ll a$, $g \ll a$, $k\,[g^2 + (b-a)^2]^{1/2} \ll 1$. The longitudinal impedance in this case is inductive. K. Bane [9] approximates the results of simulations with the code TBCI for long bunches by the formula $Z_l^{(0)} = -ikL$, where the inductance

$$L = (Z_0/2\pi)\, [g(b-a)/a]\,. \quad (74)$$

Neglecting exponents in Eq. (65), we obtain for this case

$$Z_l^0(k) = -[ikZ_0/(2(2\pi)^2)] \int_a^b dr \int_a^b dr'$$
$$\times \int d\phi \cos\phi \left\{[1/(R^2)^{1/2}] - [1/(g^2+R^2)^{1/2}]\right\} , \quad (75)$$

where $R^2 = (r-r')^2 + 4rr' \sin^2[\phi/2]$. For a shallow cavity, a significant contribution is given by small angles $\phi \simeq (b-a)/a \ll 1, \phi \simeq g/a \ll 1$. Integrals can be evaluated giving the inductance

$$L = \{Z_0(b-a)^2/[(2\pi)^2 a]\} f(\lambda) , \quad (76)$$

where $\lambda = g/(b-a)$. For small $\lambda \ll 1$

$$f(\lambda) = 4\lambda \arctan(1/\lambda) + \lambda^2(2\ln\lambda + 1) , \quad (77)$$

exactly giving K. Bane's result (74); see also the discussion after Eq. (83).

Consider now a shallow collimator

$$r(z) = b , |z| > g/2 , \qquad r(z) = a , |z| < g/2 , b > a , \quad (78)$$

where $(b-a) \ll a$, $g \ll a$, $k[g^2 + (b-a)^2]^{1/2} \ll 1$. Consideration of a shallow collimator is similar to that for a shallow cavity, with one significant difference. In deriving Eq. (65), we implied that the field inside the conductor is zero. The current $I^{(1)}$ in Eq. (60) is the current induced at the surfaces $z = \pm g/2$ by the field generated by the zero-order currents, Eq. (53). Clearly, for a collimator, $I^{(1)}$ at $z = -g/2$ cannot be generated by the zero-order current on the opposite surface $z = +g/2$, because the field inside the collimator is zero. The cross talk between surfaces $z = \pm g/2$ is generated only by the fields at the opening, and can be taken into account in the next approximations, similar to what has been done in the calculation of the impedance of a hole in a straight beam pipe. Hence, the impedance of a collimator is

$$Z_l^0(k) = -[2ikZ_0/(2\pi)] \int dr \int dr' \int d\phi \cos\phi \, G_k(r, r', z = z', \phi - \phi') . \quad (79)$$

Calculations for a shallow collimator can be performed explicitly, giving $Z_l^{(0)} = -ikL$ with the inductance

$$L = [Z_0(b-a)^2/4\pi a] \{\ln[2\pi a/(b-a)] + (3/2)\} . \quad (80)$$

K. Bane [8] describes TBCI results for this case by the formula

$$L = [Z_0(b-a)^2/(\pi a)] ,$$

which is different from Eq. (80) by a factor

$$(1/4\pi)\{\ln[2\pi a/(b-a)] + (3/2)\} .$$

Figure 2 gives TBCI results for W_{max} for an iris with an opening radius of 3.8 cm, in a beam pipe with a radius of 4.0 cm. The iris thickness g was changed in the range from 0.1 to 0.5 cm. The bunch length is 1 cm. The variation of W_{max} with g is small.

It should be noted that simulations with TBCI of a shallow cavity and a small collimator are not simple, because they

6.2 Impedance of a pill-box cavity with pipes and a step

Consider a cavity in the high-frequency limit $kg \gg 1$, $ka \gg 1$. This case has been studied before [10] and the answer is known. The real part of the impedance is

$$\mathrm{Re} Z_l = (Z_0/2\pi a)\,(g/\pi k)^{1/2}\,. \tag{81}$$

It is more convenient in this case to use Eq. (64). Integration in Eq. (64) over z can be replaced by integration over $dr = a'(z)dz$ at the boundaries $z = \pm g/2$. This gives

$$\begin{aligned} Z_l^{(0)}(k) &= (kZ_0/2\pi)\int dp \int_a^b dr \int_a^b dr' \sin^2\left[(g/2)\,(k-p)\right] \\ &\times \left[\theta(r-r')\,J_1(\Omega r')\,H_1^{(1)}(\Omega r) + \theta(r'-r)\,J_1(\Omega r)\,H_1^{(1)}(\Omega r')\right]\,. \end{aligned} \tag{82}$$

Integrals over r, r' can be calculated explicitly

$$\begin{aligned} Z_l^{(0)}(k) &= (kZ_0/2\pi)\int (dp/\Omega^2)\,\sin^2\left[(g/2)\,(k-p)\right) \\ &\times \Big\{\left[J_0(\Omega b) - J_0(\Omega a)\right] H_0^{(1)}(\Omega b) \\ &- \left[H_0^{(1)}(\Omega b) - H_0^{(1)}(\Omega a)\right] J_0(\Omega a) + (2i/\pi)\,\ln(b/a)\Big\}\,. \end{aligned} \tag{83}$$

The impedance for a shallow cavity is inductive $Z_l^{(0)} = -ikL = -ik\mathrm{L}/c$. The inductance L (in cm), calculated from Eq. (83), is depicted in Fig. 3 as a function of $g/(b-a)$. The integration has been performed for a cavity with radii $b = 4.2$ cm, $b - a = 0.2$ cm. The inductance calculated from Eq. (74) is shown by the dashes. The agreement of the results found above for small $g/(b-a)$ also takes place for a wider range of gaps, up to $g \simeq a$.

The real part of the impedance is given by the interval $-k < p < k$:

$$\begin{aligned} \mathrm{Re}\ Z_l^{(0)}(k) &= (kZ_0/2\pi)\int_{-k}^{k} \left[dp/(k^2-p^2)\right]\,\sin^2 \\ &\times \left[(g/2)(k-p)\right]\left[J_0(\Omega a) - J_0(\Omega b)\right]^2\,. \end{aligned} \tag{84}$$

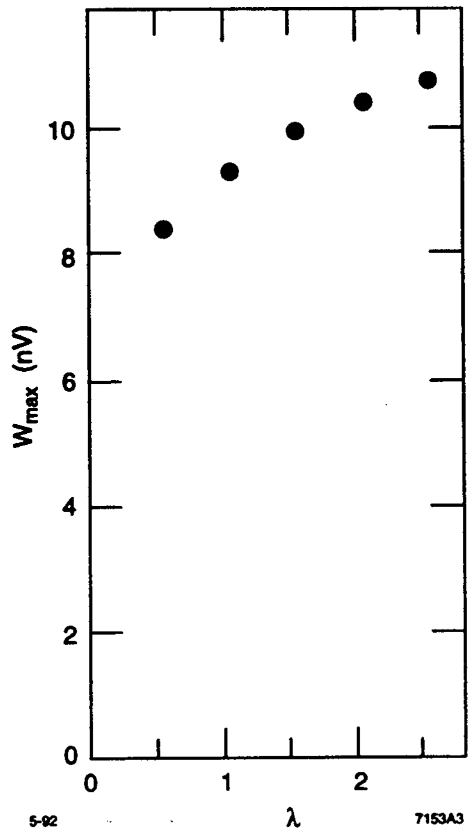

FIGURE 2: The TBCI results for the wake potential of a shallow collimator. Parameters $\sigma = 1$ cm, $b = 4$ cm, $b - a = 0.2$ cm. Perturbation theory gives the impedance constant in g.

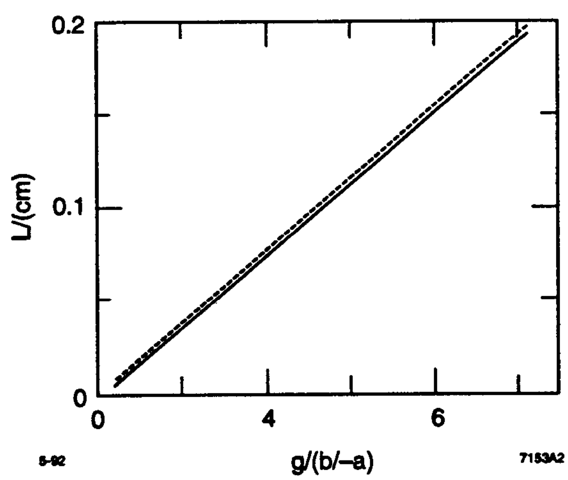

FIGURE 3: The inductance of a shallow cavity as a function of $\lambda = (b-a)/a$. The dashed line is calculated with Eq. (74), based on TBCI results. Parameters of the cavity are $a = 4$ cm, $b - a = 0.2$.

To estimate the integral Eq. (84) in the high-frequency limit $ka \gg 1$ for $b \gg a \simeq g$ we can neglect $J_0(\Omega b)$. The main contribution at high frequencies $ka \gg 1, kb \gg 1$ is given by the range of p for which $g(k-p) \geq 1$, $\Omega a \gg 1$. Using an asymptotic expression for the Bessel function, we obtain [11] the Dome-Lawson result (81).

For very large gaps g the impedance does not depend on g but depends on both radii. The impedance can be obtained from Eq. (84) in the limit $g \to \infty$, replacing $\sin^2[\ldots]$ by its average value 1/2. For the pipes of equal radii at large z,

$$\mathrm{Re} Z_l^{(0)}(k) = (kZ_0/4\pi) \int [dp/(k^2-p^2)] \, [J_0(\Omega b) - J_0(\Omega b)]^2 \, . \tag{85}$$

The main contribution for $ka \gg 1, kb \gg 1$, $b \gg a$ is given by the interval $(1/a) \gg \Omega \gg (1/b)$:

$$\mathrm{Re}\ Z_l^{(0)}(k) = (Z_0/4\pi)\ \ln ka \, . \tag{86}$$

For small b , $kb < 1$; ka should be replaced by b/a, giving an impedance similar to the impedance of a step

$$\mathrm{Re}\ Z_l^{(0)}(k) = (Z_0/4\pi) \ln(b/a) \, . \tag{87}$$

Transition from the regime of a cavity to the regime of a step occurs [14] at $g \simeq k(b-a)^2$.

The result of a numerical integration of Eq. (84) is shown in Fig. 4a. The real part of the impedance is a linear function of the Dome-Lawson parameter $(1/2\pi a)\,(g/\pi k)^{1/2}$. Results in the "cavity regime" $g \gg k(b-a)^2$ are independendent of b. Figure 4b shows that there is a smooth transition from the regime of a cavity to the regime of a step. Transition frequency depends on $(b-a)/a$.

The impedance of a collimator can be derived similarly to the impedance of a cavity, but with the correction discussed in the zeroth order approximation,

$$\begin{aligned} Z_l^{(0)}(k) &= (kZ_0/2\pi) \int (dp/\Omega^2) \\ &\times \Big\{ J_0(\Omega a)[H_0^{(1)}(\Omega a) - H^{(1)}(\Omega b)] \\ &- H_0^{(1)}(\Omega b)\ [J_0(\Omega a) - J_0(\Omega b)] + (2i/\pi)\ \ln(b/a) \Big\} \, . \end{aligned} \tag{88}$$

The real part of the impedance is given by the interval $-k < p < k$

$$\mathrm{Re} Z_l^{(0)}(k) = (kZ_0/2\pi) \int_{-k}^{k} [dp/(k^2-p^2)] \, [J_0(\Omega a) - J_0(\Omega b)]^2 \, . \tag{89}$$

The impedance calculated from this formula is shown in Fig. 5.

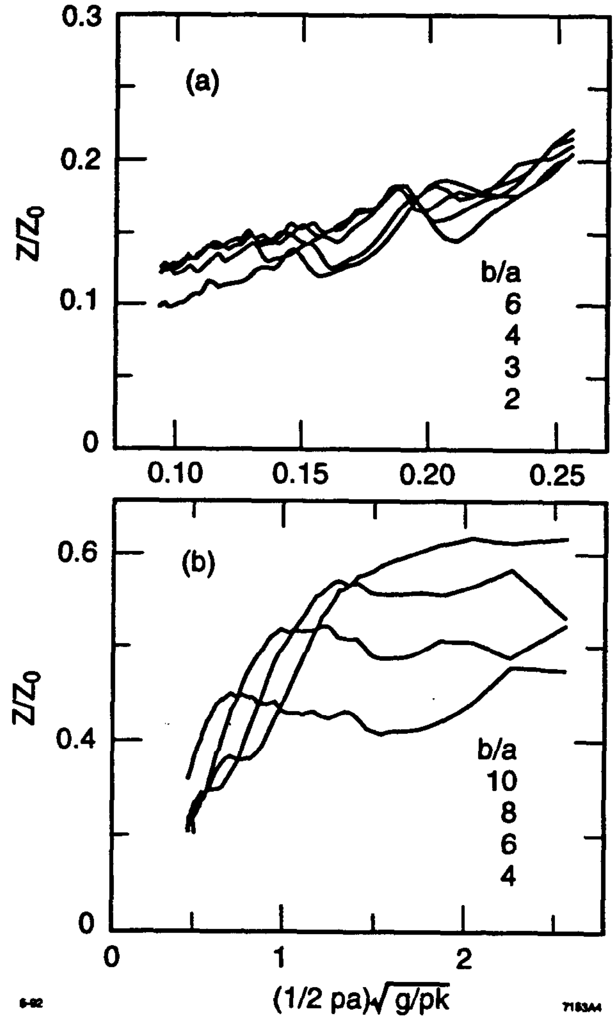

FIGURE 4: (a) The real part of the longitudinal impedance of a cavity, Eq. (84), as a function of the Dome-Lawson parameter. Results are essentially independent of the b; $g/a = 3.0$, b/a is in the range 2.0–6.0. (b) The same as in (a), but for large gaps. The transition from the regime of a cavity to the regime of a step is shown.

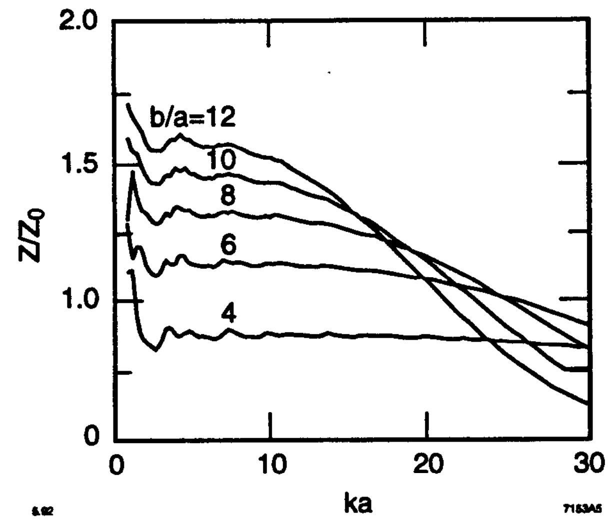

FIGURE 5: The frequency dependence of the impedance of a collimator. The impedance is constant for small ka and rolls off at large frequencies. The roll-off starts at frequencies dependent on the ratio of the radii.

6.3 Impedance of a periodic array of cavities

Consider an array of M identical cells made by irises in a straight pipe. The radius of the pipe is b, the radius of the opening in the washers is a, the thickness of the iris is l, and the length of a period is d.

The real part of the impedance can be obtained from Eq. (65):

$$\mathrm{Re} Z_l^{(0)}(k) \;=\; (kZ_0M/2\pi)\int_k^k [dp/(k^2-p^2)]\,[J_0(\Omega b)-J_0(\Omega b)]^2\;F(k,p)\;. \qquad (90)$$

This is different from the result (84) for a single cell by a form factor $F(k,p)$, describing interference of the waves generated by a bunch in different cells,

$$F(k,p) \;=\; (1/4\pi)\sum_{i,j=1}^{M}(-1)^{i-j}\;\exp\{i(k-p)(z_i-z_j)\}\;. \qquad (91)$$

Here z_i are the coordinates of the irises:

$$z_{2n+1} = nd, \ n = 0,1,\dots,M-1, \qquad z_{2n} = nd - l, \ n = 1,2,\dots M .$$

The sum (76) can be calculated. Usually $l \ll d$, and

$$F(k,p) = (1/\pi M) \sin^2 [(g/2)(k-p)] \sin^2\{[(k-p)d/2] M\} . \tag{92}$$

For small $M \simeq 1$, the impedance per cell is the same as that for a single cell and, at high frequencies, provided $ka^2/d \ggg 1$, is given by Eq. (81).

For $M \gg 1$, the impedance per cell from Eq. (90) is the convolution of two sharp functions. The first function in the integrand (90) is the same as for a single cavity and has a sharp maximum for $|k-p| \leq (1/ka^2)$. The second form factor (92) has a maximum for $|k-p| \leq (1/Md)$. The frequency behavior of the impedance depends on the parameter ka^2/Md. For a structure with fixed total length Md, there are always very large ka where the first function dominates and impedance per cell is the same as for a single cell. However, there is an intermediate range of frequencies where $M \gg ka^2/d$, where the second function is dominant. For such frequencies impedance rolls off as $k^{-3/2}$ instead of $k^{-1/2}$. More discussion on this can be found in the original paper [15]. The same result has been obtained by R. Gluckstern [16].

6.4 Impedance of a taper

The radius of a taper varies linearly from a to $b > a$ at distance L,

$$r(z) = a \ , \ z \leq 0 \ ; \quad r(z) = a + a'z \ , \ 0 < z < L \ ; \quad r(z) = a + bL \ , \ z > L \ . \tag{93}$$

The longitudinal impedance is

$$Z_l^{(0)}(k) = (Z_0/\pi)\ln(b/a) + \{[kZ_0(a')^2/8\pi\} \ S(k) \ , \tag{94}$$

where

$$S(k) = \int dp \int_0^L dz dz' \ \exp\{i(p-k)(z-z')\} \ [G_{1,p}(z,z')]|_{r=a(z), r'=a(z')} \ . \tag{95}$$

The cuts in the plane p are defined so that $\Omega = i(p^2-k^2)^{1/2}$ for $p > k$ and $p < -k$, and the contour of integration is below the cuts. Because there are no singularities in the lower plane of p, the integration over p gives zero if $z < z'$. Hence,

$$S(k) = \int dp \int_0^L dz \int_0^z dz' \ \exp\{i(p-k)(z-z')\} \ J_1[\Omega a(z')] \ H_1^{(1)}[\Omega a(z)] \ . \tag{96}$$

We will study this expression at high frequencies, which give the main contribution to the energy loss.

First, consider a taper with large angle α, $\tan\alpha = a'$, $a' \lesssim 1$. Equation (96) in this case can be simplified using asymptotic expressions for the Bessel functions and retaining the slowest oscillating exponential function:

$$S(k) = \int dq \int_0^L dz \int_0^z dz' \left(2/\{\pi\Omega[a(z)a(z')]\}^{1/2}\right) \exp\{-i(z-z')\psi(z,z',q)\} , \tag{97}$$

where $q = k - p$, and $\psi = q - a'[q(2k-q)]^{1/2}$. The integral over q can be calculated by a saddle-point method, expanding ψ around $q_0 = ka'^2/2$, provided $ka|a'| \gg 1$. That gives

$$\psi = -(ka'^2/2) + [(q-q_0)^2/4q_0] , \qquad \Omega \simeq |ka'| , \tag{98}$$

and

$$S(k) = 2(2/i\pi k) \int_0^L [dz/a(z)] \int_0^z [dz'/(z-z')^{1/2}] \exp\{i(ka'^2/2)(z-z')\} . \tag{99}$$

Significant values of $|z - z'| \sim 1/(ka'^2)$ are small provided $ka'^2L \gg 1$. This reduces the integral to

$$S(k) = (8/|ka'|) \int_0^L [dz/a(z)] = (8/ka'^2) \ln(b/a) . \tag{100}$$

The impedance obtained from Eqs. (94) and (100) is the same as the impedance of a step with the radii a, b

$$Z_l^{(0)}(k) = (Z_0/2\pi) \ln(b/a) . \tag{101}$$

The result is valid if

$$ka|a'| \gg 1 , \qquad \text{and} \qquad k|a'(b-a)| \gg 1 . \tag{102}$$

If $ka|a'| \gg 1$, but $b - a$ are so small that $k|a'(b-a)| \ll 1$, the integral (99) gives

$$\begin{aligned} S(k) &= 2(2/i\pi k)^{1/2} \int_0^L (dz/a + a'z) \int_0^z [dz'/(z-z')^{1/2}] \\ &= (8/3\pi a)(2\pi/ik)^{1/2} [(b-a)/a']^{3/2} . \end{aligned} \tag{103}$$

From this consideration, it follows that it is sufficient to study a shallow taper with $|a'| \ll 1$.

In this case it is more convenient to write the impedance in the form (65) and $S(k)$ in the form

$$S(k) = \int (d\phi/i\pi a'^2) \int_a^b dr dr' \exp\{-ik(z-z')\} [\exp\{ik|\vec{r} - \vec{r}'|\} / |\vec{r} - \vec{r}'|] \cos(\phi) . \tag{104}$$

The difference

$$|\vec{r}-\vec{r}'| = (1/|a'|)\ \left[(r-r')^2(1+a'^2)+2rr'a'^2(1-\cos\phi)\right]^{1/2}$$

can be expanded for small $|a'| \ll 1$

$$\begin{aligned} |\vec{r}-\vec{r}'| &= [(1+a'^2)/|a'|]^{1/2}\ |r-r'| \\ &+ \left\{[rr'|a'|(1-\cos\phi)]\,/\,[|r-r'|(1+a'^2)^{1/2}]\right\}\ , \end{aligned} \tag{105}$$

provided

$$(r-r')^2 \gg r^2a'^2\ .$$

Then

$$S(k) = (1/i\pi)\int_a^b (drdr'/|r-r'|)\ \int\left\{d\phi/[|a'|(1+a'^2)^{1/2}]\right\}\ \exp\{-i\phi+i\Psi\}\ , \tag{106}$$

where

$$\begin{aligned} \Psi &= -(k/a')\ (r-r')+\left\{[k(1+a'^2)^{1/2}]\,/\,|a'|\right\}\ |r-r'| \\ &+ \left\{[krr'|a'|(1-\cos\phi)]\,/\,[|r-r'|\,(1+a'^2)^{1/2}]\right\}\ . \end{aligned}$$

The integral over ϕ gives the Bessel function. For $|a'| \ll 1$, the phase Ψ is large and the exponent oscillates rapidly if $r-r'<0$ for $a'>0$, and $r-r'<0$ for $a'<0$. Consider, for example, $a'>0$. Then $r>r'$ are significant, for which

$$\Psi = (k|a'|/2)\ (r-r')+\left\{[krr'|a'|(1-\cos\phi)]\,/\,[|r-r'|\,(1+a'^2)^{1/2}]\right\}\ .$$

The first term here is negligibly small provided

$$\lambda \equiv ka|a'| \ll 1\ . \tag{107}$$

With the new variables $y=r/a$ and $x=(r-r')/a$, Eq. (106) takes the form

$$S(k) = (a/i\pi|a'|)\int(dxdy/|x|)\ J_1(\lambda y^2/|x|)\ \exp\{i\lambda(y^2/|x|)\}\ . \tag{108}$$

Here the limits of the integration are

$$1<y<p \equiv (b/a)\ , \qquad y-p<x<y-1\ .$$

One of the integrals can be calculated, introducing the new variables $\tau = y^2/x$, $\xi=y$, and changing the order of integration. The limits of integration over τ,ξ are, for $p<2$,

$$[p^2/(p-1)]<\tau<\infty\ , \qquad (\tau/2)-[(\tau^2/4)-\tau]^{1/2}<\xi<p\ ,$$

and for $p > 2$ there are two integrals with the limits

$$4 < \tau < [p^2/(p-1)]\,, \qquad (\tau/2) - [(\tau^2/4) - \tau]^{1/2} < \xi < (\tau/2) + [(\tau^2/4) - \tau]^{1/2}\,,$$

and

$$[p^2/(p-1)] < \tau < \infty\,, \qquad (\tau/2) - [(\tau^2/4) - \tau]^{1/2} < \xi < p\,.$$

Then the function $S(k)$ is given by the integral

$$\begin{aligned} S(k) &= (a/i\pi|a'|)\,\Big(\theta(p-2)\int_4^{Q(p)} (d\tau/\tau)\,[(\tau^2/4)-\tau]^{1/2} J_1(\lambda\tau)\,\exp\{i\lambda\tau\} \\ &+ \int_{Q(p)}^{\infty} (d\tau/\tau)\,\Big\{p - (\tau/2) + [(\tau^2/4)-\tau]^{1/2}\Big\}\,J_1(\lambda\tau)\,\exp\{i\lambda\tau\}\Big)\,, \end{aligned} \tag{109}$$

where $p = b/a$ and $Q(p) = [p^2/(p-1)]$.

The integral (109) can further be estimated in two extreme cases. If $\lambda Q(p) \ll 1$, impedance (94),(109) increases linearly with $|a'|$:

$$Z_l^{(0)}(k) = -i(Z_0\lambda/8\pi)\int_0^{\infty} (dx/x)\,J_1(x)\,\exp\{ix\} + o(\lambda^2)\,. \tag{110}$$

This result is confirmed by the numeric integration of Eq. (109); see Fig. 6a,b.

For very small $p - 1 = (b-a)/a \ll 1$, there is another regime where

$$\lambda \ll 1\,, \qquad \text{but} \qquad [\lambda p^2/(p-1)] \gg 1\,.$$

The impedance in this case is

$$\begin{aligned} Z_l^{(0)}(k) &= [Z_0(1+i)/(4\pi)^2](\lambda/\pi)^{1/2} \\ &\times \int_{Q(p)}^{\infty} (dx/x^{3/2})\,\{p - (x/2) + [(x^2/4) - x]^{1/2}\,\} \end{aligned} \tag{111}$$

and proportional to the $(a')^{1/2}$. This behavior previously has been found numerically [17].

Equation (111) is compared with the numerical integration of the Eq. (109) for small $(p-1) \ll 1$, see Fig. 7a,b. The transition from small $p-1$ to $p-1 \simeq 1$ is depicted in Fig. 8.

7 EXAMPLES OF TRANSVERSE IMPEDANCE

The transverse impedance is given by Eqs. (5) and (71):

$$\begin{aligned} Z_\perp(k) &= (Z_0/8\pi)\int dp\int dz dz'\,\exp\{i(k-p)(z'-z)\} \\ &\times [a'(z)a'(z')/a(z)a(z')]\,[G_{2,p}]_{r=a(z),r'=a(z')}\,. \end{aligned} \tag{112}$$

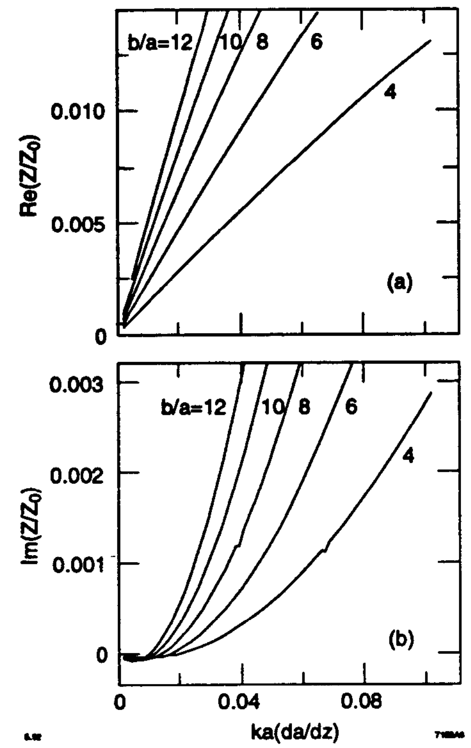

FIGURE 6: The real (a) and imaginary (b) parts of the longitudinal impedance of a taper, with large $p = b/a$.

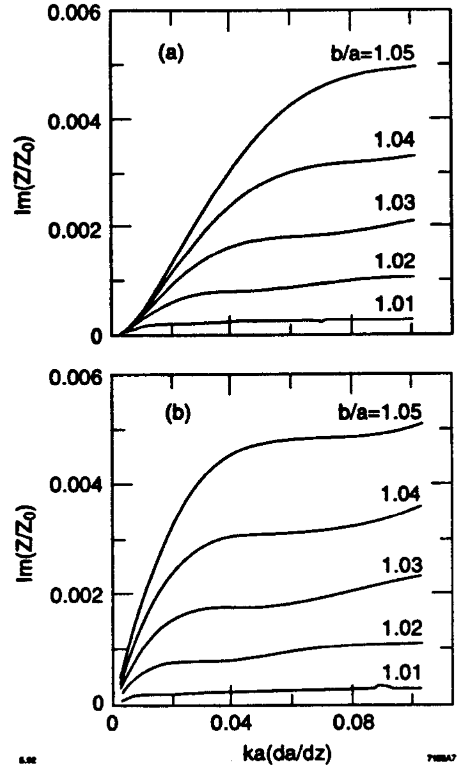

FIGURE 7: The real (a) and imaginary (b) parts of the longitudinal impedance of a taper, for small $(p-1) \ll 1$

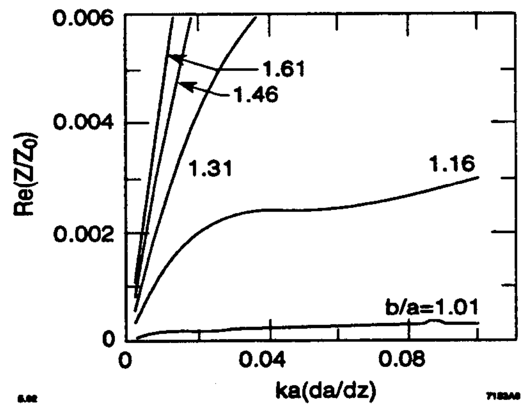

FIGURE 8: The same as in Fig. 6(a), for the transition from $(p-1) \ll 1$ to $(p-1) \simeq 1$.

or

$$Z_\perp(k) = -i(Z_0/2\pi) \int d\phi' \cos 2(\phi-\phi') \int dz dz' \exp\{-ik(z-z')\}$$
$$\times [a'(z)/a(z) [a'(z')/a(z')] [G_k(\vec{R},\vec{R}')]_{r=a(z),r'=a(z')} . \quad (113)$$

We consider here several examples.

7.1 *Impedance of a cavity with attached tubes*

We consider first Eq. (71), for a cavity with attached tubes:

$$r(z) = a , |z| > g/2 , \qquad r(z) = b , |z| < g/2 , \qquad b > a .$$

For a shallow cavity of low frequency $k(b-a) \ll 1$, the impedance is proportional to to the gap length g,

$$Z_\perp(k) = -\frac{2i\,g(b-a)}{ca^3} , \quad (114)$$

if $g \ll 2(b-a)4$. For large gaps $g \gg 2(b-a)$, dependence on the gap length is logarithmic

$$Z_\perp(k) = -\frac{2i}{\pi}\,\frac{(b-a)^2}{ca^3}\left\{\ln\frac{g}{2(b-a)}+1\right\} , \qquad \frac{kg}{2} < 1 ,$$

and is independent of g for $kg/2 > 1$.

In the high frequency limit $kg \gg 1$, $ka \gg 1$, it is more convenient to use Eq. (72), where the integrals over $dr = a'(z)dz$ can be calculated using expressions

$$\{[J_2(\Omega r)]/r\} = -(1/\Omega)\,(\partial/\partial r)\,\{[J_1(\Omega r)]/r\} ,$$

and

$$H_2^{(1)}(z)\,J_1(z) - J_2(z)\,H_1^{(1)}(z) = -(2i/\pi z) .$$

That gives

$$Z_\perp(k) = (Z_0/2\pi) \int (dp/\Omega^2)\,\sin^2[(g/2)(k-p)]\,\{(i/\pi)\,[(1/b^2)-(1/a^2)]$$
$$- \{[H_1^{(1)}(\Omega b)]/b\}\left(\{[J_1(\Omega b)]/b\} - \{[J_1(\Omega a)]/a\}\right)$$
$$+ \{[J_1(\Omega a)]/a\}\left(\{[H_1^{(1)}(\Omega b)]/b\} - \{[H_1^{(1)}(\Omega a)]/a\}\right) . \quad (115)$$

The real part of the impedance again isgiven by the interval $-k < p < k$:

$$\mathrm{Re}Z_\perp(k) = (Z_0/2\pi k)\int_{-1}^{1}[dx/(1-x^2)]\,\sin^2[(kg/2)(1-x)]$$
$$\times\left(\{J_1[kb(1-x^2)^{1/2}]/b\}-\{J_1[ka(1-x^2)^{1/2}]/a\}\right)^2 . \quad (116)$$

The behavior of the impedance (116) depends on the ratio g/a. For $g < ka^2$, $kg \gg 1$ the impedance (116) increases as k^3 for $ka \ll 1$, reaches maximum at $ka \simeq 1$, and rolls-off as $(1/ka)^{3/2}$ for large ka. For $g > ka^2$, the impedance (116) increases as k^3 for $ka \ll 1$, reaches maximum at $ka \simeq 1$, and rolls-off as $(1/ka)$ for large ka.

The results of numerical integration of Eq. (116) are shown in Fig. 9a,b.

7.2 *Transverse impedance of a taper*

For a taper, the impedance is given by Eqs. (5) and (72):

$$Z_\perp(k) = -[iZ_0/2(2\pi)^2]\int d\phi'\,\cos(\phi-\phi')\int_a^b (dr/r)\,(dr'/r')$$
$$\times\ \exp\{-ik(z-z')\}\,[(\exp\{ik|\vec{r}-\vec{r}'|\})/(|\vec{r}-\vec{r}'|)]\,, \quad (117)$$

where $z-z'=(r-r')/a'$.

Expanding $|\vec{r}-\vec{r}'|$ as in Eq. (105), we can calculate the integral over the angle ϕ. This gives the Bessel function J_2. For a shallow taper $|a'| \ll 1$, the main contribution is given by

$$r|a'| \ll |r'-r| \ll r \simeq r' .$$

With a proper choice of variables, it is possible to carry out one more integration in the same way as it was done for the longitudinal impedance of a taper. The impedance is given by the remaining integral

$$Z_\perp(k) = -\,(iZ_0|a'|/4\pi a)\{\theta(p-2)$$
$$\times\int_4^{Q(p)}(d\tau/\tau)\,J_2(\lambda\tau)\,\exp\{i\lambda\tau\}\,[(1/\xi_-)-(1/\xi_+)]$$
$$+\int_{Q(p)}^{\infty}(d\tau/\tau)\,J_2(\lambda\tau)\,\exp\{i\lambda\tau\}[(1/\xi_-)-(1/p)]\}\,, \quad (118)$$

where $p=b/a$, $Q(p)=[p^2/(p-1)]$, and

$$\xi_\pm = (\tau/2) \pm [(\tau^2/4)-\tau]^{1/2} . \quad (119)$$

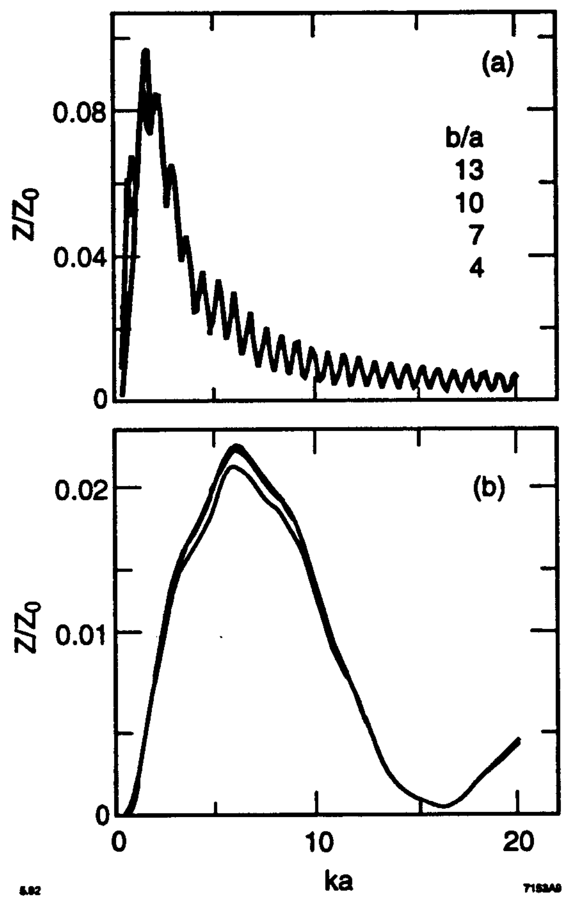

FIGURE 9: The real part of the transverse impedance of a cavity, (a) for $g/a = 4.0$, and (b) for small $g/a = 0.2$. These results are independent of b/a, which is in the range 4.0–13.0 for both (a) and (b). The curve for (b) is broader and rolls-off slower than for (a).

The result, Eq. (118), is valid if

$$|a'| \ll \lambda \equiv ka|a'| \ll 1 . \quad (120)$$

8 CONCLUSION

The perturbation method described above allows derivation of the general expressions for the longitudinal [Eqs. (65), (72)] and transverse [Eq. (113)], impedances for axially symmetric structures with arbitrary variation of the radius along the structure.

It is shown that the formulas reproduce in a systematic way, numerous previously known rsults, and obtain new results.

The longitudinal impedance is found for

Parameter	Equation
(a) Hole	(41)
(b) Slot	(43)
(c) Step	(57)
(d) Shallow cavity	(76)
(e) Shallow Collimator	(80)
(f) Cavity with pipes	(84)
(g) Taper	(99), (95) and Figs 6–8

Transverse impedance is considered for

Parameter	Equation
(a) Hole	(44)
(b) Step	(57)
(c) Shallow cavity	(116)
(d) Taper	(118)

This method allows us to obtain all these results in a unified way as extreme cases of the same formula, and to demonstrate the transition from one case to another; for example, from the regime of a cavity to the regime of a step, Fig. 4, or from a single cavity to a periodic array, Eq. (92). It seems that the method always works where the narrow band impedance is not dominant—in other words, at high frequencies—and also for low frequencies, but for shallow discontinuities. Several new results are obtained, including the longitudinal and transverse impedances of a taper. The method can be generalized to more complicated geometries; for example, for a hole in a beam pipe with finite thickness of the wall or for a structure without axial symmetry.

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