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THE CORE EMITTANCE WITH INTRABEAM SCATTERING IN e^+/e^- RINGS*

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ABSTRACT

In an e^+/e^- storage ring, intrabeam scattering will cause the equilibrium transverse and longitudinal particle distributions to be non-Gaussian. In this paper, we discuss a modification to the current theories of intrabeam scattering that more accurately describes the emittance growth of the core.

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1. INTRODUCTION

Future e^+/e^- storage rings, in order to be used as synchrotron radiation sources or damping rings for linear colliders, will need to store very dense particle bunches. One of the many effects that can limit the particle density in a storage ring is the Coulomb scattering between particles within a bunch. In this paper, we discuss the effect of the small-angle scattering, which is referred to as intrabeam scattering or the multiple Touschek effect [1,2].

Intrabeam scattering is the result of multiple small-angle Coulomb collisions between particles in the beam, leading to diffusion in the beam phase space. In e^+/e^- storage rings, where the synchrotron radiation provides a source of damping, the scattering increases the beam emittance until the additional diffusion is countered by the radiation damping. Detailed theories of intrabeam scattering have been developed in Refs. [2–5]. In this paper, we discuss a modification to the current theories that, in some cases, predicts a significant reduction in the effect of the intrabeam scattering.

The current theories calculate the rms emittance growth. The rms value is a useful quantity for describing the width of a Gaussian distribution—but intrabeam scattering yields a non-Gaussian beam since the Coulomb scattering has a non-Gaussian distribution. Specifically, the beam distribution due to the scattering will have a nearly Gaussian core, with long tails that can have a significant contribution to the rms width; the core is due to the multiple soft scattering, while the tails arise from the infrequent hard scatterings.

The situation in a storage ring is analogous to the theory of scattering in a material [6] except that in an e^+/e^- ring it is necessary to include the betatron motion and the damping due to the synchrotron radiation. To calculate the equilibrium beam distribution, the scattering can be analyzed as a filtered Poisson process similar to shot noise [8,9]. However, the full distribution function is unduly complicated, so instead we calculate the width of the Gaussian beam core; this is the value that is relevant for the luminosity in a collider or the brilliance in a synchrotron radiation source.

As mentioned, the core of the distribution is due to the multiple soft scattering, while the tails are due to the infrequent hard scattering. Thus, we calculate a boundary, in terms of the momentum transfer in the collision, that separates the contributions to the beam core from those to the tails. All collisions with momentum transfer q greater than the boundary $q_\star$ are assumed to contribute to the tails and will be neglected when calculating the width of the core.

We approximate the boundary $q_\star$ as the point where the integrated rate of scattering, with momentum transfers greater than $q_\star$, is equal to the damping rate,

$$\nu(q > q_\star) \sim \tau_{SR}^{-1} \quad , \tag{1}$$

where τ_{SR} is the synchrotron radiation damping time. This choice is justified in Refs. [7-9] and is intuitively appealing. The Gaussian core is generated by multiple scattering, within the time scale set by the damping, so that the Central Limit Theorem applies. In contrast, the tails are generated by infrequent scatterings, within the damping time, and thus have a distribution similar to the Coulomb scattering distribution $f(x) \sim x^{-3}$.

This paper is an extension (and correction) of work that appeared in Ref. [11,12]. We start by giving a very brief introduction to the intrabeam scattering emittance growth process, and then we calculate the scattering rate. Next, we apply these results to two examples, and further illustrate with a simulation of the beam distribution.

2. INTRABEAM SCATTERING

In a reference frame co-moving with an ultra-relativistic particle beam, the beam usually has an anisotropic momentum distribution. The longitudinal momentum tends to be much smaller than the horizontal or vertical [13]. Coulomb scattering between particles in the beam will redistribute the beam momenta in an approach to "thermal" equilibrium. Since the longitudinal direction is "cooler" than the transverse, one would expect the longitudinal momentum spread to increase at the expense of the transverse momenta. Unfortunately, this is complicated by the energy dependence of the particle orbit, i.e., the dispersion function; the dispersion couples a change in the longitudinal momentum to the transverse planes. Thus, a scattering event that transfers transverse momentum to the longitudinal plane has both a cooling and a heating effect on the transverse phase space.

In high energy e^+/e^- rings, the heating is far more important than the cooling effect [2]; the transverse cooling is roughly proportional to $1/2\gamma^2$, while the heating is proportional to the dispersion invariant $\mathcal{H}_x \equiv \gamma_x\eta_x^2 + 2\alpha_x\eta_x\eta'_x + \beta_x{\eta'_x}^2$, where α_x, β_x, γ_x are the storage ring lattice parameters and η_x is the dispersion [11]. Thus, the intrabeam scattering causes both the longitudinal and transverse equilibrium emittances to grow until the growth is countered by the damping due to the synchrotron radiation.

The emittance growth depends upon the second moment of the momentum exchange [2–5]

$$\Delta\epsilon_z \propto \int_{q_{\min}}^{q_{\max}} q_z^2 \, \frac{d\sigma}{dq} \, dq \qquad\qquad \Delta\epsilon_x \propto \int_{q_{\min}}^{q_{\max}} \mathcal{H}_x q_z^2 \, \frac{d\sigma}{dq} \, dq \quad , \tag{2}$$

where q_z is the longitudinal component of the exchange. Because the Coulomb cross-section is inversely proportional to q^3, the emittance increase depends upon the "Coulomb log," which can be written in terms of the limits on the momentum exchange, the impact parameter, or the scattering angle,

$$\ln \frac{q_{\max}}{q_{\min}} \approx \ln \frac{b_{\max}}{b_{\min}} \approx \ln \frac{\theta_{\max}}{\theta_{\min}} \quad . \tag{3}$$

Our calculation will determine a new value for $q_{\max}$, denoted $q_\star$, that approximates the boundary between the contributions to the core and the tails. This corrects the Coulomb log so that it only includes the contributions to the core emittance.

3. SCATTERING RATES

To solve for the value of $q_\star$, we calculate the integrated rate of the hard scattering as a function of the boundary $q_\star$, and set this equal to the damping rate, Eq. (1). To calculate the scattering rate, we essentially follow Piwinski's derivation in Refs. [2] and [5]; we calculate the scattering rate of two particles and then integrate over the beam distribution. In the center-of-mass reference frame of two scattering particles, the differential cross section can be written

$$d\,\bar{\sigma} = 2\pi \, \frac{r_0^2}{q^4 \, \bar{\beta}^2} \, q \, dq \quad , \tag{4}$$

where q is the (normalized) momenta exchange: $q = 2\bar{\beta}\sin(theta/2)$, r_0 is the classical electron radius, $\bar{\beta}$ is the velocity of the particles in units of c, and we have assumed that the particles are nonrelativistic in the center-of-mass frame.

Further assuming an ultra-relativistic beam, the velocity of the two colliding particles in the center-of-mass frame is [2]

$$\bar{\beta} = \frac{\gamma}{2} \sqrt{(x_1' - x_2')^2 + (y_1' - y_2')^2 + \frac{(\delta_1 - \delta_2)^2}{\gamma^2}} \quad , \tag{5}$$

where the bar denotes the center-of-mass reference frame, and x', y', and δ are the angular deviations, $p_\perp/p_0$, and relative energy deviation, $(E-E_0)/E_0$, of the two particles in the laboratory frame; the factors of γ come from the relativistic transformation from the laboratory frame.

Now the integrated scattering rate, in the laboratory reference frame, can be written

$$\nu(q_{\max}, q_\star) = \frac{4\pi}{\gamma^2} \int \bar{\beta} c \, P(\vec{x}_1, \vec{x}_2) \int_{q_\star}^{q_{\max}} \frac{d\bar{\sigma}}{dq} \, dq \, d\vec{x}_1 \, d\vec{x}_2 \quad , \tag{6}$$

where $P(\vec{x}_1, \vec{x}_2) \, d\vec{x}_1 \, d\vec{x}_2$ is the phase space density factor, and $q_{\max}$ is the maximum momentum transfer.

The phase space density factor $P \, d\vec{x}_1 \, d\vec{x}_2$ is

$$P \, d\vec{x}_1 \, d\vec{x}_2 = \rho_{x1} \, \rho_{x2} \, \rho_{y1} \, \rho_{y2} \, \rho_{z1} \, \rho_{z2} \, \delta(x_1 - x_2) \, \delta(y_1 - y_2) \, \delta(z_1 - z_2) \, d\vec{x}_1 \, d\vec{x}_2 \quad , \tag{7}$$

where $\rho_{x,y,z}$ are the distributions in the three planes and $\vec{x} = (x_\beta, x_\beta', y_\beta, y_\beta', z, \delta)$. Here, we have assumed that the particles are in the same region of space when scattering. This is the approximation made in Refs. [2] and [5]; it is not completely valid when discussing the small-angle scattering, but in our case, where we are interested in hard scatterings, it is a valid assumption since the hard scatterings occur at small impact parameters.

Assuming a Gaussian beam, the distributions in the transverse betatron and longitudinal phase spaces have the forms

$$\rho_x(x_\beta, x'_\beta) = \frac{1}{2\pi\epsilon_x} \exp\left[-\frac{(\gamma_x x_\beta^2 + 2\alpha_x x_\beta x'_\beta + \beta_x x'^2_\beta)}{2\epsilon_x}\right] \quad , \tag{8}$$

and

$$\rho_z(z, \delta) = \frac{1}{2\pi\sigma_z\sigma_\delta} \exp\left[-\frac{z^2}{2\sigma_z^2} - \frac{\delta^2}{2\sigma_\delta^2}\right] \quad , \tag{9}$$

where $\alpha_{x,y}$, $\beta_{x,y}$, and $\gamma_{x,y}$ are the storage ring lattice functions and $\epsilon_{x,y}$ is the rms emittance. Because of the dispersion, the actual transverse coordinates depend upon both the transverse betatron coordinates and the energy deviation: $x = x_\beta + \delta\eta_x$ and $x' = x'_\beta + \delta\eta'_x$.

At this point, we can calculate the integrals. To simplify the result, we use the integral identity

$$\frac{1}{\sqrt{\bar{\beta}^2}} = \int_0^\infty \frac{dx \exp\{-x\bar{\beta}^2\}}{\sqrt{\pi x}} \quad . \tag{10}$$

to remove the factor of $1/\bar{\beta}$. This yields

$$\nu(q > q_\star) = \frac{1}{q_\star^2} \frac{Ncr_0^2}{4\pi\gamma^2\, \epsilon_x\epsilon_y\, \sigma_z\sigma_\delta} \int_0^\infty \frac{dx}{\sqrt{x}} \left(x^3 + ux^2 + vx + w\right)^{-1/2} \tag{11}$$

where we have assumed that $q_{\max} \gg q_\star$ and

$$\begin{aligned} u &= \gamma^2\left(\frac{\mathcal{H}_x}{\epsilon_x} + \frac{1}{\sigma_\delta^2} + \frac{\beta_x}{\gamma^2\,\epsilon_x} + \frac{\beta_y}{\gamma^2\,\epsilon_y}\right) , \\ v &= \gamma^2\left(\frac{\mathcal{H}_x\beta_y}{\epsilon_x\epsilon_y} + \frac{\beta_x\beta_y}{\gamma^2\,\epsilon_x\epsilon_y} + \frac{\beta_x}{\sigma_\delta^2\,\epsilon_x} + \frac{\beta_y}{\sigma_\delta^2\,\epsilon_y} + \frac{\eta_x^2}{\epsilon_x^2}\right) , \\ w &= \gamma^2\left(\frac{\eta_x^2\,\beta_y}{\epsilon_x^2\epsilon_y} + \frac{\beta_x\beta_y}{\sigma_\delta^2\,\epsilon_x\epsilon_y}\right) \quad . \end{aligned} \tag{12}$$

The integral in Eq. (11) can be expressed in terms of the elliptic integral of the first kind $F(\phi, x)$. In general, the integrand does not factor simply, and the parameters are complex expressions of u, v, and w. In the case of flat beams, where $\epsilon_x \gg \epsilon_y$ and $\epsilon_x \sim \sigma_\delta^2 \eta_x^2/\beta_x$, we can approximate the elliptic function to find

$$\nu(q > q_\star) \sim \frac{1}{q_\star^2} \frac{cr_0^2}{3\gamma^3} \frac{N}{\sigma_x \, \sigma_y \, \sigma_z \, \sigma_{x'}} \quad . \tag{13}$$

We now need to consider the variation of the lattice parameters with azimuth. This will have two effects:

- first, the scattering rate will vary with position, and
- second, the variation will increase the contribution to the tails for a given average scattering rate.

The latter effect is not very significant in the longitudinal, but it can be important in the horizontal, because $\mathcal{H}_x$ can vary substantially. In lattices with large variation in $\mathcal{H}_x$, calculating $q_\star$ from the average rate will err on the conservative side, i.e., it will overestimate the growth of the core emittance. We can improve the result to some degree by including only regions where $\mathcal{H}_x \neq 0$ when calculating the average. Thus,

$$\overline{\nu_{\epsilon_x}} \, (q > q_\star) = \oint \frac{ds}{C} \, \nu(q > q_\star) \, , \quad \text{and} \quad \overline{\nu_{\epsilon_x}} \, (q > q_\star) = \int\limits_{\mathcal{H}_x \neq 0} \frac{ds}{C} \, \nu(q > q_\star) \quad , \tag{14}$$

where C is the ring circumference.

Finally, we use Eqs. (1) and (14) to find the boundary $q_\star$:

$$q_\star = \left[\frac{\tau_{SR} N c r_0^2}{4\pi\gamma^2 \, \epsilon_x \epsilon_y \, \sigma_z \sigma_\delta} \int \frac{ds}{C} \int\limits_0^\infty \frac{dx}{\sqrt{x}} \, \left(x^3 + ux^2 + vx + w\right)^{-1/2} \right]^{1/2} \quad , \tag{15}$$

Table 1. Examples of correction to IBS.

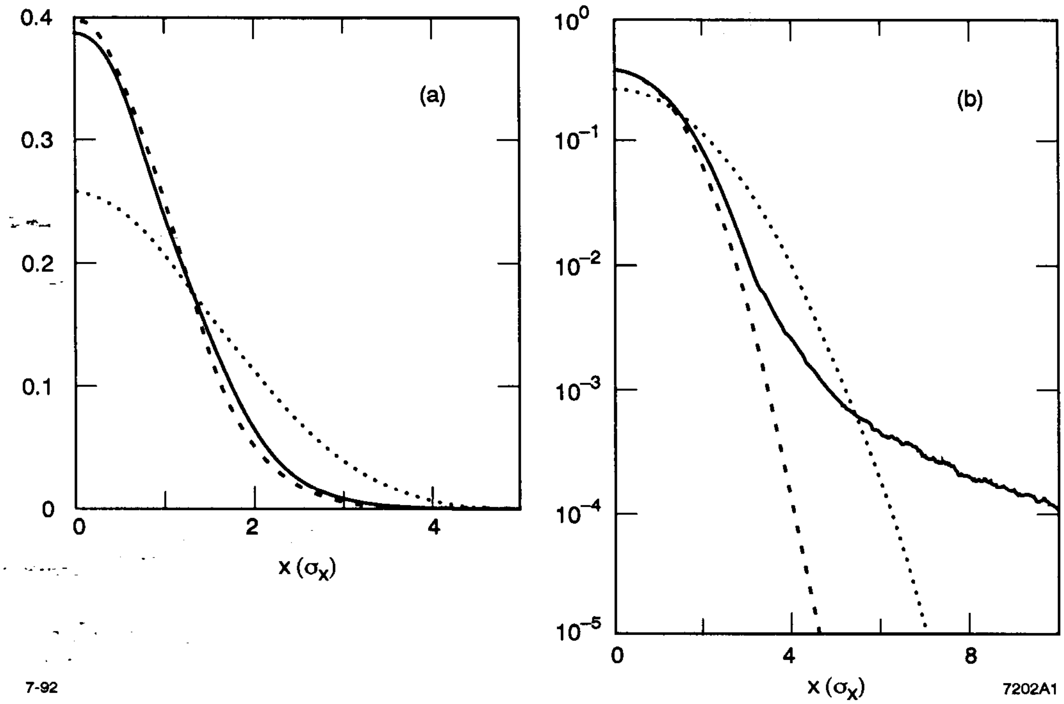
	NLC DR	ATF DR
$\gamma\epsilon_0$	2.0×10^{-6} m-rad	3.4×10^{-6} m-rad
$\Delta\epsilon_{IBS}/\epsilon_0$	37%	47%
$\Delta\epsilon_{IBS}/\epsilon_0$ with correction $\theta_{\rm max}^{\rm core}$	21%	30%

where the integral over the ring is taken around the entire ring in the longitudinal case, or just over the nondispersive regions in the transverse case. Unfortunately, neither the formalism of Piwinski nor that of Bjorken and Mtingwa allows us to simply use this limit. Instead, both are expressed in terms of the maximum angle or the minimum impact parameter. Since these limits only appear logarithmically, the calculation is not very sensitive, and we can calculate the maximum scattering angle using an rms value for $\bar{\beta}$: $\theta_{\rm max}^{\rm core} \sim q_\star/\bar{\beta}_{\rm rms}$.

4. EXAMPLES

We have used these results to calculate the correction to the intrabeam emittance growth in the Next Linear Collider (NLC) damping ring design [14] and the ATF damping ring design at KEK [15]; the results are listed in Table 1. In both cases, the intrabeam scattering calculation was performed using the formalism of Bjorken and Mtingwa [3] and the correction was calculated separately from Eq. (15). The intrabeam emittance growth is reduced by almost a factor of two, which occurs because the maximum scattering angle that contributes to the core is actually very small: $10^{-4} \sim 10^{-3}$.

Finally, we have simulated the beam distribution due to the intrabeam scattering in the NLC damping ring. In the simulations, synchrotron radiation damping



Figures 1(a) and (b). Simulation of the beam distribution with intra-beam scattering (solid), and the corrected (dashed) and uncorrected (dotted) Gaussian approximations. The rms of the simulated distribution equals that of the uncorrected Gaussian approximation, but the corrected Gaussian approximation describes the core of the beam well.

was included, but the excitation due to the radiation was not. We simulated 10^9 scattering events (ten particles for ten damping times). To keep the simulation times reasonable, the synchrotron radiation damping rate was artificially increased by a factor of 30. This slightly exaggerates the tails so that they contribute 52% of the uncorrected emittance, while with the actual NLC parameters the tails should contribute 44% of the uncorrected emittance.

The results are illustrated in a linear-linear plot in Fig. 1(a) and in a log-linear plot in Fig. 1(b). In both figures, the solid line is the simulated result while the dashed line is the Gaussian distribution calculated by neglecting the single scattering events, and the dotted curve is the uncorrected Gaussian distribution. The rms width of the simulation and the uncorrected Gaussian approximation are equal. One can see that the uncorrected result seriously over estimates the core emittance, while the corrected distribution describes the core of the beam very well. For these parameters, the corrected rms emittance growth is a factor of 2.1 smaller than the uncorrected result.

5. DISCUSSION

The current theories of intrabeam scattering neglect the distinction between the single scattering regime and the multiple scattering regime; the hard single scatterings cause tails on the beam distribution that heavily bias the rms emittance calculation. We have approximated the location of the transition between the two regimes and have described the necessary modifications to the current theories. Finally, we used these results to calculate the intrabeam scattering in two damping ring designs for future linear colliders. In both examples, we found substantially smaller equilibrium emittance increases, roughly a factor of two, when neglecting the tail contributions.

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