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Field Theory Without Feynman Diagrams: One-Loop Effective Actions

MATTHEW J. STRASSLER★

Stanford Linear Accelerator Center

Stanford University, Stanford, California 94309

ABSTRACT

In this paper the connection between standard perturbation theory techniques and the new Bern-Kosower calculational rules for gauge theory is clarified. For one-loop effective actions of scalars, Dirac spinors, and vector bosons in a background gauge field, Bern-Kosower-type rules are derived without the use of either string theory or Feynman diagrams. The effective action is written as a one-dimensional path integral, which can be calculated to any order in the gauge coupling; evaluation leads to Feynman parameter integrals directly, bypassing the usual algebra required from Feynman diagrams, and leading to compact and organized expressions. This formalism is valid off-shell, is explicitly gauge invariant, and can be extended to a number of other field theories.

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1. Introduction

In the past year significant advances have been made in techniques for calculating one-loop scattering amplitudes in gauge theories. Following on the successes of several authors at applying string theory and various technical innovations to tree-level gauge theory calculations[1,2,3,4], Z. Bern and D. A. Kosower have derived new rules from string theory for one-loop gauge theory scattering amplitudes. In reference 5, they present the derivation of the rules and apply them to the computation of two-to-two gluon scattering at one loop, which previously was difficult enough to challenge the most expert calculators.[9] In reference 6, they present their rules in a compact form and work a simple example. Although obtained from string theory, the Bern-Kosower rules do not refer to string theory in any way, but as they also bear little resemblance to Feynman rules, it is of interest to derive them directly from field theory. Bern and Dunbar[14] showed how to map the Bern-Kosower rules onto Feynman diagrams and demonstrated that the background field method plays an important role; in this paper I take the opposite route, deriving Bern-Kosower rules from the field theory path integral with the use of the background field method.

The main result of this paper is that calculational rules similar to those of Bern and Kosower can be derived from first-quantized field theory. Unlike the "connect-the-dots" approach of Feynman diagrams, first-quantized field theory (particle theory) views a particle in a loop as a single entity, acted on by operators representing the effects of external fields. We are all well-accustomed to this approach in atomic physics, where electromagnetic fields are treated as operators acting on quantum mechanical electrons, but to my knowledge it rarely been used for calculations with relativistic particles. (Feynman presented formulas similar to those discussed in this paper but did not use them to develop perturbation theory.[11]) In any case, it will not surprise those familiar with first-quantized strings that just as string theory amplitudes are evaluated as two-dimensional path integrals, so particle theory amplitudes can be calculated using one-dimensional path integrals — the path integrals of quantum mechanics.

In this paper I address the issue of the effective action at one loop. In section 2, I construct the one-loop effective action of a scalar particle in a background gauge field, and derive rules almost identical to those of Bern and Kosower. In sections 3 and 4 I generalize this approach to Dirac spinors and vector bosons. Section 5 contains a study of the integration-by-parts procedure involved in the Bern-Kosower rules, and an illustration of its relation to manifest gauge invariance. After a short comment (section 6) on an alternative organization of color traces in this formalism, I conclude in section 7 with some extensions of this approach to other field theories.

2. The Effective Action of a Scalar in a Background Field

In this section, I will show that the one-loop effective action of a particle in a background field, when written as a one-dimensional path integral, is calculable at any order in the coupling constant g. A particle in a loop can be described as a simple quantum mechanical system existing for a finite, periodic time, or, alternatively, as a one-dimensional field theory on a compact space; external fields act as operators on the particle Hilbert space, just as in usual quantum mechanics. At any order in the external field, the effective action is a correlation function of these operators in a free and therefore soluble theory, and can be expressed in a compact form. By writing the effective action as a one- rather than a four-dimensional path integral I employ quantum mechanics instead of quantum field theory; as string theory in its present form is a first-quantized theory, it is not especially surprising that the expressions found from string theory by Bern and Kosower are of the same form as those found in this paper.

Working initially in Euclidean spacetime, let us first consider the one-loop vacuum energy of a free scalar field, with Lagrangian

$$\mathcal{L} = -(\partial\phi^\dagger)\cdot(\partial\phi) - m^2\phi^\dagger\phi \; . \tag{2.1}$$

First represent it in terms of Schwinger proper time τ:[5,10]

$$\begin{aligned}\log Z &= \log\Big[\int \mathcal{D}\phi \; e^{i\int d^4x\mathcal{L}}\Big] = -\log\big[\det(-\partial^2+m^2)\big]\\ &= -\operatorname{Tr}\log(-\partial^2+m^2) = \int_0^\infty \frac{dT}{T}\int \frac{d^4p}{(2\pi)^4}\;\exp\big[-\frac{1}{2}\mathcal{E}T(p^2+m^2)\big] \; .\end{aligned} \tag{2.2}$$

The parameter $\mathcal{E}$ (the einbein) is an arbitrary constant. Next convert this result into a path integral over $x^\mu(\tau)$:

$$\begin{aligned}\log Z &= \int_0^\infty \frac{dT}{T}\int \mathcal{D}p\,\mathcal{D}x\;\exp\Big[\int_0^T d\tau\; ip\cdot\dot{x}\,\Big]\;\exp\Big[-\frac{1}{2}\mathcal{E}\int_0^T d\tau(p(\tau)^2+m^2)\,\Big]\\ &= \int_0^\infty \frac{dT}{T}\,\mathcal{N}\int_{x(T)=x(0)} \mathcal{D}x\;\exp\Big[-\int_0^T d\tau\big(\frac{1}{2\mathcal{E}}\dot{x}^2+\frac{\mathcal{E}}{2}m^2\big)\Big] \; ,\end{aligned} \tag{2.3}$$

where the normalization constant $\mathcal{N}$ is

$$\mathcal{N} = \int \mathcal{D}p\; e^{-\frac{1}{2}\int_0^T d\tau\mathcal{E}p^2} \tag{2.4}$$

and satisfies

$$\mathcal{N} \int \mathcal{D}x \; e^{-\int_0^T d\tau \frac{1}{2\mathcal{E}} \dot{x}^2} = \int \frac{d^D p}{(2\pi)^D} \; e^{-\frac{1}{2}\mathcal{E}Tp^2} = [2\pi \mathcal{E} T]^{-D/2} \; . \tag{2.5}$$

The result of (2.3) is a one-dimensional field theory: the particle position $x^\mu(\tau)$ is a set of four fields living in the one-dimensional space of proper time, called the worldline. Eq. (2.3) contains the well-known first-order form of the action for a free particle[12], which, unlike the usual Einstein action, is well defined in the massless limit:

$$L = \frac{1}{2\mathcal{E}} \dot{x}^2 \; . \tag{2.6}$$

(Since a massless particle has no internal clock, τ is not actually proper time in this case, though I will loosely continue to refer to it as such.) Classically, the action is reparametrization invariant (that is, invariant under $\tau \to \tau'(\tau)$) when the einbein, the square root of the one-dimensional metric, is chosen to transform in the proper way. On the other hand, the functional integral in (2.3) is not invariant unless one integrates over the einbein as well. In the present work I will keep $\mathcal{E}$ constant and ignore the reparametrization invariance, since it is not needed for practical results.

Now let us consider the same system (massless, for simplicity) in a classical background Abelian gauge field $A_\mu(x)$:

$$\mathcal{L} = \phi^\dagger D^2 \phi \tag{2.7}$$

where $D_\mu = \partial_\mu - igA_\mu$. The object of interest is the one-loop effective action generated by (2.7), as a function of A_μ. In analogy to eqs. (2.2)–(2.3),

$$\begin{aligned}
\Gamma[A] &= -\log\,[\det\,(-D^2)\,] \\
&= +\int_0^\infty \frac{dT}{T} \int \frac{d^4p}{(2\pi)^4}\ \langle p|\exp[-\frac{1}{2}\mathcal{E}T(p+gA)^2]\,|p\rangle \\
&= \int_0^\infty \frac{dT}{T}\,\mathcal{N} \int \mathcal{D}x\ \exp\Big[-\int_0^T d\tau(\frac{1}{2\mathcal{E}}\dot{x}^2\ +\ igA[x(\tau)]\cdot\dot{x})\Big]\ .
\end{aligned} \tag{2.8}$$

Continuing this result to Minkowski spacetime and redefining $\mathcal{E} \to -\mathcal{E}$ gives

$$\begin{aligned}
\Gamma[A] &= \int_0^\infty \frac{dT}{T}\,\mathcal{N} \int \mathcal{D}x\ \exp\Big[-\int_0^T d\tau(\frac{1}{2\mathcal{E}}\dot{x}^2\ -\ igA[x(\tau)]\cdot\dot{x})\Big] \\
&= \int_0^\infty \frac{dT}{T}\mathcal{N} \int \mathcal{D}x\ e^{-\int_0^T d\tau(\frac{1}{2\mathcal{E}}\dot{x}^2)}\ \exp[ig\oint dx\cdot A(x)]\ .
\end{aligned} \tag{2.9}$$

This expression is immediately recognizable as the expectation value of a Wilson loop of the background field, in a certain ensemble of loops. It is therefore explicitly gauge invariant with respect to the background gauge field, as it should be.

The non-Abelian generalization of this structure is easy to guess; one merely inserts a trace over color states:

$$\Gamma[A] = \int_0^\infty \frac{dT}{T}\,\mathcal{N} \int \mathcal{D}x\ \mathrm{Tr}_{\mathrm{R}} \exp\Big[-\int_0^T d\tau(\frac{1}{2\mathcal{E}}\dot{x}^2\ -\ igA[x(\tau)]\cdot\dot{x})\Big]\ , \tag{2.10}$$

where the gauge field is a matrix $A_\mu^a T^a$ in the gauge group representation R of the scalar. Notice that the usual path-ordering in the Wilson loop appears here as proper-time–ordering, implicit in the path integral construction.

Let us now consider the expansion of this effective action to order g^N, which is equivalent to studying the one-particle-irreducible (1PI) Feynman diagrams with N background gluons and one scalar loop. (By "gluon" I mean any non-abelian vector boson.) In the standard Feynman graph technique there are a number of such diagrams, involving both the one-gluon/two-scalar vertex and the two-gluon/two-scalar vertex. Here, there is only *one* computation. We expand the Wilson loop to order g^N:

$$\Gamma_N[A] = \frac{(ig)^N}{N!} \int_0^\infty \frac{dT}{T} \mathcal{N} \int \mathcal{D}x \; e^{-\int_0^T d\tau \frac{1}{2\varepsilon}\dot{x}^2} \mathrm{Tr}\Big(\prod_{i=1}^N \int_0^T dt_i A[x(t_i)] \cdot \dot{x}(t_i) \Big) \, . \quad (2.11)$$

Up to this point the background field is completely arbitrary. To compute $\Gamma_N[A]$ as a function of momentum eigenstates, we insert for A_μ a sum of classical modes of definite (outgoing) momentum k_i, polarization ϵ_i, and gauge charge T^{a_i}:

$$A^\mu(x) = \sum_{i=1}^N T^{a_i} \epsilon_i^\mu e^{ik_i \cdot x} \quad (2.12)$$

Again T^{a_i} is a matrix in the representation of the scalar. Inserting this function into (2.11) and keeping only the terms in which each mode appears precisely once, we find:

$$\begin{aligned} \Gamma_N(k_1, \dots, k_N) = (ig)^N \int_0^\infty \frac{dT}{T} \mathcal{N} \int \mathcal{D}x \; e^{-\int_0^T d\tau \frac{1}{2\varepsilon}\dot{x}^2} \\ \mathrm{Tr}(T^{a_N} \dots T^{a_1}) \prod_{i=1}^N \int_0^{t_{i+1}} dt_i \; \epsilon_i \cdot \dot{x}(t_i) e^{ik_i \cdot x(t_i)} \end{aligned} \quad (2.13)$$

plus terms with all other orderings of the t_i and T^{a_i}. (Here $t_{N+1} \equiv T$.) Notice that for a given integration ordering (= path-ordering around the loop = proper-time-ordering = color-trace-ordering), the color information factors out, as is well-known in open string theory and tree-level Feynman diagrams. In string theory the color trace is known as a Chan-Paton factor.[13] The utility of computing color-ordered tree-level partial amplitudes using color-ordered Feynman diagrams was emphasized by Mangano, Parke and Xu[2]; a study of color-ordering in loop graphs was performed by Bern and Kosower.[7] For pure vector field backgrounds, only one color-ordering is actually necessary, as all other orderings are related to it by permutation of labels; because of this, I will consider for the remainder of this paper only one color ordering at a time, leaving the sum over color orderings implicit.

String theorists will immediately recognize eq. (2.13); the string theory version of this formula gives the expectation value of N "vertex operators", which in string theory can be interpreted as a *scattering amplitude* of N strings. For strings, duality of the s and t channels implies that not only the one-particle irreducible loop but also the *trees* which are sewn onto the loop are calculated in this way. In particle theory, however, eq. (2.13) computes *only the effective action*, the one particle irreducible graphs with a scalar loop, at order g^N. Still, it has the advantage of being well-defined even for off-shell external gauge fields, unlike usual string theory.

To calculate this expectation value I use the standard path integral methods of string perturbation theory.[12] First, disregard the polarization vectors, and notice that the momenta k_i in (2.13) serve as sources of the four fields $x^\mu(\tau)$:

$$J^\mu(\tau) = \sum_{j=1}^{N} i\ k_j^\mu \delta(\tau - t_j) \tag{2.14}$$

Using eq. (2.5), we find

$$
\begin{aligned}
&\Gamma_N(k_1,\dots,k_N)\\
&=\frac{(ig)^N}{(4\pi)^2(\mathcal{E}/2)^2}\mathrm{Tr}(T^{a_N}\dots T^{a_1})\int_0^\infty \frac{dT}{T^3}\prod_{i=1}^{N}\int_0^{t_{i+1}} dt_i\\
&\qquad \exp[\int_0^T d\tau\int_0^T d\tau' - \frac{1}{2}J^\mu(\tau)G_B(\tau,\tau')J_\mu(\tau')]\\
&=\frac{(ig)^N}{(4\pi)^2(\mathcal{E}/2)^2}\mathrm{Tr}(T^{a_N}\dots T^{a_1})\int_0^\infty \frac{dT}{T^3}\prod_{i=1}^{N}\Big(\int_0^{t_{i+1}} dt_i\Big)\\
&\qquad \exp\Big[\sum_{i,j=1}^{N}\frac{1}{2}k_i\cdot k_j G_B(t_i,t_j)\Big]\;.
\end{aligned}
\tag{2.15}
$$

Here $G_B(t,t')$ is the one-dimensional propagator on a loop, which I will discuss later. (The B indicates that G_B is the Green function of the Bosonic field x^μ.)

The standard method for including the polarization vectors is to exponentiate them, with the understanding that the only terms to be used are those which contain one ϵ_i:

$$
\dot{x}\cdot A_i^\mu(x(t_i)) = T^{a_i}\exp\left[\epsilon_i\cdot\partial_{t_i}x(t_i)+ik_i\cdot x(t_i)\right]\Big|_{\text{linear in }\epsilon_i}
\tag{2.16}
$$

This leads to a new source for x^μ:

$$
J^\mu(\tau)=\sum_{1}^{N}\delta(\tau-t_i)\,(\epsilon_i^\mu\partial_{t_i}+ik_i^\mu)\;.
\tag{2.17}
$$

Integration over $x(\tau)$ gives

$$\Gamma_N(k_1,\ldots,k_N)=\frac{(ig)^N}{(4\pi)^2(\mathcal{E}/2)^2}\mathrm{Tr}(T^{a_N}\ldots T^{a_1})\int_0^\infty\frac{dT}{T^3}\Big(\prod_{i=1}^N\int_0^{t_{i+1}}dt_i\Big)$$
$$\exp\Big[\frac{1}{2}\sum_{i,j=1}^N\big(k_i\cdot k_jG_B(t_j-t_i)\big]$$
$$-\ 2ik_i\cdot\epsilon_j\frac{\partial}{\partial t_j}G_B(t_j-t_i)$$
$$-\ \epsilon_i\cdot\epsilon_j\frac{\partial^2}{\partial t_i\partial t_j}G_B(t_j-t_i))\Big]\Big|_{\text{linear in each }\epsilon}\ ; \tag{2.18}$$

again only terms in which each polarization vector appears exactly once are to be used. String theorists and those familiar with the work of Bern and Kosower[5] will recognize this form for the amplitude.

Now let us study the Green function (one-dimensional propagator), which satisfies the equation

$$\frac{1}{\mathcal{E}}\partial_t^2G_B(t,t')=\delta(t-t') \tag{2.19}$$

with appropriate boundary conditions. If we were studying this Green function on the real line, the solution would be

$$G_B(t,t')=\frac{\mathcal{E}}{2}|t-t'|\ +\ A\ +\ Bt\ . \tag{2.20}$$

Notice that the Green function is finite as t approaches t', which is not true for higher dimensions; thus there are no operator singularities when x fields come together. This naturally simplifies many discussions.

To find the Green function on a circle of circumference T, one must first note that eq. (2.19) has no solution on the loop; it is equivalent to solving Poisson's equation for a charge in a compact space, for which the potential is infinite unless there is a background charge that makes the total space neutral. Since we have one unit of charge at t', we should add a uniform background charge of density $-1/T$. The new Green function equation is

$$\frac{1}{\mathcal{E}}\partial^2 G_B(t,t') = \delta(t-t') - \frac{1}{T} \ , \tag{2.21}$$

which has a solution when the condition of periodicity in $t \to t+T$ is imposed:

$$G_B(t,t') = \frac{\mathcal{E}}{2}\Big(|t-t'| - (t-t')^2/T\Big) \ + \ \text{constant} \ . \tag{2.22}$$

It is convenient to take the arbitrary constant to be zero, as any additive constant in G_B cancels out of eq. (2.18). This function has as its derivative

$$\partial_t G_B(t,t') = \frac{\mathcal{E}}{2}(\text{sign}(t-t') - 2(t-t')/T) \ , \tag{2.23}$$

and its second derivative is given in eq. (2.21). Note that G_B and $\partial_t^2 G_B$ are symmetric in their arguments, while $\partial_t G_B$ is antisymmetric. These functions (up to a multiplicative constant) were found by Bern and Kosower[5] from the one-loop string theory bosonic Green function and its derivatives, in the limit where $t-t'$ is large compared to the width of the string theory torus. Roughly adhering to their conventions, I shall use the notation $G_B^{ji} \equiv G_B(t_j - t_i)$, $\dot{G}_B^{ji} \equiv \partial_{t_j} G_B^{ji}$, and $\ddot{G}_B^{ji} \equiv \partial_{t_j}^2 G_B^{ji}$.

It is useful to transform eq. (2.18) into a simpler form. First, through the use of the crucial relations $G_B(t,t) = 0$ and (by antisymmetry in t and t') $\partial_t G_B(t,t) \equiv 0$, the terms in (2.18) with $\epsilon_i \cdot k_i$ and k_i^2 are removed *without* the use of on-shell conditions. Second, it is useful to replace $t_i \to u_i T$, where u_i is dimensionless; N powers of T are thereby factored out. Next, observe that the integral over u_N is trivial; after the first $N-1$ integrals no dependence on the u_i remains, and so the last integral, which contributes a factor of unity, can be dropped. It is useful to choose the origin of proper time by fixing $t_N \equiv T$, and as a consequence we should sum only over color traces which are not related by cyclic permutation. A further advantage is gained by choosing the (dimensionless) gauge $\mathcal{E} = 2$. Lastly, anticipating the use of dimensional regularization, I redo the integral over momentum in $4-\epsilon$ dimensions as in eq. (2.5). (For the remainder of this paper, the conventions chosen above will be used except where explicitly noted.)

The result of all these changes is

$$
\begin{aligned}
\Gamma_N(k_1,\ldots,k_N) = & \frac{(ig\mu^{\epsilon/2})^N}{(4\pi)^{2-\epsilon/2}} \mathrm{Tr}(T^{a_N}\ldots T^{a_1}) \\
& \int_0^\infty \frac{dT}{T^{3-N-\epsilon/2}} \int_0^1 du_{N-1} \int_0^{u_{N-1}} du_{N-2} \cdots \int_0^{u_2} du_1 \\
& \exp\Big[\sum_{i<j=1}^{N} k_i \cdot k_j G_B^{ji}\Big] \\
& \exp\Big[\sum_{i<j=1}^{N} \big(-\ i(k_i\cdot\epsilon_j - k_j\cdot\epsilon_i)\ \dot{G}_B^{ji} +\ \epsilon_i\cdot\epsilon_j\ \ddot{G}_B^{ji}\big)\Big]\Big|_{\text{linear in each } \epsilon}\ ;
\end{aligned} \tag{2.24}
$$

plus all other proper-time-orderings. Meanwhile the Green functions have become

$$
\begin{aligned}
G_B(t,t') &\equiv T\Big(|u-u'| - (u-u')^2\Big)\ ; \\
\partial_t G_B(t,t') &\equiv \big(\mathrm{sign}(u-u') - 2(u-u')\big)\ ; \\
\partial_t^2 G_B(t,t') &\equiv \frac{2}{T}\big(\delta(u-u') - 1\big)\ .
\end{aligned} \tag{2.25}
$$

Comparison with reference 5 or 6 shows that the correspondence between eqs. (2.24) and (2.25) and the Bern-Kosower rules for the one-particle-irreducible scalar loop diagram with N gluons is exact, up to differences in conventions.

Following Bern and Kosower[6], let us study the result of (2.24). The overall constant factor, the color trace and the integrals are easy to understand. The exponential

$$\exp\Big[\sum_{i<j=1}^{N} k_i\cdot k_j G_B^{ji}\Big] = \exp\Big[T\sum_{i<j=1}^{N} k_i\cdot k_j\Big(|u_j-u_i|-(u_j-u_i)^2\Big)\Big] \tag{2.26}$$

is a ubiquitous factor which, after the integration over T, becomes the usual Feynman parameterized denominator for a scalar loop integral (notice it contains no polarization vectors, and is thus spin-independent):

$$\int_0^\infty dT\, T^\alpha \exp\Big[\sum_{i<j=1}^{N} k_i\cdot k_j G_B^{ji}\Big] = \frac{\Gamma(\alpha+1)}{\Big[-\sum_{i<j=1}^{N} k_i\cdot k_j\Big(|u_j-u_i|-(u_j-u_i)^2\Big)\Big]^{\alpha+1}}\,. \tag{2.27}$$

The remaining term,

$$\exp\Big[\sum_{i<j=1}^{N}\big(-\,i(k_i\cdot\epsilon_j-k_j\cdot\epsilon_i)\,\dot G_B^{ji}+\,\epsilon_i\cdot\epsilon_j\,\ddot G_B^{ji}\big)\Big]\Big|_{\text{linear in each }\epsilon}\,, \tag{2.28}$$

which I shall call the "generating kinematic factor", provides the numerator of the Feynman parameter integral. It is the only part of (2.24) (other than the overall normalization) which has any information about the type of particle in the loop or the nature of the external field. It is also the only part of the result which cannot be guessed on general grounds; we undergo the usual struggles with Feynman diagrams and loop momentum integrals in order to obtain precisely this piece of information.

However, the form of the generating kinematic factor causes some practical problems. At first glance (2.24) appears to have expressed the entire result in such a way that one has exactly one set of Feynman parameter integrals for each color trace, but this is not quite true. The difficulties stem from the $\ddot{G}_B$ functions. The first problem is that each term with M $\ddot{G}_B$'s has M fewer powers of T than terms without $\ddot{G}_B$'s, so a number of different integrals over T must be performed. The second problem is that hiding inside each $\ddot{G}_B^{ji}$ is a delta function in $t_j - t_i$. The evaluation of this delta function gives the contribution of the Feynman diagram in which gluons i and j come onto the loop *via* a four-point vertex. Thus the expression in eq. (2.24) contains all of the 1PI Feynman diagrams, in fact, and each one generates slightly different integrals and integrands. (Fortunately, these problems can be dealt with[6], as I will discuss in section 5.)

There is a subtle factor of two concerning the delta function in $\ddot{G}_B^{ij}$. Consider smoothing out the singularity slightly; then, in order to maintain the symmetries of G_B and its derivatives only half of the delta function actually contributes to a given color trace. In other words, one must be careful to assign half of the delta function in $\ddot{G}_B^{ij}$ to $\mathrm{Tr}(\cdots T^{a_i}T^{a_j}\cdots)$ and the other half to $\mathrm{Tr}(\cdots T^{a_j}T^{a_i}\cdots)$.

I now present the simplest possible example, the contribution of a massless scalar to the gluon vacuum polarization. There are two Feynman diagrams, the first of which involves two three-point vertices, the other of which involves a single four-point vertex. The former is given by

$$(ig)^2\mathrm{Tr}(T^aT^b)\int \frac{d^Dp}{(2\pi)^D}\frac{(i)^2\epsilon_1\cdot(2p-k_1)\epsilon_2\cdot(2p-k_1)}{p^2(p-k_1)^2} \tag{2.29}$$

where k_1 is the momentum flowing out along gluon 1. The second diagram is given by

$$2ig^2\mathrm{Tr}(T^aT^b)\int \frac{d^Dp}{(2\pi)^D}\frac{i\epsilon_1\cdot\epsilon_2}{p^2}\ . \tag{2.30}$$

I now use the Schwinger trick[10] to evaluate (2.29) in a form conducive to comparison with the expression in (2.24).

$$\begin{aligned}
&\int \frac{d^Dp}{(2\pi)^D}\frac{-\epsilon_1\cdot(2p-k_1)\epsilon_2\cdot(2p-k_1)}{p^2(p-k_1)^2}\\
&\quad=\int_0^\infty TdT\int_0^1 da\int\frac{d^Dp}{(2\pi)^D}\\
&\qquad\Big[-\epsilon_1\cdot(2\partial_v-k_1)\epsilon_2\cdot(2\partial_v-k_1)\Big]e^{-T[p^2+a(k_1^2-2p\cdot k_1)]}e^{v\cdot p}\Big|_{v=0}\\
&\quad=\int_0^\infty TdT\int_0^1 da\Big[-\epsilon_1\cdot(2\partial_v-k_1)\epsilon_2\cdot(2\partial_v-k_1)(e^{ak_1\cdot v+v^2/4T})\Big]\Big|_{v=0}\\
&\qquad\times\ e^{-Tk_1^2(a-a^2)}\int\frac{d^Dp'}{(2\pi)^D}e^{-Tp'^2}\ .
\end{aligned} \tag{2.31}$$

Carrying out the derivatives and the integral over momentum, and adding to this expression the contribution of (2.30), we are left with

$$\begin{aligned}
\Pi =&\frac{-(g\mu^{\epsilon/2})^2}{(4\pi)^{2-\epsilon/2}}\mathrm{Tr}(T^aT^b)\int_0^\infty\frac{dT}{T^{1-\epsilon/2}}\\
&\Big\{\Big[\int_0^1 da\Big(-\frac{2}{T}\epsilon_1\cdot\epsilon_2-(1-2a)^2\epsilon_1\cdot k_1\epsilon_2\cdot k_1\Big)e^{-Tk_1^2(a-a^2)}\Big]\\
&\qquad+\frac{2}{T}\epsilon_1\cdot\epsilon_2\Big\}\ ,
\end{aligned} \tag{2.32}$$

where $\epsilon = 4-D$.

Alternatively we may write down the result of (2.24) for $N = 2$:

$$\Gamma_2(k_1, k_2) = \frac{(ig\mu^{\epsilon/2})^2}{(4\pi)^{2-\epsilon/2}} \mathrm{Tr}(T^a T^b) \int_0^\infty \frac{dT}{T^{1-\epsilon/2}} \int_0^1 du \; e^{k_1 \cdot k_2 G_B(1-u)} \\ \left[k_2 \cdot \epsilon_1 k_1 \cdot \epsilon_2 [\dot{G}_B(1-u)]^2 + \; \epsilon_1 \cdot \epsilon_2 \; \ddot{G}_B(1-u))\right] . \tag{2.33}$$

Define $a = 1 - u$, plug in the functions in (2.25), and the result appears:

$$\Pi = \frac{-(g\mu^{\epsilon/2})^2}{(4\pi)^{2-\epsilon/2}} \mathrm{Tr}(T^a T^b) \int_0^\infty \frac{dT}{T^{1-\epsilon/2}} \int_0^1 da \; e^{Tk_1 \cdot k_2 (a-a^2)} \\ \left[\frac{2}{T}(\delta(a) - 1)\epsilon_1 \cdot \epsilon_2 + (1-2a)^2 \epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1\right] . \tag{2.34}$$

Note that, as advertised, the diagram involving a four-point vertex (eq. (2.30)) is found by evaluating the delta function in (2.34); since $\mathrm{Tr}(T^a T^b) = \mathrm{Tr}(T^b T^a)$ this trace receives the full contribution of the delta function. This example also makes clear that, as explained by Bern and Kosower[5], the differences $u_i - u_j$ are directly related to the usual Feynman parameters.

3. The Effective Action of a Spinor Particle in a Background Field

The case of a spinning particle is a simple generalization of the particle theory used in section 2. The one-loop action of a Dirac spinor with a vector-like coupling to a background field is

$$S = \int d^4x \; \overline{\chi}(i \not{D} - m)\chi \tag{3.1}$$

where $D_\mu = \partial_\mu - igA_\mu$. The one-loop effective action as a function of A_μ is therefore

$$\begin{aligned}\Gamma[A] &= \log\left[\det\ (i\not{D} - m)\right] \\ &= \frac{1}{2}\log\left[\det(i\not{D} - m)\ \det(-i\not{D} - m)\right] \\ &= \frac{1}{2}\log\left[\det\ \left(D^2\mathbf{1}\ -\ \frac{ig}{4}F_{\mu\nu}[\gamma^\mu,\gamma^\nu] + m^2\right)\right].\end{aligned} \tag{3.2}$$

where I use $\det(\not{D}) = \det(\gamma^5\ \not{D}\gamma^5) = \det(-\ \not{D})$. This expression for the effective action is also associated with the second order action for a Dirac spinor

$$S = \int d^4x - \frac{1}{m}\chi_L^\dagger(\not{D}^2 + m^2)\chi_R \tag{3.3}$$

where the $\frac{1}{2}$ in (3.2) appears because $\chi_{L,R}$ are two-component Weyl spinors. The relevance of these formulas to the Bern-Kosower rules was noted by Bern and Dunbar.[14]

Since the gamma matrices are anticommuting operators, it is natural to introduce worldline fermions to represent them. This technique has long been employed to introduce spin[15,17,18], and even color[19], into quantum mechanics. There is nothing mysterious about this; finite representations of compact groups can be generated by a set of fermionic operators.

One may therefore implement a supersymmetric generalization of the procedure outlined in eq. (2.8), introducing Grassmann fields $\psi^\mu(\tau)$ as partners of the fields $x^\mu(\tau)$. I will want the usual fermionic anticommutation relations

$$\{\psi^\mu, \psi^\nu\} = g^{\mu\nu}\ , \tag{3.4}$$

which imply that as operators the ψ^μ fields are just constants equal to $\sqrt{\frac{1}{2}}\gamma_\mu$, and I take as the Hilbert space of the theory the four components $|\alpha\rangle$ of the Dirac fermion, which are acted on in the usual way by the ψ fields:

$$\psi^\mu \left|\alpha\right\rangle = \frac{1}{\sqrt{2}}\gamma^\mu_{\alpha\beta}\left|\beta\right\rangle \ . \tag{3.5}$$

I will now evaluate (3.2) (in the massless case) as in section 2, taking the worldline fermions to have the usual antiperiodic boundary conditions. (One need consider periodic boundary conditions only for chiral fermions[20].) Direct construction of the particle path integral leads to

$$\begin{aligned}
\Gamma[A] &= \frac{1}{2}\mathrm{Tr}\log\,\Big[D^2\mathbf{1} \ - \ \frac{ig}{4}F_{\mu\nu}[\gamma^\mu,\gamma^\nu]\Big] \\
&= -\frac{1}{2}\int_0^\infty \frac{dT}{T}\sum_\alpha \int \frac{d^4p}{(2\pi)^4} \\
&\quad \langle\alpha,p|\exp\big[-\frac{1}{2}\mathcal{E}T\{(p+gA)^2 + igF_{\mu\nu}\psi^\mu\psi^\nu\}\big]\,|\alpha,p\rangle \\
&= -\frac{1}{2}\int_0^\infty \frac{dT}{T}\mathcal{N}\int \mathcal{D}x\ \mathcal{D}\psi \\
&\quad \mathrm{Tr}\ \exp\big[-\int_0^T d\tau(\frac{1}{2\mathcal{E}}\dot{x}^2 \ + \frac{1}{2}\psi\cdot\dot{\psi} - igA_\mu\dot{x}^\mu + ig(\mathcal{E}/2)\psi^\mu F_{\mu\nu}\psi^\nu)\big] \ .
\end{aligned} \tag{3.6}$$

The abelian version of this action was first presented by Brink, Di Vecchia, and Howe[15]; the nonabelian case was discussed by several authors.[16]

In this way, the effective action for a spinor is expressed as a *supersymmetric* Wilson loop, in a free supersymmetric theory. The particle action is invariant under the transformation

$$\delta_\eta x^\mu = -\mathcal{E}\eta\psi^\mu \ ; \ \delta_\eta\psi^\mu = \eta\dot{x}^\mu \ . \tag{3.7}$$

This supersymmetry and the superfield formulation of this theory have been addressed by many authors, for example in reference 15; I will not discuss it further in this work.

Now let us consider the effective action (3.6) at order g^N. For the moment I shall ignore the $[A_\mu, A_\nu]$ term in $F_{\mu\nu}$; I will return to it at the end of this section. Expanding for the moment only the terms with a single power of the gauge field to order N, and inserting the momentum eigenstates of eq. (2.12), one finds

$$
\begin{aligned}
\Gamma^0[A] &= -\frac{1}{2}\frac{(ig)^N}{N!}\int_0^\infty \frac{dT}{T}\,\mathcal{N}\int \mathcal{D}x\;\mathcal{D}\psi\,\exp[-\int_0^T d\tau(\frac{1}{2\mathcal{E}}\dot{x}^2+\frac{1}{2}\psi\cdot\dot{\psi})]\\
&\qquad \mathrm{Tr}\prod_{i=1}^{N}\int_0^T dt_i\,\Big\{A_\mu[x(t_i)]\cdot\dot{x}^\mu(t_i)\ -\mathcal{E}\psi^\mu(t_i)\partial_\mu A_\nu[x(t_i)]\cdot\psi^\nu(t_i)\Big\}\\
&= -\frac{1}{2}(ig)^N\int_0^\infty \frac{dT}{T}\,\mathcal{N}\int \mathcal{D}x\;\mathcal{D}\psi\,\exp[-\int_0^T d\tau(\frac{1}{2\mathcal{E}}\dot{x}^2+\frac{1}{2}\psi\cdot\dot{\psi})]\\
&\qquad \mathrm{Tr}\prod_{i=1}^{N}\int_0^T dt_i\,T^{a_i}\,\left[\epsilon_i\cdot\partial_i x(t_i)\ +i\mathcal{E}\epsilon_i\cdot\psi(t_i)\ k_i\cdot\psi(t_i)\right]e^{ik_i\cdot x(t_i)}
\end{aligned}
\tag{3.8}
$$

(I write Γ^0 to remind the reader that I have left out the commutator term in $F_{\mu\nu}$.) Here string theorists will find the vertex operators for vector fields used in the superstring.

Again we can put the polarization vectors in the exponentials; using Grassmann variables θ and $\overline{\theta}$, we may write

$$
\begin{aligned}
\mathcal{V} &\equiv igT^a\,\left[\epsilon\cdot\dot{x}+i\mathcal{E}\epsilon\cdot\psi\ k\cdot\psi\right]e^{ik\cdot x}\\
&= igT^a\int d\theta d\overline{\theta}\exp\left[\overline{\theta}\theta\epsilon\cdot\dot{x}+\theta\sqrt{\mathcal{E}}\epsilon\cdot\psi+i\overline{\theta}\sqrt{\mathcal{E}}k\cdot\psi+ik\cdot x\right]\,.
\end{aligned}
\tag{3.9}
$$

This leads to sources for x^μ

$$
J^\mu(\tau)=\sum_1^N\ \delta(\tau-t_i)(\overline{\theta}_i\theta_i\epsilon_i^\mu\partial_{t_i}+ik_i^\mu)\,.
\tag{3.10}
$$

and ψ^μ

$$\eta^\mu(\tau,\theta,\bar\theta) = \sum_1^N \delta(\tau - t_i)\sqrt{\mathcal{E}}(\theta_i\epsilon_i + i\bar\theta_i k_i) \ . \tag{3.11}$$

The result of carrying out the x and ψ integrals (in the gauge $\mathcal{E} = 2$) is

$$\begin{aligned}
\Gamma_N^0(k_1,\dots,k_N) = -4\,\frac{(ig)^N}{2(4\pi)^2}\mathrm{Tr}(T^{a_N}\dots T^{a_1})\int_0^\infty \frac{dT}{T^{3-N}}\Big(\prod_{i=1}^N \int_0^{u_{i+1}} du_i\Big) \\
\exp\Big(\sum_{i<j=1}^N k_i\cdot k_j G_B^{ji}\Big)\Big\{\Big(\prod_{i=1}^N\int d\theta_i d\bar\theta_i\Big) \\
\exp\Big(\sum_{i<j=1}^N \big(-i\,(\bar\theta_j\theta_j k_i\cdot\epsilon_j - \bar\theta_i\theta_i k_j\cdot\epsilon_i)\dot G_B^{ji} \\
+\bar\theta_i\theta_i\bar\theta_j\theta_j\epsilon_i\cdot\epsilon_j\ddot G_B^{ji}\big)\Big) \\
\exp\Big(\sum_{i<j=1}^N\big[-\bar\theta_i\bar\theta_j k_i\cdot k_j + i\bar\theta_i\theta_j k_i\cdot\epsilon_j \\
+ i\theta_i\bar\theta_j\epsilon_i\cdot k_j + \theta_i\theta_j\epsilon_i\cdot\epsilon_j\big]G_F^{ji}\Big)\Big\}\ ,
\end{aligned} \tag{3.12}$$

plus terms involving all other proper-time/color orderings. The overall factor of four comes from

$$\int \mathcal{D}\psi\; e^{-\int_0^T d\tau\frac{1}{2}\psi\cdot\dot\psi} = \mathrm{Tr}_\psi\mathbf{1} = \sum_{\alpha=1}^4\langle\alpha|\alpha\rangle \ . \tag{3.13}$$

The generating kinematic factor (in braces) has a bosonic part identical to (2.28), as well as terms that contain the one-loop Green functions G_F ($G_F^{ji} = -G_F^{ij} \equiv G_F(t_j - t_i)$) of the fermionic ψ fields. In addition to implementing the constraint that every polarization vector appears exactly once, the Grassmann integrations over θ and $\bar\theta$ ensure that in any term of the generating kinematic factor in which $\epsilon_i^\mu G_F^{ij}$ appears, $k_i^\nu G_F^{ik}$ must also appear. This implies that the G_F functions always occur in closed chains of the form

$$\prod_{k=1}^{d} G_F^{i_{k+1},i_k} \; ; \; (i_{d+1} \equiv i_1) \; . \tag{3.14}$$

(As the G_F's are antisymmetric in their arguments, a term like $G_F^{12}G_F^{13}G_F^{23}$ is not ruled out; on the contrary, it is equal to $-G_F^{12}G_F^{23}G_F^{31}$ which is of the form (3.14).)

The bosonic part of the action in (3.6) is the same as in section 2, so the G_B functions are again given by eq. (2.25). The G_F functions satisfy

$$\frac{1}{2}\partial_t G_F(t,t') = \delta(t-t'). \tag{3.15}$$

Since the fermions also satisfy antiperiodic boundary conditions

$$\psi(t \to T) = -\psi(t \to 0) \; , \tag{3.16}$$

we take the antiperiodic solution of eq. (3.15):

$$G_F(t,t') = \text{sign}(t-t') = \text{sign}(u-u'). \tag{3.17}$$

This function is double-valued, since it changes sign only at $t = t'$:

$$G_F(t,t') = -G_F(t+T,t'). \tag{3.18}$$

If the theory is abelian, then the single expression (3.12) contains the entire one-loop effective action (which is also the full photon one-loop S-matrix.) However, if we are working in a non-abelian gauge theory, then in addition to the expression given in (3.12) for the effective action we must include terms involving the quadratic term in $F_{\mu\nu}$,

$$-\frac{1}{2}g^2\mathcal{E}\psi^\mu[A_\mu, A_\nu]\psi^\nu \ , \tag{3.19}$$

which generates two-gluon vertex operators of the form

$$\mathcal{O}_{i,j} = -g^2\mathcal{E}(T^{a_j}T^{a_i})\ \epsilon_j\cdot\psi\ \epsilon_i\cdot\psi\ e^{i(k_i+k_j)\cdot x} \ . \tag{3.20}$$

In the second-order formalism for spinors in gauge fields, the usual three-point vertices are replaced by vertices with not only one gluon but also two, in analogy with scalars in gauge fields. This can be inferred from eq. (3.3). As in the previous section, a part of the two-gluon/two-spinor vertex is associated with the delta function in $\ddot{G}_B$, but because of the particle's spin this vertex contains a new piece generated by the operator $\mathcal{O}_{i,j}$.

The contribution of this operator can be evaluated through a process known as "pinching", which is related to the Bern-Kosower rules for trees attached to loops. In this process gluons i and j are brought to the same point on the loop ("pinched"), and a subsidiary "pinched kinematic factor", containing the contribution of $\mathcal{O}_{i,j}$, is extracted from the generating kinematic factor in a systematic way. The reader may wish to review the Bern-Kosower rules[6], which serve as motivation for the following unusual manipulation of (3.20):

$$
\begin{aligned}
\mathcal{O}_{i,j} = & -g^2(T^{a_j}T^{a_i})\int d\theta_i d\bar{\theta}_i d\theta_j d\bar{\theta}_j \\
& \qquad (-\bar{\theta}_i\bar{\theta}_j)\exp\left[\theta_i\sqrt{\mathcal{E}}\epsilon_i\cdot\psi(t_i)+\theta_j\sqrt{\mathcal{E}}\epsilon_j\cdot\psi(t_j)+i(k_i+k_j)\cdot x\right]\Big|_{t_i=t_j} \\
= & \,(ig)^2(T^{a_j}T^{a_i})\int d\theta_i d\bar{\theta}_i d\theta_j d\bar{\theta}_j \\
& \qquad \left.\frac{\exp\left[\theta_i\sqrt{\mathcal{E}}\epsilon_i\cdot\psi+\theta_j\sqrt{\mathcal{E}}\epsilon_j\cdot\psi+i(k_i+k_j)\cdot x-\bar{\theta}_i\bar{\theta}_j k_i\cdot k_j G_F^{ji}\right]}{k_i\cdot k_j G_F^{ji}}\right|_{t_i=t_j}
\end{aligned}
\tag{3.21}
$$

Insertion of this operator into (3.8) to replace two operators of the type (3.9) gives the pinched kinematic factor. Comparison with (3.12) shows that the *pinched* factor consists of all the terms in the *generating* kinematic factor which contain $k_i\cdot k_j G_F^{ji}$, with the replacement

$$
k_i\cdot k_j G_F^{ji}\to\begin{cases}+1, & \text{if } t_j>t_i;\\ -1, & \text{if } t_j<t_i\ ,\end{cases}
\tag{3.22}
$$

and with t_i set equal to t_j. Notice that if a term contains $\epsilon_i\cdot\epsilon_j G_F^{ji}$ as well, it vanishes since $G_F^{ji}(0)=0$ by antisymmetry.

In order to keep track of the different pinch contributions, it is useful to write down a simple mnemonic rule based on Bern-Kosower diagrams. While this could be done in many ways, the particular choice presented here will eventually permit a smoother transition from effective actions to scattering amplitudes.

Draw all (planar) ϕ^3 graphs with one loop, N external legs and any number $N_T\le N/2$ of trees with *one* vertex. Consider a particular graph and a particular color(path)-ordering; label the external legs clockwise from 1 to N following the path-ordering. Now examine the generating kinematic factor of (3.12) term by term. Two external gluons flow into each tree vertex; let j be the gluon lying most clockwise, and call the other gluon i. If a given term does not contain a factor $k_i \cdot k_j G_F^{ji}$ for *each* tree vertex in the graph, then it vanishes. Even then, it must contain exactly *one* G_F^{ji} at each vertex; otherwise it vanishes. If it survives, then replace each $k_i \cdot k_j G_F^{ji}$ by $+1$, replace $t_i \to t_j$ in all Green functions, and eliminate the t_i integral.

As an application of the formalism of this chapter, let us consider the contribution of a Dirac spinor to the gluon vacuum polarization. In the usual first-order formalism of Dirac, the single diagram has the form

$$g^2 \mathrm{Tr}(T^a T^b) \int \frac{d^D p}{(2\pi)^D} \frac{-\mathrm{Tr}[\not\epsilon_1 (\not p - \not k_1) \not\epsilon_2 \not p]}{p^2 (p-k_1)^2} \tag{3.23}$$

Usually this diagram is evaluated by writing

$$\mathrm{Tr}[\not\epsilon_1 (\not p - \not k_1) \not\epsilon_2 \not p] = 4[\epsilon_1 \cdot (p-k_1) \epsilon_2 \cdot p + \epsilon_1 \cdot p \, \epsilon_2 \cdot (p-k_1) - p \cdot (p-k_1) \epsilon_1 \cdot \epsilon_2] \,, \tag{3.24}$$

after which the momentum integral is performed. One may also use

$$\not\epsilon_i (\not p - \not k_i) = 2\epsilon_i \cdot p - \not p \not\epsilon_i - \not\epsilon_i \not k_i \tag{3.25}$$

and write (after some algebra)

$$\begin{aligned}
&2\mathrm{Tr}[\not\epsilon_1 (\not p - \not k_1) \not\epsilon_2 \not p] \\
&\quad = \mathrm{Tr}[- \not\epsilon_1 \not\epsilon_2](p^2 + (p-k_1)^2) + \mathrm{Tr}[(2\epsilon_1 \cdot p - \not\epsilon_1 \not k_1)(2\epsilon_2 \cdot (p-k_1) - \not\epsilon_2 \not k_2)] \\
&\quad = -4\epsilon_1 \cdot \epsilon_2 \big(p^2 + (p-k_1)^2\big) + 4\epsilon_1 \cdot (2p-k_1) \, \epsilon_2 \cdot (2p-k_1) \\
&\qquad - 4\big(\epsilon_1 \cdot \epsilon_2 k_1 \cdot k_2 - \epsilon_1 \cdot k_2 k_1 \cdot \epsilon_2\big)
\end{aligned} \tag{3.26}$$

which puts the amplitude in a second-order form. The first and second term yield the contribution of (2.31) times a factor of -2; the last term is independent of the loop momentum. The result is

$$
\begin{aligned}
\Pi = 2\frac{(g\mu^{\epsilon/2})^2}{(4\pi)^{2-\epsilon/2}}\mathrm{Tr}(T^aT^b)\int_0^\infty \frac{dT}{T^{1-\epsilon/2}} \\
\Big\{\Big[\int_0^1 da\Big(-\frac{2}{T}\epsilon_1\cdot\epsilon_2-(1-2a)^2\epsilon_1\cdot k_1\epsilon_2\cdot k_1 \\
+\big(\epsilon_1\cdot\epsilon_2 k_1\cdot k_2-\epsilon_1\cdot k_2 k_1\cdot\epsilon_2\big)\Big)e^{-Tk_1^2(a-a^2)}\Big] \\
+\frac{2}{T}\epsilon_1\cdot\epsilon_2\Big\}\ ,
\end{aligned}
\tag{3.27}
$$

where $\epsilon = 4 - D$.

By contrast, evaluation of (3.12) at order g^2 immediately yields

$$
\begin{aligned}
\Gamma_2(k_1,k_2) = -2\frac{(ig\mu^{\epsilon/2})^2}{(4\pi)^{2-\epsilon/2}}\mathrm{Tr}(T^aT^b)\int_0^\infty \frac{dT}{T^{1-\epsilon/2}}\int_0^1 du\ e^{k_1\cdot k_2 G_B(1-u)} \\
\Big[\epsilon_1\cdot k_2\epsilon_2\cdot k_1[\dot{G}_B(1-u)]^2+\ \epsilon_1\cdot\epsilon_2\ \ddot{G}_B(1-u)\big) \\
+\big(\epsilon_1\cdot\epsilon_2 k_1\cdot k_2-\epsilon_1\cdot k_2\epsilon_2\cdot k_1\big)[G_F(1-u)]^2\Big]\ ,
\end{aligned}
\tag{3.28}
$$

which is identical to (3.27). There are no pinches to perform, since the integrand contains no terms with a single power of G_F^{12}.

4. The Effective Action of a Vector Particle in a Background Field

Now let us consider the case of a massless spin-one particle. There are many ways to proceed, and among them are several directly inspired by the methods of string theory. In a model inherited from the bosonic string, one would introduce a single oscillator mode with a vector index, whose sole purpose would be to excite an unphysical scalar "vacuum" (which would eventually be removed by hand) to a vector boson state. One could then imagine projecting out all higher spin states, either by hand or by tricks ranging from adding large masses (as in the string) or by adding complex phases to the oscillators (along the lines of string orbifold constructions). Another possibility is to use a supersymmetric construction; as in the superstring, a fermionic oscillator with a vector index can be used to excite a "vacuum" (which one projects away) to a state with vector indices. Extra states can again be projected out in a number of ways. I will use this latter construction, following closely both the usual superstring methodology[12] and the work of Brink, Di Vecchia and Howe.[15]

The action of a Yang-Mills particle Q_μ, expressed in Feynman gauge, in a classical background A^μ is well-known to be

$$\begin{aligned} S = \int d^4x \; \{ & Q^{a\mu}[(D^2)^{ab} g_{\mu\nu} - g(F^c_{\rho\sigma} J^{\rho\sigma})_{\mu\nu} f^{cab}] Q^{b\nu} + \overline{\omega}(D^2)^{ab}\omega \\ & + \text{order}(Q^3, Q^4, \overline{\omega}Q\omega, etc.)\} \; , \end{aligned} \tag{4.1}$$

where $D_\mu = \partial_\mu - igA_\mu$ and $gF_{\mu\nu} = i[D_\mu, D_\nu]$ are functions *only* of the background field, ω is the ghost of background field Feynman gauge, and $J_{\mu\nu}$ is the spin-one (hermitean) generator of Lorentz transformations:

$$(J_{\mu\nu})^{\rho\sigma} = i(\delta^\rho_\mu \delta^\sigma_\nu - \delta^\rho_\nu \delta^\sigma_\mu) \ . \tag{4.2}$$

(Feynman gauge for Q_μ is appropriate in that the propagator is $\Box^{-1}$, as we had for scalars and spinors; background field gauge is essential since the result must be gauge invariant with respect to the classical field A_μ. The appearance of background field gauge in this context and the following expression for the effective action were discussed in the work of Bern and Dunbar.[14] A useful introduction to background field gauge is given in reference 21.) The one-loop effective action is found from the part of (4.1) which is quadratic in the quantum fields:

$$\Gamma[A] = -\frac{1}{2} \log \left[\det \left(D^2 - gF^{\mu\nu} J_{\mu\nu} \right) \right] + \log \left[\det \left(D^2 \right) \right] . \tag{4.3}$$

Again the structure of the effective action suggests the use of Grassmann variables, and turning to Brink, Di Vecchia, and Howe[15], we find that they have discussed the relevant theory.

Let us consider a particle with coordinates $(x^\mu, \psi^\mu_+, \psi^\mu_-)$. We will find it useful to consider also the real field $\psi^\mu = (\psi^\mu_- + \psi^\mu_+)$. The worldline fermions satisfy

$$\begin{aligned} \{\psi^\mu_+, \psi^\nu_-\} &= g^{\mu\nu} = \frac{1}{2}\{\psi^\mu, \psi^\nu\} \ ; \\ \{\psi^\mu_+, \psi^\nu_+\} &= \{\psi^\mu_-, \psi^\nu_-\} = 0 \ . \end{aligned} \tag{4.4}$$

If we define a vacuum $|0\rangle$ as the state such that $\psi^\mu_- \left|0\right\rangle = 0$ for all μ, then the full set of sixteen states (for a given momentum) is

$$|0\rangle \ ; \ \psi^\mu_+ |0\rangle \ ; \ [\psi^\mu_+, \psi^\nu_+] \, |0\rangle \ ; \ \epsilon_{\mu\nu\rho\sigma} \psi^\nu_+ \psi^\rho_+ \psi^\sigma_+ |0\rangle \ ; \ \epsilon_{\mu\nu\rho\sigma} \psi^\mu_+ \psi^\nu_+ \psi^\rho_+ \psi^\sigma_+ |0\rangle \ . \tag{4.5}$$

These are antisymmetric tensors; in four-dimensions the (0,1,2,3,4)-index antisymmetric tensors have (1,4,6,4,1) components of which only (1,2,1,0,0) are physical degrees of freedom. This model therefore describes a scalar, a vector boson, and a pseudoscalar. However, if we can implement a projection onto states with odd fermion number, then the truncated Hilbert space

$$\psi_+^\mu \,|0\rangle \quad \text{and} \quad \epsilon_{\mu\nu\rho\sigma}\psi_+^\nu\psi_+^\rho\psi_+^\sigma \,|0\rangle \ . \tag{4.6}$$

will contain only the spin-one states as physical modes.

In a complete analysis of this truncated model, one must study the super-reparametrization ghosts in order to derive the Bern-Kosower rules; however, I have chosen to skirt the issue of ghosts in this article. For the present paper it will be sufficient to use a trick borrowed from string theory, in which the gluon ghosts of field theory are accounted for by hand, and in which the three-index tensor is given a mass which is sent to infinity at the end of the calculation.

Derivation of the superparticle Lagrangian is straightforward when one observes the following:

$$\langle\rho|\,\frac{i}{2}[\psi_\mu,\psi_\nu]\,|\sigma\rangle = \frac{i}{2}\left\langle\psi_-^\rho[\psi_\mu,\psi_\nu]\psi_+^\sigma\right\rangle = (J_{\mu\nu})^{\rho\sigma}. \tag{4.7}$$

Remembering that we will eventually do away with the spurious states, let us extend the theory to the full set of sixteen states in (4.5). As in (3.6), we are led to the particle Lagrangian

$$L = \frac{1}{2\mathcal{E}}\dot{x}^2 + \psi_+\cdot\dot{\psi}_- - igA_\mu\dot{x}^\mu + ig\frac{\mathcal{E}}{2}\psi^\mu F_{\mu\nu}\psi^\nu \ . \tag{4.8}$$

However, in order to carry out the trick described above we will want to make the three-index tensor heavy. We must therefore break the degeneracy of the sixteen states by adding a harmonic oscillator potential:

$$L \to L - C(\psi_+ \cdot \psi_- - 1) \ . \tag{4.9}$$

For positive C the ψ's form a fermionic harmonic oscillator whose states are spaced by $\Delta m^2 = C$ and whose vacuum is a tachyon with $m^2 = -C$. Fortunately this tachyon is unphysical; it will be removed from the theory by truncation as discussed above, and so causes no difficulties. All other states except the vector boson will vanish as a result of the truncation or because their masses will be taken to infinity. (This construction is taken directly from the superstring.[12])

One can proceed straightforwardly with the computation of the effective action in direct analogy to the spinor and scalar cases. The field theory ghosts in background field gauge contribute a factor of $\log\det D^2$; as noted by Bern and Dunbar[14], and as expected from string theory[12], this is exactly the negative of the effective action of a complex scalar in the adjoint representation (see eq. (2.10)):

$$\Gamma[A]_{\mathbf{ghosts}} = -\int_0^\infty \frac{dT}{T}\,\mathcal{N} \int \mathcal{D}x\ \mathrm{Tr}\exp\Big[-\int_0^T d\tau\big(\frac{1}{2\mathcal{E}}\dot{x}^2 \ - \ igA\cdot\dot{x}\big)\Big] \ . \tag{4.10}$$

The gauge boson contribution may be calculated by projecting out the even fermion states in the theory and by letting $C \to \infty$. The projection, which is the GSO projection well-known from string theory[22], is implemented by the operator

$$P_{GSO} = \frac{1}{2}\left[1 - (-1)^F\right] \ , \tag{4.11}$$

where $F = (\psi_+)^\mu \cdot (\psi_-)_\mu$ is the fermion number of a state. Clearly only the states of (4.6) survive. It is well-known[12] that the operator $(-1)^F$ is implemented in the path-integral by choosing periodic boundary conditions for fermions:

$$\psi(t \to T) = \psi(t \to 0) \; . \tag{4.12}$$

We may therefore write

$$\begin{aligned}
\Gamma[A] &= -\frac{1}{2}\mathrm{Tr}\log\left[D^2\mathbf{1} \; - \; gF^{\mu\nu}J_{\mu\nu}\right] \\
&= \frac{1}{2}\lim_{C\to\infty}\int_0^\infty \frac{dT}{T} \sum_{s_0,s_1,s_2,s_3=0}^{1} \int \frac{d^4p}{(2\pi)^4} \\
&\quad \langle s_\rho, p|\, P_{GSO} \exp\Big[-\frac{1}{2}\mathcal{E}T\{(p+gA)^2 \\
&\qquad\qquad - C(\psi_+\cdot\psi_- - 1) + igF_{\mu\nu}\psi^\mu\psi^\nu\}\Big]\,|s_\rho, p\rangle \\
&= \frac{1}{2}\lim_{C\to\infty}\int_0^\infty \frac{dT}{T}\,\mathcal{N}\int \mathcal{D}x\,\frac{1}{2}\Big[\int_{(\frac{1}{2})}\mathcal{D}\psi \; - \int_{(0)}\mathcal{D}\psi\;\Big] \\
&\quad \mathrm{Tr}\;\exp\Big[-\int_0^T d\tau\;\Big(\frac{1}{2\mathcal{E}}\dot{x}^2 \; + \psi_+\cdot\dot{\psi}_- - \frac{\mathcal{E}}{2}C(\psi_+\cdot\psi_- - 1) \\
&\qquad\qquad - igA_\mu\dot{x}^\mu + ig\frac{\mathcal{E}}{2}\psi^\mu F_{\mu\nu}\psi^\nu\Big)\Big]\;,
\end{aligned} \tag{4.13}$$

where the subscripts $(\frac{1}{2})$ and (0) indicate antiperiodic and periodic boundary conditions on the worldline fermions.

Proceeding as in the previous section (eqs. (3.6)–(3.12)), we find (in the gauge $\mathcal{E} = 2$)

$$\Gamma_N^0(k_1,\ldots,k_N) = \frac{(ig)^N}{2(4\pi)^2}\mathrm{Tr}(T^{a_N}\ldots T^{a_1})\int\limits_0^\infty \frac{dT}{T^{3-N}}\Big(\prod_{i=1}^N \int\limits_0^{u_{i+1}} du_i\Big)$$
$$\exp\Big(\sum_{i<j=1}^N k_i\cdot k_j G_B^{ji}\Big)\Big\{\Big(\prod_{i=1}^N\int d\theta_i d\bar\theta_i\Big)$$
$$\exp\Big(\sum_{i<j=1}^N \big(-i\,(\bar\theta_j\theta_j k_i\cdot\epsilon_j - \bar\theta_i\theta_i k_j\cdot\epsilon_i)\dot G_B^{ji}$$
$$+\bar\theta_i\theta_i\bar\theta_j\theta_j\epsilon_i\cdot\epsilon_j\ddot G_B^{ji}\big)\Big) \tag{4.14}$$
$$\sum_{p=0}^1(-)^{p+1}\frac{Z_{\frac{p}{2}}}{2}\exp\Big(2\sum_{i<j=1}^N\big[-\bar\theta_i\bar\theta_j k_i\cdot k_j + i\bar\theta_i\theta_j k_i\cdot\epsilon_j$$
$$+i\theta_i\bar\theta_j\epsilon_i\cdot k_j + \theta_i\theta_j\epsilon_i\cdot\epsilon_j\big]G_F^{(\frac{p}{2})ji}\Big)\Big\};$$

again the symbols $(\frac{1}{2})$ and (0) indicate antiperiodic and periodic fermions. Notice the factor of two relative to (3.12) in the exponential of the fermionic Green functions. The Z factors are given (in Minkowski spacetime) by

$$\begin{Bmatrix} Z_{(\frac{1}{2})} \\ Z_{(0)}\end{Bmatrix} = \int\limits_{\psi(T)=(\mp)\psi(0)} \mathcal{D}\psi\ e^{-\int_0^T d\tau[\psi_+\cdot\dot\psi_- - C(\psi_+\cdot\psi_- -1)]} = \mathrm{Tr}_\psi(\pm1)^F e^{-H[\psi]T}$$
$$= e^{-CT}\Big(\sum_{s=0}^1\langle s|\,(\pm e^{CT})^s\,|s\rangle\Big)^4 = 16e^{CT}\begin{Bmatrix}\cosh^4\\ \sinh^4\end{Bmatrix}(-CT/2)$$
$$= e^{-CT}\pm 4 + 6e^{CT}+\ldots \tag{4.15}$$

When continued to Euclidean spacetime, the exponentials change sign, and cancellations remove all growing exponentials; I will explain this below.

The bosonic green functions are identical to those used for the scalar and spinor particle (eq. (2.25)), since the free bosonic action

$$L_B = \frac{1}{2\mathcal{E}}\dot x^2 \tag{4.16}$$

is independent of the particle's spin. The free fermionic action is

$$L_F = \psi_+ \cdot \dot{\psi}_- - C\psi_+ \cdot \psi_- \ ; \tag{4.17}$$

however, this leads in Minkowski spacetime to Green functions which blow up as $C \to \infty$. It is therefore necessary to analytically continue to Euclidean spacetime to study this limit.

Moving to Euclidean spacetime, and being careful to define the number operator properly, we have

$$L_F^{Eucl} = \psi_+^\mu g_{\mu\nu}(\partial_t + C)\psi_-^\nu \ , \tag{4.18}$$

Let us first compute the Green functions on the line. Define

$$G_F^{+-}(t,t') = \langle \psi_+(t)\psi_-(t') \rangle \ ; \tag{4.19}$$

this function satisfies

$$(\partial_t + \theta(t-t')C)G_F^{+-}(t,t') = \delta(t-t') \ . \tag{4.20}$$

where $\theta(t)$ is a step function which is zero for negative t. This equation implies

$$G_F^{+-}(t,t') = \theta(t-t')\exp(-C|t-t'|) \ . \tag{4.21}$$

Similarly

$$G_F^{-+}(t,t') = -\theta(t'-t)\exp(-C|t-t'|) \ . \tag{4.22}$$

Since G_F^{++} and G_F^{--} both vanish,

$$G_F(t,t') = \langle \psi(t)\psi(t') \rangle = \mathrm{sign}(t-t')\exp(-C|t-t'|) \ . \tag{4.23}$$

On the circle of circumference T, we will need to find functions, one periodic ($G_F^{(0)}$), another antiperiodic ($G_F^{(\frac{1}{2})}$) in $t \to t+T$, which reduce to eq. (4.23) in the limit that $T \to \infty$. An analysis analogous to the above yields

$$\begin{aligned} G_F^{(\frac{1}{2})}(t-t') &= 2\ \mathrm{sign}(t-t')e^{-\frac{1}{2}CT}\ \cosh\left[C(\frac{1}{2}T - |t-t'|)\right]\ ; \\ G_F^{(0)}(t-t') &= 2\ \mathrm{sign}(t-t')e^{-\frac{1}{2}CT}\ \sinh\left[C(\frac{1}{2}T - |t-t'|)\right]\ . \end{aligned} \tag{4.24}$$

Again these are precisely the functions found by Bern and Kosower in the derivation of their field theory rules[5].

The next task is to discard the three-index tensor by sending C to infinity. We must carefully analyze the effective action (4.14) to see what terms remain in this limit. The following discussion is almost identical to that of Bern and Kosower[5]; I repeat it here for the sake of completeness.

It is necessary to study separately terms with and without G_F chains. For terms that contain no G_F's, the only dependence on C is given in the prefactors $Z_{\frac{\epsilon}{2}}$, which in Euclidean spacetime take the form

$$\left\{ \begin{array}{c} Z_{(\frac{1}{2})} \\ Z_{(0)} \end{array} \right\} = 16e^{CT} \left\{ \begin{array}{c} \cosh^4 \\ \sinh^4 \end{array} \right\} (CT/2) = e^{CT} \pm 4 + 6e^{-CT} + \ldots \tag{4.25}$$

The first term, associated with the propagation of the tachyon, blows up as $C \to \infty$; fortunately it cancels in the expression

$$\frac{1}{2}\left(Z_{(\frac{1}{2})} - Z_{(0)}\right) = 4 + \mathcal{O}(e^{-2CT})\ , \tag{4.26}$$

leaving us with an overall factor of 4. This factor stems from the sum over the four states $\psi_+^\mu\,|0\rangle$ which can propagate around the loop. These purely bosonic terms are partially cancelled by the contribution of the ghosts (eq. (4.10)); the removal of the timelike and longitudinal modes of the vector boson reduces the number of states, and the overall factor, from 4 to 2. (In the usual dimensional regularization schemes, this number becomes $2 - \frac{1}{2}\epsilon$; however it is natural in this formalism to use dimensional reduction or the variant of it developed by Bern and Kosower[5,6], in which the number of states is left at 2.)

Consider next the expansion in powers of e^C of a chain product of antiperiodic $G_F^{(\frac{1}{2})}$'s, minus the same chain of periodic $G_F^{(0)}$'s. This is precisely the sort of expression we obtain from (4.14) as a result of the GSO projection. From (4.24) we find that

$$
\begin{aligned}
\frac{1}{2}\Big[\prod_{k=1}^{d} G_F^{(\frac{1}{2})}(t_{i_{k+1}}, t_{i_k}) - \prod_{k=1}^{d} G_F^{(0)}(t_{i_{k+1}}, t_{i_k})\Big] \\
= [\prod_{1}^{d}\operatorname{sign}(t_{i_{k+1}} - t_{i_k})] e^{-CT} \exp\big(- C \sum_{k=1}^{d} |t_{i_{k+1}} - t_{i_k}|\big) \times \qquad (4.27) \\
[\sum_{n=1}^{d} \exp\big(2C|t_{i_{n+1}} - t_{i_n}|\big) + \mathcal{O}(e^{-CT})] \ .
\end{aligned}
$$

(Here $i_{d+1} \equiv i_1$.) The leading term in (4.27) is of the form

$$
[\prod_{1}^{d} \operatorname{sign}(t_{i_{k+1}} - t_{i_k})] e^{-CT} \sum_{n=1}^{d} \exp\big(- C f(t_i; t_n)\big) \qquad (4.28)
$$

where

$$
f(t_i; t_n) = \sum_{k=1}^{d} |t_{i_{k+1}} - t_{i_k}| - 2|t_{i_{n+1}} - t_{i_n}| \geq 0 \ . \qquad (4.29)
$$

Unless $f(t_i; t_n) = 0$ for some n, (4.28) will contribute too strong a power of e^{-C}, and a term containing it will vanish in the limit $C \to \infty$.

Since the expressions above are cyclic in k, one can rotate the k's to make $t_{i_d} = t_{\max} \equiv \max[t_{i_k}]$; let $t_{\min} \equiv \min[t_{i_k}]$. Then

$$2|t_{i_{n+1}} - t_{i_n}| \leq 2(t_{\max} - t_{\min}) \leq \sum_{k=1}^{d} |t_{i_{k+1}} - t_{i_k}| \ . \tag{4.30}$$

For $f(t_i;t_n) = 0$, both equalities must obtain. Notice that the second equality can be satisfied only when

$$t_{\max} = t_{i_d} > t_{i_{d-1}} > \cdots > t_{i_2} > t_{i_1} = t_{\min} \tag{4.31}$$

or

$$t_{\max} = t_{i_d} > t_{i_1} > t_{i_2} > \cdots > t_{i_{d-1}} = t_{\min} \ . \tag{4.32}$$

(I will call a chain satisfying (4.31) or (4.32) a path-ordered chain since the ordering is with respect to proper time. I remind the reader that the color trace is ordered in the same way.) The first equality in (4.30) can only hold when $t_n = t_{\max}$ and $t_{n+1} = t_{\min}$, or *vice versa*. Thus the condition $f(t_i;t_n) = 0$ can only occur either when (4.31) holds and $n = d$, or when (4.32) holds and $n = d-1$. (In the case $d = 2$, both (4.31) and (4.32) hold.) It follows that a path-ordered chain of G_F's contributes

$$[\prod_1^d \mathrm{sign}(t_{i_{k+1}} - t_{i_k})]e^{-CT} = e^{-CT} \begin{cases} -1, \text{ if (4.31) holds;} \\ -(1)^{d-1}, \text{ if (4.32) holds;} \\ -2, \text{ if } d = 2; \end{cases} \tag{4.33}$$

Of course, as this derivation is essentially the same as that of reference 5, the result (4.33) agrees with that of Bern and Kosower.

The exponential in (4.33) cancels the overall factor of e^{CT} which was found in eq. (4.25), leaving only the numerical factor -2 or ± 1. All other terms from such a chain, as well as those from chains which are not path-ordered, have additional decaying exponentials which vanish in the limit $C \to \infty$. Using the above argument twice, it is easy to see that a term with more than one G_F chain will always vanish in the limit $C \to \infty$. We therefore find that out of the expression (4.14), only terms with single path-ordered chains of G_F's of length 0 to N contribute, and then are simply replaced by the factor ± 1 or ± 2. At this point all dependence on C has vanished and we may return to Minkowski spacetime.

How should one interpret these rules? It is easiest to do so from an operator standpoint. Since we are throwing away all states of (4.5) except the spin-one tensor, we require that the application of a ψ_+ operator, which moves us out of the space of spin-one states, be accompanied by the simultaneous application of a ψ_- operator in order to bring us back to it. This translates into a requirement that the Wick contractions which generate the Green functions do not overlap one another; hence the G_F's must be path-ordered.

We now have enough information to write down a set of rules for the unpinched diagram, starting with the same formula we had in the spinor case (eq. (3.12)). To obtain the generating kinematic factor of the vector boson, manipulate the kinematic factor of (3.12): throw away all terms except those with no G_F's and those with a single G_F chain, and multiply terms without G_F's by 2. Next, replace the G_F chains by

$$[\prod_1^d G_F^{i_{k+1},i_k}] \to \begin{cases} -2^d, \text{ if (4.31) holds;} \\ -(-2)^d, \text{ if (4.32) holds;} \\ -8, \text{ if } d=2; \\ 0 \ \text{ otherwise.} \end{cases} \tag{4.34}$$

where the powers of two account for the slight differences between equations (4.14) and (3.12). Finally, substitute the bosonic Green functions of (2.25), plug the result back into (3.12), multiply by $-\frac{1}{4}$ and evaluate the integral.

The non-Abelian part of $F_{\mu\nu}$ contributes to amplitudes for vectors just as it does for spinors. The resulting pinch rules are almost as described in the previous section, but one must decide whether to perform pinches before or after requiring that all chains be path-ordered. The relevant consideration is that the pinch technique is just a trick to generate the correct set of G_F's; one could drop the trick and calculate directly the pinched kinematic factor by inserting $\mathcal{O}_{i,j}$ (eq. (3.20)) into the path integral, just as is done in (3.8) with the usual $\mathcal{V}$'s (eq. (3.9)). Only *after* the whole set of G_F chains in the pinched kinematic factor is known should one apply the analysis of eqs. (4.27)–(4.33) to determine which chains survive in the limit $C \to \infty$. Therefore, one should perform all pinches *before* requiring that G_F chains be path ordered; for example, the chain

$$G_F^{12} G_F^{2,i+1} k_i \cdot k_{i+1} G_F^{i+1,i} G_F^{i1} \tag{4.35}$$

for $t_N > t_{N-1} > \cdots > t_1$ will contribute to the diagram in which gluons $i+1$ and i are pinched, even though in the evaluation of the unpinched Bern-Kosower diagram it is discarded. (Notice that pinching cannot change the number of G_F chains in a given term, and so one may safely discard from the original generating kinematic factor any term with more than one such chain.)

Thus, the rule for pinched diagrams is the following: Return to the generating kinematic factor for the vector boson, and carry out the pinches as explained in section 3. Next, apply the path-ordering requirement to G_F chains, replacing them with the factors in eq. (4.34). Finally, substitute the usual functions for the G_B's, insert the kinematic factor into (3.12), multiply by $-\frac{1}{4}$ and compute the integrals.

As an example, consider the pure $SU(N)$ Yang-Mills vacuum polarization in background field gauge. The reader may check that if the algebra of Feynman diagrams is organized as explained by Bern and Dunbar[14], it is straightforward to obtain

$$\begin{aligned}\Pi =& \frac{(g\mu^{\epsilon/2})^2 f^{acd} f^{bdc}}{(4\pi)^{2-\epsilon/2}} \int_0^\infty \frac{dT}{T^{1-\epsilon/2}} \\ &\Big\{\Big[\int_0^1 da\Big(-\frac{2}{T}\epsilon_1\cdot\epsilon_2 - (1-2a)^2\epsilon_1\cdot k_1\epsilon_2\cdot k_1 \\ &\quad + 4\big(\epsilon_1\cdot\epsilon_2 k_1\cdot k_2 - \epsilon_1\cdot k_2 k_1\cdot\epsilon_2\big)\Big)e^{-Tk_1^2(a-a^2)}\Big] \\ &\qquad + \frac{2}{T}\epsilon_1\cdot\epsilon_2\Big\}\ ,\end{aligned} \tag{4.36}$$

where $\epsilon = 4 - D$. I have included the ghosts in this expression, using dimensional reduction in which the number of physical helicity states is exactly 2.

According to the above rules for vector bosons, this result can be extracted from the result of (3.28) by replacing $(G_F^{21})^2 = -G_F^{21}G_F^{12}$ with $+8$, multiplying the terms with $(\dot G_B^{21})^2$ and $\ddot G_B^{21}$ by 2, and multiplying the entire expression by $-\frac{1}{4}$. Indeed this gives

$$\Pi = -\frac{(g\mu^{\epsilon/2})^2}{(4\pi)^{2-\epsilon/2}}\mathrm{Tr}(T^aT^b)\int_0^\infty \frac{dT}{T^{1-\epsilon/2}}\int_0^1 du\ e^{k_1\cdot k_2 G_B(1-u)}$$
$$\left[\epsilon_1\cdot k_2\epsilon_2\cdot k_1[\dot{G}_B(1-u)]^2+\ \epsilon_1\cdot\epsilon_2\ \ddot{G}_B(1-u)\right.$$
$$\left.+4\big(\epsilon_1\cdot\epsilon_2 k_1\cdot k_2-\epsilon_1\cdot k_2\epsilon_2\cdot k_1\big)\right]\ , \tag{4.37}$$

which is identical to (4.36) (recall that $(T^a_{adj})^{cd} = -if^{acd}$.) There are no pinches to perform; this is the complete result.

It is amusing to combine the results of (2.33), (3.28) and (4.37). Consider the gluon vacuum polarization in a theory with n_f Dirac fermions and n_s complex scalars in the adjoint representation:

$$\Pi = -\frac{(g\mu^{\epsilon/2})^2}{2(4\pi)^{2-\epsilon/2}}\mathrm{Tr}(T^aT^b)\int_0^\infty \frac{dT}{T^{1-\epsilon/2}}\int_0^1 du\ e^{k_1\cdot k_2 G_B(1-u)}$$
$$\left\{(2-4n_f+2n_s)\left[\epsilon_1\cdot k_2\epsilon_2\cdot k_1[\dot{G}_B(1-u)]^2+\ \epsilon_1\cdot\epsilon_2\ \ddot{G}_B(1-u)\right]\right.$$
$$\left.+4(2-n_f)\big(\epsilon_1\cdot\epsilon_2 k_1\cdot k_2-\epsilon_1\cdot k_2\epsilon_2\cdot k_1\big)\right\}\ , \tag{4.38}$$

(Since γ^5 does not play a role in vacuum polarizations, the contribution of a chiral fermion to the above expression is exactly half that of a Dirac fermion.) Notice that the factor multiplying the bosonic Green functions counts degrees of freedom, and therefore cancels for all supermultiplets. With appropriate choices of matter supermultiplets in various representations, it is possible to make the remainder of (4.38) vanish, leaving the theory one-loop finite. When all particles are in the adjoint representation, complete cancellation occurs for the case $n_f = 2$ and $n_s = 3$; this is the famous $N = 4$ spacetime supersymmetric Yang-Mills theory, which is known to be finite.[23] Notice that this result requires no integrations; it follows directly from the rules for obtaining the generating kinematic factors from (3.12) and from the overall normalizations.

5. Integration by Parts and Manifest Gauge Invariance

Bern and Kosower[5,7] showed that there are benefits associated with performing an integration-by-parts (IBP) on all terms involving a $\ddot{G}_B$; when the $\ddot{G}_B$'s are completely eliminated, it is possible to derive a much simpler set of rules for scattering amplitudes. As discussed by Bern and Dunbar[14], this IBP causes an interesting and intricate reshuffling of terms. Essentially, the delta-functions which produce the four-point vertices of field theory are removed by the IBP, allowing a scattering amplitude to be expressed in terms of Bern-Kosower graphs, which have only ϕ^3 vertices. Each Bern-Kosower graph is related to the "unpinched diagram" – the one with all gluons attached directly to the loop – through the systematic pinch prescription.

In the effective action, the reorganization from the IBP is not much of a simplification, as it leads to as many or more diagrams than Feynman graphs. Nonetheless it is worthwhile in many cases: the additional diagrams are much easier to calculate than usual Feynman graphs due to the systematic "pinch" rules, and the number of types of Feynman parameter integrals is reduced. Furthermore, and perhaps most importantly, it makes possible a direct analysis of individual gauge invariant contributions to the effective action. Still, the IBP is not essential for effective actions, and the casual reader may safely skip this section at a first reading.

The reader intending to study this section should be warned that the IBP, while necessary for a complete picture of the possibilities opened by the work of Bern and

Kosower, represents the weakest link in the present paper. A full understanding of the IBP requires a clarification of the role of string duality, which permits the reorganization which I will outline below. In the absence of this clarification it is only possible to present the IBP and the associated pinch rules as a trick, motivated by the Bern-Kosower rules for scattering amplitudes[5,6] and the work of Bern and Dunbar.[14] I will demonstrate the validity of this trick in a simple case; however, while I have checked that it works in general, I will not present a proof, as the only proof I know proceeds case by case and is both tedious and uninformative. If a simple and enlightening proof is found, it will be published separately. With or without proof, the pinch rules appear completely *ad hoc* at the present time, and the reader is urged to familiarize herself with the Bern-Kosower rules outlined in reference 6 to help put the present section in context.

To illustrate the trick, I present the simplest case. Consider a term from the generating kinematic factor of (3.12) of the form

$$\epsilon_i \cdot \epsilon_j \ddot{G}_B^{ij} \times F(\epsilon_m, k_n) \ , \tag{5.1}$$

where F contains neither k_i nor k_j and therefore has no dependence on either t_i or t_j. The IBP of (5.1) can be done with respect to t_i, t_j, or $t_i - t_j$; different results will be found in the different cases, the variations among them being total derivatives. For simplicity let us IBP with respect to t_i; for a particular color ordering, the initial expression from (3.12) is

$$\int_0^T dt_{N-1} \cdots \int_0^{t_{i+1}} dt_i \int_0^{t_i} dt_{i-1} \cdots \int_0^{t_2} dt_1 \ \epsilon_i \cdot \epsilon_j \ \ddot{G}_B(t_i - t_j) \ F \exp\Big[\sum_{r<s} k_r \cdot k_s G_B^{sr}\Big] \tag{5.2}$$

which becomes

$$\int_0^T dt_{N-1}\cdots\int_0^{t_{i+1}} dt_i \int_0^{t_i} dt_{i-1}\cdots\int_0^{t_2} dt_1\ \epsilon_i\cdot\epsilon_j\ \dot{G}_B(t_i-t_j)\ F\exp\big[\sum_{r<s} k_r\cdot k_s G_B^{sr}\big]$$
$$\times\big[\delta(t_{i+1}-t_i)-\delta(t_i-t_{i-1})\ -\sum_{m\neq i} k_i\cdot k_m \dot{G}_B(t_i-t_m)\big]\ . \tag{5.3}$$

The last term now fits in neatly with the terms in the generating kinematic factor which lack $\ddot{G}_B$'s, but the delta functions — the surface terms from the IBP — are an annoyance. (These delta functions contribute only to one color trace, so there are no subtle factors of two associated with them.) Essentially they are color commutators; they would cancel against surface terms from other proper-time orderings were the theory abelian, but cannot do so here since different proper-time orderings have independent color traces. Fortunately these surface terms bear a simple relationship to the last term in (5.3). Specifically, take the terms in the sum over m with $m=i\pm 1$:

$$-\int_0^T dt_{N-1}\cdots\int_0^{t_{i+1}} dt_i \int_0^{t_i} dt_{i-1}\cdots\int_0^{t_2} dt_1$$
$$\epsilon_i\cdot\epsilon_j\dot{G}_B(t_i-t_j)\ \sum_{m=i\pm 1} k_i\cdot k_m\dot{G}_B(t_i-t_m)\ F\exp\big[\sum_{r<s}k_r\cdot k_s G_B^{sr}\big]\ . \tag{5.4}$$

Now, motivated by the pinch rules of section 3 and the work of Bern and Dunbar[14], replace $k_i\cdot k_{i\pm 1}\dot{G}_B^{i,i\pm 1}$ with ∓ 1 and set $t_i=t_{i\pm 1}$; in this way the surface terms are reproduced.

The case $j=i\pm 1$ is special: one of the surface terms contains $\dot{G}_B^{jj}\equiv 0$, and so the pinch $t_i=t_j$ does *not* get a contribution from the IBP. This leads to a modification of the rule for "pinching": the pinch of a term containing $(\dot{G}_B^{i,i\pm 1})^2$ vanishes.

(Again this matches with Bern and Kosower[6] and with section 3.) Recall that $\ddot{G}_B^{ij}$ contains a delta function, which accounts for the Feynman graph in which a four-point vertex connects gluons i and j; the missing surface term is cancelled by the half of this delta function that contributes to the color trace under consideration.

In addition to terms like (5.1), the kinematic factor of eq. (3.12) has terms in which $F(\epsilon_m, k_n)$ contains $\dot{G}_B$ functions dependent on t_i and t_j, or in which there are several $\ddot{G}_B$'s; these cases must be dealt with in turn. It can be shown that there is a simple rule governing the resulting pinches which is similar to the Bern-Kosower pinch rules for scattering amplitudes. However, as mentioned above, no useful or interesting proof is known at the present time, and so for now I will simply state without proof the IBP and pinch rules for effective actions.

The first stage of the IBP reorganization involves the elimination of all $\ddot{G}_B$'s in analogy to eqs. (5.2)–(5.3). Specifically, carry out the IBP of the generating kinematic factor, *dropping all surface terms*, until no $\ddot{G}_B$'s remain. (Bern and Kosower have proven that this is always possible.[7]) The result is the "improved generating kinematic factor", associated with the unpinched diagram. Every term in this improved kinematic factor contains a certain number of factors of $k_i \cdot k_j$, where i and j are arbitrary. The number of these factors cannot exceed $N/2$, since the maximum number of $\ddot{G}_B^{ij}$'s and $k_i \cdot k_j G_F^{ij}$'s in any term in the original generating kinematic factor is also $N/2$. Each pinch absorbs one of these factors, as well as one of the integrals over t_i, and so the maximum number of pinches which must be performed simultaneously is $N/2$.

Now I present the pinch rules, which are necessary to account for the IBP surface terms. The procedure is closely related to the Bern-Kosower rules for scattering amplitudes; the reader is again urged to review reference 6.

Draw all (planar) ϕ^3 graphs with one loop, N external legs and any number N_T of trees, such that although each tree may have several vertices, the total number of tree vertices N_V is at most $N/2$. (Diagrams with trees may seem out of place in the construction of a 1PI object like an effective action, but the trees used here, unlike those for scattering amplitudes, do not contribute the usual propagator poles; they serve only as a mnemonic for ensuring all surface terms are accounted for.) The gluons which flow into a tree before entering the loop are said to be pinched; the number of these is $N_V + N_T$. Consider a particular graph and a particular color(path)-ordering; label the external legs clockwise from 1 to N following the path-ordering. Each tree vertex, since it is a three-point vertex, is characterized by one line pointing toward the loop and two outward pointing lines I and J, with two sets of external legs $i_1, ..., i_m$ and $j_1, ..., j_n$ that flow into them. Let J be the line lying most clockwise. Now examine the improved generating kinematic factor term by term. If a given term does not contain a factor $k_i \cdot k_j \dot{G}_B^{ji}$ or $k_i \cdot k_j G_F^{ji}$ for *each* tree vertex, where i belongs to the set of gluons flowing into line I and j flows into J, then it vanishes. Even then, it must contain exactly *one* $\dot{G}_B^{ji}$ or G_F^{ji} at each vertex; otherwise it vanishes. If it survives, then replace $k_i \cdot k_j \dot{G}_B^{ji}$ or $k_i \cdot k_j G_F^{ji}$ by $+1$, replace $t_i \to t_j$ in all Green functions, and eliminate the t_i integral.

It is useful to review the arguments of Bern and Kosower for carrying out the IBP.[5,6,8] After the IBP, the improved generating kinematic factor is made up of only $\dot{G}_B$'s and G_F's; it has no singularities and contains no dependence on T. This simpler form leads to fewer separate integrations, and also allowed Bern and Kosower to construct a formalism in which one needs only ϕ^3 graphs to compute scattering amplitudes. In addition, since the kinematic factor is independent of T, the overall power of T is given by the number of t_i integrations; a diagram with N gluons and k pinches has an integral $\int dT/T^{3-N+k}$. As a consequence, the ultraviolet infinities of gauge theory appear only in terms with $N-2$ pinches, since $\int dT/T$ is the only possible source of ultraviolet divergences. Indeed one may interpret this reorganized amplitude using gauge invariant structures. I will illustrate this in a simple example below, and will discuss this further in later work.

To see the IBP in action, let us apply it to the vacuum polarization in (2.33):

$$\Pi = \Gamma_2(k_1,k_2) = \frac{(g\mu^{\epsilon/2})^2}{(4\pi)^{2-\epsilon/2}} \mathrm{Tr}(T^aT^b)\Big[\epsilon_1\cdot\epsilon_2\; k_1\cdot k_2 - \epsilon_1\cdot k_2\epsilon_2\cdot k_1\Big]$$
$$\int_0^\infty \frac{dT}{T^{1-\epsilon/2}}\int_0^1 du\; e^{k_1\cdot k_2 G_B(1-u)}[\dot{G}_B(1-u)]^2\;. \tag{5.5}$$

This expression has the remarkable property of being explicitly transverse. In usual techniques this property is not visible until the full set of integrations is complete. (This is the full result; since the integrand contains two powers of $\dot{G}_B^{12}$, there is no pinch contribution. Of course this will always be true for a two-point function.) In fact, (5.5) represents precisely the $(A^\mu)^2$ piece of $F^{\mu\nu}F_{\mu\nu}$, which appears as the only infinite term in the unrenormalized effective action. In light of the previous paragraph, it will not surprise the reader that other infinities, namely the one-pinch piece of the $(A_\mu)^3$ term and the two-pinch piece of the $(A_\mu)^4$ term of the effective action, reproduce explicitly the remaining pieces of $F^{\mu\nu}F_{\mu\nu}$. Additionally, since one may perform at most $N/2$ pinches, there are no infinities beyond $N=4$ in the effective action. Thus, even though the complicated process of pinching replaces the many diagrams of Feynman rules, the IBP and the Bern-Kosower-type pinch rules allow for a clearer separation of the different types of contributions to the effective action. This may prove useful in the analysis of the divergence structure of more complex theories.

Another interesting feature of this reorganization is illustrated through the IBP of (3.28):

$$
\begin{aligned}
\Pi = -2\frac{(g\mu^{\epsilon/2})^2}{(4\pi)^{2-\epsilon/2}}\mathrm{Tr}(T^aT^b)\Big[\epsilon_1\cdot\epsilon_2 k_1\cdot k_2 - \epsilon_1\cdot k_2\epsilon_2\cdot k_1\Big] \\
\int_0^\infty \frac{dT}{T^{1-\epsilon/2}}\int_0^1 du\; e^{k_1\cdot k_2 G_B(1-u)}\Big([\dot{G}_B(1-u)]^2 - [G_F(1-u)]^2\Big)\ .
\end{aligned}
\tag{5.6}
$$

As pointed out by Bern and Kosower[5,12], the IBP allows use of worldline supersymmetry in a clever way. Were the system truly worldline supersymmetric, the effective action would vanish. Supersymmetry would require that both x^μ and ψ^μ satisfy periodic boundary conditions, so that $\dot{G}_B^{ij}$ and G_F^{ij} would be equal. It follows that every supersymmetric amplitude expressed as a function of only $\dot{G}_B$ and G_F would vanish under the formal replacement $\dot{G}_B^{ij} \to G_F^{ij}$. However, in (3.12) the only dependence on boundary conditions is hidden in the Green functions themselves; the functional dependence on the Green functions is the same in all cases. As a result, even when x^μ and ψ^μ have different boundary conditions the replacement $\dot{G}_B^{ij} \to G_F^{ij}$ everywhere in the improved kinematic factor (and use of momentum conservation) leads to a complete cancellation. In particular, the result of (5.6) has this property. This trick can be used as a check on the algebra of the IBP.

To find the vacuum polarization for a vector boson loop, follow the rules in section 4. Specifically, take eq. (5.6), replace $(G_F^{21})^2 = -G_F^{21}G_F^{12}$ by $+8$, multiply the term with $(\dot{G}_B^{21})^2$ by 2, and multiply the entire expression by $-\frac{1}{4}$:

$$\Pi = \frac{(g\mu^{\epsilon/2})^2}{(4\pi)^{2-\epsilon/2}} \mathrm{Tr}(T^a T^b) \Big[\epsilon_1 \cdot \epsilon_2 k_1 \cdot k_2 - \epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1\Big]$$
$$\int_0^\infty \frac{dT}{T^{1-\epsilon/2}} \int_0^1 da\ e^{T k_1 \cdot k_2 (a-a^2)} \Big((1-2a)^2 - 4\Big) \ . \tag{5.7}$$

The reader may check that the same result is obtained by integrating (4.37) by parts, and that the integration over a yields the usual factor of 11/3 associated with the Yang-Mills beta function.

6. Colorful Comments

It is often desirable to use a formulation in which only group matrices in the fundamental representation appear; the usefulness of this approach for scattering amplitudes is detailed in the literature.[1,2,7] In this case one should write the effective action as a product of parallel or antiparallel Wilson loops. Since in $U(N_c)$ the $U(1)$ photon decouples from the $SU(N_c)$ gluons, one-loop amplitudes for $SU(N_c)$ can be calculated using $U(N_c)$[1]; working with the full unitary group allows the use of a number of useful tricks.[1,7] If the particle in the loop lies in the adjoint representation of $U(N_c)$, one may consider it as a sort of "bound state" of a fundamental N_c and an antifundamental $\overline{N_c}$ representation; some of the external vector bosons couple to the N_c while others couple, independently, to the $\overline{N_c}$. For a scalar particle, the effective action is

$$\Gamma[A] = \int_0^\infty \frac{dT}{T} \mathcal{N} \int \mathcal{D}x \, \exp\Big[-\int_0^T d\tau(\frac{1}{2\mathcal{E}}\dot{x}^2)\Big]$$
$$\mathrm{Tr}_{N_c} \, \exp\Big[\int_0^T d\tau(igA\cdot\dot{x})\Big] \, \mathrm{Tr}_{N_c} \left\{\exp\Big[\int_0^T d\tau(igA\cdot\dot{x})\Big]\right\}^\dagger . \tag{6.1}$$

The first trace is path-ordered, while the second is anti-path-ordered. In such an expression it becomes immediately obvious that one expects contributions with one or two group traces at the one loop-level, as is well-known to those familiar with the double line formalism of 't Hooft[24] or with open string theory.[7] Rewriting (3.12) in this form changes only the trace structure: letting $X^a(T^a)$ be the group matrices in the adjoint (fundamental) representation, we replace

$$\mathrm{Tr}(X^{a_N}\cdots X^{a_1}) \to \sum_{m=1}^{N}(-1)^m \mathrm{Tr}(T^{b_{N-m}}\cdots T^{b_1})\mathrm{Tr}(T^{c_1}\cdots T^{c_m}) \tag{6.2}$$

where $t_{b_{i+1}} > t_{b_i}$ and $t_{c_{j+1}} > t_{c_j}$. Thus we divide the gluons into two sets, writing down a path-ordered trace for one and an anti-path-ordered trace for the other, and sum over all sets and all orderings. If $m = 0$ or N the trace of the unit matrix yields a factor of N_c. Notice that for $N = 2$ the traces with $m = 0$ and $m = 2$ are equal, while the case $N = 4$, $m = 2$ appears twice in this sum since it is invariant under proper-time-reversal; this accounts for the factors of two which appear for these traces in the Bern-Kosower rules.[5]

Each color trace in (6.2) is internally path-ordered, but operators in *different* traces may be integrated past each other without altering the color structure. As a result, surface terms from the IBP and the operator $\mathcal{O}_{i,j}$ (eq. (3.19)) only appear for gluons lying adjacent to each other in the same color trace; we must therefore only pinch gluons in the *same* trace. Again this is in agreement with the Bern-Kosower rules.[5] (For vector particles, the rules for G_F chains are unaffected by changes in the organization of color; for a chain to contribute it must still be path-ordered as in (4.31) or (4.32).)

It may have occurred to the reader educated in string theory that although I treated color using a Wilson-loop formalism related to the open string, I might have introduced color via the use of internal currents as in the closed string. This has been discussed in the literature.[19] Such a treatment can easily be implemented, and rules can be derived using an approach very similar to that of Bern and Kosower[5]; however this is somewhat more complicated than the technique used in this paper.

7. Some Extensions

There are a number of additional theories that are simple to construct. For example, to study massive scalars or spinors in a background gauge field, add a mass term to the particle Lagrangian, as in eq. (2.3):

$$L \to L - \frac{1}{2}\mathcal{E}m^2 \tag{7.1}$$

where $\mathcal{E}$ is the einbein, and I work in Minkowski spacetime. From the point of view of one-dimensional general relativity, this is just a cosmological constant. In the gauge $\mathcal{E} = 2$, the scalar effective action becomes

$$\begin{aligned}\Gamma[A] &= -\int_0^\infty \frac{dT}{T}\,\mathcal{N}\int \mathcal{D}x\;\exp\Big[-\int_0^T d\tau(\frac{1}{4}\dot{x}^2\; - m^2 - igA\cdot\dot{x})\Big]\\ &= -\int_0^\infty \frac{dT}{T}\,\mathcal{N}\;e^{+m^2T}\int \mathcal{D}x\;\exp\Big[-\int_0^T d\tau(\frac{1}{4}\dot{x}^2\; - igA\cdot\dot{x})\Big]\,.\end{aligned} \tag{7.2}$$

Thus the effect is merely to add a factor of e^{+m^2T} to the integrand of the integral over T. Exactly the same factor occurs for massive spinors. In Euclidean spacetime the factor is e^{-m^2T}, which illustrates the decoupling of particles as $m \to \infty$.

Another straightforward modification is the inclusion of background scalars. Consider the theory

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi) \tag{7.3}$$

The one-loop particle Lagrangian of a scalar particle in a background scalar field can be found by letting $\phi = \Phi + \delta\phi$, where $\delta\phi$ is a quantum fluctuation around the classical field Φ, and keeping only the terms quadratic in $\delta\phi$.

$$L = \frac{1}{2\mathcal{E}}\dot{x}^2 - \frac{1}{2}\mathcal{E}V''(\Phi). \tag{7.4}$$

A prime denotes a derivative with respect to Φ. Notice that mass terms for the scalar arise correctly from this formula.

Spinors interact with this field in a slightly more complex way; the Yukawa interaction $h\Phi\overline{\Psi}\Psi$ is easily incorporated in analogy to eq. (3.2):

$$\begin{aligned}\Gamma[A] &= \log\;\Big[\det\;\big(i\not{D} - h\Phi\big)\Big]\\ &= \frac{1}{2}\log\;\Big[\det\;\big(i\not{D} - h\Phi\big)\big(-i\not{D} - h\Phi\big)\Big]\\ &= \frac{1}{2}\log\;\Big[\det\;\big(\not{D}^2\mathbf{1}\; - ih\not{D}\Phi + h^2\Phi^2\big)\Big]\end{aligned} \tag{7.5}$$

The associated spinor particle has Lagrangian

$$L = \frac{1}{2\mathcal{E}}\dot{x}^2 + \frac{1}{2}\psi\dot{\psi} - h^2\Phi^2 + ih\psi^\mu D_\mu \Phi \ . \tag{7.6}$$

Notice that the one-scalar vertex operator for $\Phi = e^{ik\cdot x}$ is $\mathcal{V}_\Phi = -ih(ik\cdot\psi)e^{ik\cdot x}$, as in string theory. If we let the scalar field have a vacuum expectation value v, and let $\Phi' = \Phi - v$, then (7.6) becomes

$$L = \frac{1}{2\mathcal{E}}\dot{x}^2 + \frac{1}{2}\psi\dot{\psi} - (hv)^2 - 2h^2 v\Phi' - h^2\Phi'^2 + ih\psi^\mu \partial_\mu \Phi' \ . \tag{7.7}$$

Of course the particle picks up a mass $m_\Psi = hv$, and the scalar vertex operator becomes $\mathcal{V}_\Phi = -ih(ik\cdot\psi - 2im_\Psi)e^{ik\cdot x}$.

More interesting is the interaction of a vector boson with a scalar. At this point we should remember that a single background scalar can change the particle in the loop from a vector into a scalar! We must therefore build a theory which consistently describes a particle that can be either scalar or vector. Again string theory is a guide; simply use dimensional reduction. Extend the vector theory of section 4 to a fifth dimension (add fields $x^4, \psi^4_\pm$) but insist that the fifth component of all momenta of all particles or fields must vanish. Since the momentum of the particle must lie in the usual spacetime, a polarization vector pointing solely in the x^4 direction will always satisfy the physical condition $\epsilon\cdot k = 0$; thus the particle's new physical mode is a Lorentz scalar, while its others are unchanged. In short, we have a theory of gauge bosons and a Higgs boson in the adjoint representation.

The reduction of (4.1) from five to four dimensions, with $\Phi \equiv A_4$ and $\phi \equiv Q_4$, is

$$S = \int d^4x \; \{Q^{a\mu}[(D^2 + g^2\Phi\Phi)^{ab} g_{\mu\nu} - g(F^c_{\rho\sigma} J^{\rho\sigma})_{\mu\nu} f^{cab}] Q^{b\nu}$$
$$+ gQ^{a\mu}(D_\mu\Phi)^c \phi^b f^{abc} - g\phi^a(D_\mu\Phi)^c Q^{b\mu} f^{abc}$$
$$- \phi^a[(D^2 + g^2\Phi\Phi)^{ab}\phi^b + \overline{\omega}^a(D^2 + g^2\Phi\Phi)^{ab}\omega^b \tag{7.8}$$
$$+\text{order}(Q^3, Q^4, DQ\phi^2, \overline{\omega}Q\omega, etc.)\} \ .$$

This formula stems from the gauge $D^\mu Q_\mu + ig[\Phi, \phi] = 0$, called background 't Hooft-Feynman gauge. Notice that this gauge contains a new, gauge dependent $\Phi^2\phi^2$ interaction, different from the $\langle\Phi^2\rangle\,\phi^2$ interaction present in usual 't Hooft-Feynman gauge[25], in which $\partial^\mu Q_\mu + ig[\langle\Phi\rangle\,, \phi] = 0$. It is clear from (7.8) that if Φ acquires a vacuum expectation value the gluons, ghosts, and Goldstone bosons associated with spontaneously broken generators have the same mass matrix:

$$(M^2)^{ab} = g^2 \langle\Phi\Phi\rangle^{ab} = g^2 f^{ace} f^{bde} \left\langle \Phi^c \Phi^d \right\rangle \ . \tag{7.9}$$

It is straightforward to add in the symmetry breaking potential for the Higgs boson, and to extend this approach to Higgs bosons in other representations.

The particle Hamiltonian for this theory is

$$H = (p_\mu - gA_\mu)^2 - (p_4 - g\Phi)^2 - ig\psi^\mu F_{\mu\nu}\psi^\nu \; + 2ig\psi^\mu D_\mu \Phi \psi^4 \ ; \tag{7.10}$$

when p_4 is set to zero, the resulting Lagrangian is

$$L = \frac{1}{2\mathcal{E}}\dot{x}^2 \; + \psi_+ \cdot \dot{\psi}_- - \psi^4_+ \dot{\psi}^4_- - g^2\Phi^2 - igA^\mu \dot{x}_\mu$$
$$+ ig\psi^\mu F_{\mu\nu}\psi^\nu \; + 2ig\psi^\mu D_\mu \Phi \psi^4 \ . \tag{7.11}$$

The last term is the one that turns a scalar in the loop into a vector, and *vice versa.* When $\langle\Phi\rangle$ is non-zero the mass matrix of (7.9) is clearly generated. To add in a Higgs potential $V(\Phi)$, use

$$L \to L - V''(\Phi)(-\psi_+^4\psi_-^4 - 1) \ ; \tag{7.12}$$

the oscillator potential for ψ^4 assures that of the physical states only $\psi_+^4 |0\rangle$, the scalar, will feel the potential. This sort of theory can be used — perhaps profitably — for calculations in the standard model; a set of rules is in preparation.

Adding gravity as a background is also straightforward. Consider a theory of a scalar boson in a background metric $G_{\mu\nu}$:

$$L = \frac{1}{2\mathcal{E}} G_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - \frac{\mathcal{E}}{2}m^2 \ . \tag{7.13}$$

This is generally covariant with respect to both worldline and spacetime coordinate redefinitions. One may extend this theory to particles with spin. The relevant Lagrangians were again written down by Brink, Di Vecchia and Howe[15], and I shall not repeat them here. The technique for constructing internal gravitons appears (in part) in the same paper: instead of one complex set of worldline fermions, use two. Define a particle with an N=4 worldline supersymmetry, described by coordinates $(x^\mu, \psi_\pm^\mu, \chi_\pm^\mu)$. The allowed states can be written down as in (4.6); projections onto odd ψ and χ number and onto states which are even under $\psi \to \chi$ leaves a rank-two symmetric tensor as the propagating modes of the theory. While not particularly elegant, this example illustrates that it is straightforward to construct a tensor of any arbitrary rank and symmetry. It may be hoped that useful rules can be obtained from this theory as well.

Finally, I should point out that every theory described in this paper is part of the mode expansion of a string in a background string field.[26] The possible connection of this construction to the Bern-Kosower rules was noted by Bern and Dunbar.[14]

8. Conclusion

In this paper, I have shown that it is possible to construct one-loop effective actions perturbatively without the use of Feynman diagrams, and with a method that has certain conceptual and practical advantages over the standard technique. By viewing a one-loop computation as a system of a particle (or superparticle) in a background field, one can construct formulas and rules valid to all orders in the background field which closely match the string-derived rules of Bern and Kosower for gauge theory. It is now evident that one reason for the simplicity of the Bern-Kosower rules compared to Feynman diagrams is that string theory is a first-quantized system; the ease of one-dimensional as opposed to four-dimensional calculations is clearly demonstrated both in this paper and in the work of Bern and Kosower. The formalism developed in this paper represents a technical and conceptual shift away from the standard techniques of path integral perturbative field theory and back to basic quantum mechanics and the background field method.

Appendix: Conventions

In this paper I have used conventions which are appropriate for particles and Wilson loops and which generate expressions that are simple to compare with those of Feynman diagrams. Unfortunately they are not the most convenient from all points of view, and indeed Bern and Kosower have chosen a very different set of conventions. It is straightforward to convert from one to the other, and in this appendix I explain how to do so.

First, let me review my conventions. I use

$$g_{\mu\nu} = \mathrm{diag}\{+ - - -\} \quad ; \quad \mathrm{Tr}[T^a T^b] = \frac{1}{2} \; ; \\ D_\mu = \partial_\mu - igA_\mu \quad ; \quad gF_{\mu\nu} = i[D_\mu, D_\nu] \tag{8.1}$$

and for Grassmann integrations

$$\int d\theta d\bar{\theta} \; \bar{\theta}\theta = 1 \; . \tag{8.2}$$

To convert my expressions to those of Bern and Kosower:[6]

1. Reverse the order of the color trace.
2. Write the Grassmann integral of (3.12) as $\int d\bar{\theta}_i d\theta_i$ (but keep eq. (8.2).
3. Replace $\dot{G}_B$ with $-\dot{G}_B$.
4. Divide all $\dot{G}_B$ and G_F functions by 2.
5. Multiply all group matrices by $\sqrt{2}$.
6. Account for these factors of two by multiplying the entire amplitude by $2^{N/2}$.

As a result,

7. The improved kinematic factor vanishes under $\dot{G}_B \to -G_F$.
8. Pinches at a vertex with gluons j and i, j the most clockwise, result in the replacement $k_i \cdot k_j \dot{G}_B^{ji}(G_F^{ji}) \to +(-)\frac{1}{2}$.

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