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The Newton–Wigner Position Operator and the Domain of Validity of One-Particle Relativistic Theory (*)

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Summary.—It is suggested that the minimal localization of the relativistic system in terms of the Newton–Wigner mean size can be used as the end criteria for one-particle interpretation of relativistic wave equations.

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1. – Introduction

The notion of localizability of particles in relativistic quantum theory has a long history [1]. The Newton–Wigner approach [2] to this problem stimulated a lot of theoretical research on this subject [3], but only a few examples can be found in the literature [4] in which relativistic position operators were used to investigate actual physical phenomena (of experimental interest). This is due to the fact that in scattering experiments, a very precise determination (of the order of Compton wavelength) of the position

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is not required. However, it seems that we now have interesting results from heavy ion collision experiments [5] where, it is believed, new physics occurs in the very small spatial regions. These experiments were partially inspired by the prediction [6] of vacuum decay in strong external fields. Theoretical investigations of this phenomena suggest the existance of very localized bound states. For example, the mean radius of the ground state electron orbital in a hydrogen-like atom with critical charge $Z \approx 173$ is several times smaller than the electron's Compton wavelength [7] . The question naturally arises as to whether one can actually attribute a real physical meaning to such a small radius, and to what extent the use of the Dirac equation in handling such problems is justified.

The idea of this note is that the mean value of the Newton–Wigner position operator, which is believed to be the correct one-particle position operator, can be used to define the domain of validity of one-particle relativistic theory: a one-particle interpretation is no longer possible beyond a minimal localization given in terms of the Newton–Wigner mean size of the system.

2. – The Newton–Wigner position operator

For convenience, we sketch the derivation of the Newton–Wigner position operator for Dirac particles. The Newton–Wigner operator $\vec{Q}$ can be defined to be the operator whose eigenstates are the "most localized" wave-packets formed from only positive-energy solutions of the Dirac equation. Let $\psi^{(s)}_{(\vec{y})}(\vec{x})$ be such a wave-function describing an electron with spin projection s localized at the space point $\vec{y}$ at the time $t = 0$. The natural normalization condition for these states is

$$\left(\psi^{(s)}_{(\vec{y})}\,,\ \psi^{(r)}_{(\vec{z})}\right) \;=\; \delta_{rs}\,\delta\,(\vec{z}-\vec{y}) \quad , \tag{1}$$

where the scalar product is defined by

$$(\psi,\,\varphi) \;=\; \int d\,\vec{x}\,\psi^{+}(x)\,\varphi(x) \;=\; \int d\,\vec{p}\,\psi^{+}\,(\vec{p})\,\varphi\,(\vec{p}) \quad , \tag{2}$$

and, for momentum space wave-functions, we have

$$\psi\,(x) \;=\; [1/(2\pi)^{3/2}]\int d\,\vec{p}\,\psi\,(\vec{p})\,\exp\{-ip\cdot x\} \quad ,$$

$$p\cdot x \;=\; p_0\,t-\vec{p}\cdot\vec{x} \;\equiv\; -\vec{p}\cdot\vec{x} \quad ,$$

and

$$p_0 \;=\; \sqrt{\vec{p}^{\,2}+m^2} \quad .$$

Translational invariance imposes some restrictions on $\psi^{(s)}_{(\vec{y})}$, namely,

$$\psi^{(s)}_{(\vec{y})}\,(\vec{x}) \;=\; \psi^{(s)}_{(\vec{y}+\vec{a})}\,(\vec{x}+\vec{a}) \quad ,$$

or, in the momentum space,

$$\psi^{(s)}_{(\vec{y})}\,(\vec{p}) \;=\; \exp\{i\vec{p}\cdot\vec{a}\}\,\psi^{(s)}_{(\vec{y}+\vec{a})}\,(\vec{p}) \quad . \tag{3}$$

With Eq. (3), Eq. (1) gives

$$\psi^{+(s)}_{(\vec{y})}\,(\vec{p})\,\psi^{(r)}_{(\vec{y})}\,(\vec{p}) \;=\; (2\pi)^{-3}\,\delta_{rs} \quad . \tag{4}$$

On the other hand,

$$\psi^{(s)}_{(\vec{y})}\,(\vec{p}) \;=\; f_{\vec{y}}\,(\vec{p})\,U\,(\vec{p},s) \quad , \tag{5}$$

where $U(\vec{p},1/2)$ and $U(\vec{p},-1/2)$ are two independent positive-energy solutions of the free particle Dirac equation. For them, we take

$$U\,(\vec{p},s) = \Lambda_+\,(\vec{p})\;U\,(0,s) \quad ,$$

where the rest state four-component spinors are

$$U\,(0,1/2 \;=\; \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \;, \quad \text{and} \quad U\,(0,-1/2) \;=\; \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \;,$$

and $\Lambda_+(\vec{p})$ is a positive-energy projection operator

$$\Lambda_+\,(\vec{p}) \;=\; \frac{1}{2}\;\left[1+(\vec{\alpha}\cdot\vec{p}+\beta m)\,/p_0\right] \;=\; \left[(\hat{p}+m)\;\gamma_0/2p_0\right] \quad ,$$

$$\Lambda_+^+\,(\vec{p}) \;=\; \Lambda_+\,(\vec{p})\;, \qquad \Lambda_+^2\,(\vec{p}) \;=\; \Lambda_+\,(\vec{p}) \quad .$$

It is easy to check that the normalization of $U(\vec{p},s)$ is

(6) $\quad U^+(\vec{p},s)\;U(\vec{p},s') = U^+(0,s)\;\Lambda_+(\vec{p})\;U(0,s') = \left[(p_0+m)\,/\,2p_0\right]\delta_{ss'}$.

Eqs. (4–6) imply

$$\left|f_{\vec{y}}\,(\vec{p})\right|^2 = (2\pi)^{-3}\;\left[2p_0\,/\,(p_0+m)\right] \quad .$$

To fix the phase of $f_{(\vec{y})}(\vec{p})$, we adopt

$$f_{\vec{0}}^*\,(\vec{p}) \;=\; f_{\vec{0}}\,(\vec{p}) \quad .$$

When the momentum space wave-function for the electron localized at $\vec{y}$ is at time $t = 0$, Eq. (7) reads

$$(7) \quad \psi^{(s)}_{(\vec{y})}(\vec{p}) = \left[1/(2\pi)^{3/2}\right] \sqrt{2p_0 / (p_0 + m)} \; \exp\{-i\vec{p}\cdot\vec{y}\} \; U(p,s) \quad .$$

Now we construct the Q^k operator for which (7) is the eigenfunction with eigenvalue y^k:

$$Q^k \, \psi^{(s)}_{(\vec{y})}(\vec{p}) = y^k \, \psi^{(s)}_{(\vec{y})}(\vec{p}) \quad .$$

To get an explicit momentum space form of $\vec{Q}$, we use the following trick: for any positive-energy wave-function $\psi(\vec{p})$,

$$\begin{aligned} Q^k \, \psi(\vec{p}) &= Q^k \, \textstyle\sum_s \int d\vec{x} \left(\psi^{(s)}_{(\vec{x})}, \psi\right) \psi^{(s)}_{(\vec{x})}(\vec{p}) \\ &= \textstyle\sum_S \int d\vec{x} \cdot x^k \left(\psi^{(s)}_{(\vec{x})}, \psi\right) \psi^{(s)}_{(\vec{x})}(\vec{p}) \quad . \end{aligned}$$

Here we assume that the $\psi^{(s)}_{(\vec{x})}$ eigenfunctions form a complete system for positive-energy solutions. When substituting Eq. (2) and Eq. (7) we get, after x-integration,

$$\begin{aligned} Q^k \, \psi(\vec{p}) &= \int d\vec{q} \left(2\sqrt{q_0 p_0} \Big/ \sqrt{(q_0 + m)(p_0 + m)}\right) \\ &\times \left[\textstyle\sum_s U(\vec{p},s)\, U^+(\vec{p},s)\right] \left[-i \left(\partial/\partial q^k\right) \delta(\vec{q} - \vec{p})\right] \psi(\vec{q}) \quad . \end{aligned}$$

But

$$\begin{aligned} \textstyle\sum_s U(\vec{p},s)\, U^+(\vec{q},s) &= \Lambda_+(\vec{p}) \left(\textstyle\sum_s U(O,s)\, U^+(O,s)\right) \Lambda_+(\vec{q}) \\ &= (1/2)\, \Lambda_+(\vec{p})\, (1+\gamma^0)\, \Lambda_+(\vec{q}) \quad , \end{aligned}$$

so

$$Q^k \, \psi\,(\vec{p}) \;=\; \Lambda_+\,(\vec{p})\,(1+\gamma^0) \\ \times \sqrt{p_0\,(p_0+m)}\,\left[i\,\partial/(\partial\,p^k)\right]\,\sqrt{p_0/(p_0+m)}\,\Lambda_+\,(\vec{p})\,\psi\,(\vec{p}) \quad .$$

From this, we conclude that

(8)
$$Q^k = \Lambda_+\,(\vec{p})\,(1+\gamma_0)\,\sqrt{p_0/(p_0+m)} \\ \times \left[i\partial/(\partial\,p^k)\right]\,\sqrt{p_0/(p_0+m)}\,\Lambda_+\,(\vec{p}) \quad .$$

Some other forms of the Newton–Wigner (N-W) position operator can be presented. First of all, let us see the effect of the Foldy–Wouthuysen (F-W) transformation [8] on Eq. (8).

In the F-W representation, the eigenfunctions of Eq. (7) become

$$\phi^{(s)}_{(\vec{y})}\,(\vec{p}) \;=\; \exp\{i\,W\}\,\psi^{(s)}_{(\vec{y})}\,(\vec{p}) \quad ,$$

where the Foldy–Wouthuysen unitary operator is

$$\exp\{i\,W\} \;=\; \sqrt{2p_0/(p_0+m)} \\ \times\;\left[(1/2)\,(1+\gamma_0)\,\Lambda_+(\vec{p}) + (1/2)\,(1-\gamma_0)\,\Lambda_-(\vec{p})\right] \quad .$$

Using the identity

$$(1/2)\,(1+\gamma_0)\,\Lambda_+(\vec{p})\,U(0,s) \;=\; (1/2)\,(1+\gamma_0)\,\Lambda_+(\vec{p}) \\ \times\,(1/2)\,(1+\gamma_0)\,U(0,s)\,\Lambda_+(\vec{p})\,U(0,s) \\ =\; [(p_0+m)\,/\,2p_0]\,U(0,s) \quad ,$$

we get

$$\phi^{(s)}_{(\vec{y})}\,(\vec{p}) \;=\; [1/(2\pi)^{3/2}]\;\exp\{-i\vec{p}\cdot\vec{y}\}\;U(0,s) \quad .$$

These are N-W position operator eigenstates in the Foldy–Wouthuysen representation and the corresponding form of the position operator itself can be derived from them in the manner described above. The result is (in the momentum space)

$$\vec{Q} \;=\; (1/2)\,(1+\gamma_0)\;i\;(\partial/\partial\vec{p}) \quad ,$$

or in configuration space

$$\vec{Q} \;=\; (1/2)\,(1+\gamma_0)\,\vec{x} \equiv (1/2)\,(1+\gamma_0)\;\vec{x}\;(1/2)\,(1+\gamma_0) \quad .$$

So the Newton–Wigner position operator is just the positive-energy projection of the Foldy–Wouthuysen "mean position operator" [8] and in the Dirac representation, its momentum space form can be presented as

$$\vec{Q} \;=\; \Lambda_+\,(\vec{p})\;\Big(i\;[\partial/(\partial\vec{p})] + i\;[(\beta\vec{\alpha})(2p_0)]$$

$$-\,\{[i\beta\;(\vec{\alpha}\cdot\vec{p})\;\vec{p} + [\vec{\sigma}\times\vec{p}]\;p_0]/2p_0^2\;(p_0+m)\}\Big)\;\Lambda_+\,(\vec{p}) \quad ,$$

where $\vec{\sigma} = (1/2i)\vec{\alpha}\times\vec{\alpha}$.

Note that the Newton–Wigner position operator can be expressed in a representation-independent manner in terms of the Poincaré group generators [9].

The $P_\mu, M_{\mu\nu}$ generators of the Poincaré group are determined by the unitary transformations

$$U = 1 - i\,\epsilon^\mu\, P_\mu - i\,(1/2)\, w_{\mu\nu}\, M^{\mu\nu} \quad , \tag{9}$$

corresponding to the infinitesimal inhomogeneous Lorentz transformation

$$x'_\mu = x_\mu + w_{\mu\nu}\, x^\nu + \epsilon_\mu \quad . \tag{10}$$

These generators have the following physical meanings:

$H = P_0$	is the generator of an infinitesimal time translation.
$J^i = (1/2)\epsilon_{ijk} M^{jk}$	is the generator of an infinitesimal rotation about the i^{th} axis.
$K^i = M^{i0}$	is the generator of an infinitesimal Lorentz transformation along the i^{th} axis.

As we have seen above, the N–W position operator has the simplest form in the Foldy–Wouthuysen representation. Therefore, we begin with Foldy's form [10] for the generators of rotations and Lorentz transformations

$$\vec{J} = \vec{q} \times \vec{P} + \vec{S}$$

(11)

$$\vec{K} = (1/2)\,(H\vec{q} + \vec{q}H) + (H+m)^{-1}\, \vec{P} \times \vec{S} - t\vec{P} \quad ,$$

where $\vec{S}$ is a spin operator and $\vec{q}$ is a "mean position" operator. The second equation in Eq. (11) requires some explanation. To explain its origin, we first derive the expression for the Lorentz transformation generators in the more familiar Dirac representation, where the transformation law for a wave-function is [11]:

$$\psi'(x) = S(\Lambda)\, \psi(\Lambda^{-1}x) \quad ,$$

and for infinitesimal Lorentz transformations

$$(12) \qquad S = 1 - (i/4)\, \sigma_{\mu\nu}\, w^{\mu\nu} \ , \qquad \sigma_{\mu\nu} = (i/2)\,(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) \quad .$$

From the general form [12] for the Lorentz transformation

$$\vec{r'} = \vec{r} - \vec{\beta}t + \left[\left(1/\sqrt{1-\beta^2}\,\right) - 1\right]\ (\vec{\beta}/\beta^2)\ \left[\left(\vec{\beta}\cdot\vec{r}\right) - \beta^2 t\right]$$

$$t' = \left(1/\sqrt{1-\beta^2}\,\right)\ \left\{t - \vec{\beta}\cdot\vec{r}\right\} \quad ,$$

we derive, for the case of the infinitesimal velocity $\vec{\beta}$:

$$\vec{r'} = \vec{r} - \vec{\beta}t \ , \qquad t' = t - \vec{\beta}\cdot\vec{r} \quad .$$

Comparing with Eq. (10), we conclude that the only nonzero $w_{\mu\nu}$ are

$$w_{i0} = -w_{0i} = \beta^i \quad ,$$

so Eqs. (9) and (12) read

$$U = 1 - i\vec{\beta}\cdot\vec{K} \ , \qquad S = 1 - (1/2)\ \vec{\alpha}\cdot\vec{\beta} \quad .$$

Therefore,

$$\psi' \; (\vec{r},t) \equiv (1 - i\vec{\beta}\cdot\vec{K}) \; \psi \; (\vec{r},t) = \left(1 - (1/2)\vec{\alpha}\cdot\vec{\beta}\right) \; \psi \; (\vec{r} + \vec{\beta}t, \; t + \vec{\beta}\cdot\vec{r})$$

$$\approx \left(1 - (1/2) \; \vec{\alpha}\cdot\vec{\beta}\right) \; \left(1 + t\vec{\beta}\cdot\nabla + \vec{\beta}\cdot\vec{r} \, (\partial/\partial t)\right) \; \psi \; (\vec{r},t)$$

$$\approx \left[1 - i\vec{\beta}\cdot\left(\vec{r}H - (i/2) \; \vec{\alpha} - t\vec{P}\right)\right] \; \psi \; (\vec{r},t) \quad ,$$

and the generator for the infinitesimal Lorentz transformation in the Dirac representation is [10]:

$$\vec{K} = \vec{r}H - (i/2) \; \vec{\alpha} - t\vec{P} = (1/2) \; (\vec{r}H + H\vec{r}) - t\vec{P} \quad .$$

Now we can replace $\vec{r}$ by the "mean position" operator $\vec{q}$ according to

$$\vec{r} = \vec{q} - i \; (\beta\vec{\alpha}/2p_0) + \left[\left\{i\beta \, (\vec{\alpha}\cdot\vec{p}) \; \vec{p} + [\vec{\sigma}\times\vec{p}] \; p_0\right\} \Big/ \left\{2p_0^2 \; (p_0 + m)\right\}\right] \; ,$$

and after some algebra, using

$$\beta\vec{\alpha}H + H\beta\vec{\alpha} \; = \; 2i\beta \; (\vec{p}\times\vec{\sigma}) \; ,$$

$$\vec{\sigma}H + H\vec{\sigma} = 2m\beta\vec{\sigma} + 2\vec{p} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

$$\vec{S} \; = \; (1/2) \; \beta\vec{\sigma} \; , \qquad \text{and} \qquad H \; = \; p_0 = \sqrt{\vec{p}^{\,2} + m^2} \; ,$$

.

for positive-energy manifold, we recover Eq. (11).

To express the "mean position" operator in terms of the Poincaré group generators, let us consider [13] the Pauli–Lubanski four-vector

$$W_\mu = (1/2)\, \epsilon_{\mu\nu\lambda\sigma}\, M^{\nu\lambda}\, P^\sigma \quad .$$

For time and space components, we have, in terms of $\vec{J}$ and $\vec{K}$:

$$W_0 = \vec{P} \cdot \vec{J}\, , \qquad \vec{W} = H\vec{J} + \vec{P} \times \vec{K} \quad . \tag{13}$$

If we insert $\vec{J} = \vec{q} \times \vec{P} + \vec{S}$, we get

$$W_0 = \vec{P} \cdot \vec{S} \quad , \tag{14}$$

and after more algebra, using identities which follow from the canonical commutator $[q_i, P_j] = i\delta_{ij}$:

$$[H, \vec{q}] \;=\; -i\,(\vec{P}/H)\, , \qquad \vec{q} \times \vec{P} + \vec{P} \times \vec{q} = 0$$

$$H\vec{q} \times \vec{P} + \vec{P} \times \vec{q}H \;=\; [H, \vec{q}] \times \vec{P} \;=\; 0 \quad ,$$

the expression for the space components becomes

$$\vec{W} = m\vec{S} + (H+m)^{-1}\, (\vec{P} \cdot \vec{S}) \cdot \vec{P} \quad . \tag{15}$$

From Eqs. (13–15), the expression for the spin operator $\vec{S}$ in terms of the Poincaré group generators can be easily derived

$$\vec{S} = m^{-1}\,(H\vec{J} + \vec{P}\times\vec{K}) - m^{-1}\,(H+m)^{-1}\,(\vec{P}\times\vec{J})\,\vec{P} \quad .$$

This should be substituted into

$$\vec{q} = H^{-1}\,\left\{\vec{K} + t\vec{P} - (i/2)\,(\vec{P}/H) - (H+m)^{-1}\,\vec{P}\times\vec{S}\right\} \quad ,$$

which follows from [11] and

$$(1/2)\,(H\vec{q} + \vec{q}H) = H\vec{q} + (1/2)\,[\vec{q}, H] \,.$$

The final result is

$$\vec{q} = H^{-1}\,\left[\vec{K} + t\vec{P} - i\,(\vec{P}/2H)\right] - m^{-1}\,H^{-1}\,(H+m)^{-1}\,\vec{P}\times(H\vec{J} + \vec{P}\times\vec{K})\,, \tag{16}$$

and for the Newton–Wigner position operator

$$\vec{Q} = \Lambda_+\,(\vec{p})\,\vec{q}\Lambda_+\,(\vec{p}) \quad .$$

Actually, Eq. (16) can be viewed [13] as a definition of the position operator in the case of any spin. In particular, for spin zero particles, it reduces to

$$\vec{q} = H^{-1}\,\left(\vec{K} + t\vec{P} - i\,(\vec{P}/2H)\right) \quad .$$

To derive $\vec{K}$, we rewrite the Klein–Gordon equation $(\Box + m^2)\ \varphi\ (x) = 0$ in the Hamiltonian form [14,11]

$$i\ (\partial\phi/\partial t) \ = \ H\ \phi\ , \qquad\qquad \phi = (1/2)\ \begin{bmatrix} \varphi + (i/m)\ (\partial\varphi/\partial t) \\ \varphi - (i/m)\ (\partial\varphi/\partial t) \end{bmatrix}$$

$$H = (\tau_3 + i\ \tau_2)\ (\vec{p}^{\ 2}/2m) + m\ \tau_3 \quad .$$

For infinitesimal Lorentz transformations

$$\varphi'\ (\vec{r},t) \ = \ \varphi\ (\vec{r} + \vec{\beta}t, t + \vec{\beta}\cdot\vec{r})\ ,$$

and

$$\dot{\varphi}'\ (\vec{r},t) \ = \ \vec{\beta}\cdot\nabla\ \varphi\ (\vec{r} + \vec{\beta}t, t + \vec{\beta}\cdot\vec{r}) + \dot{\varphi}\ (\vec{r} + \vec{\beta}t, t + \vec{\beta}\cdot\vec{r})\ ,$$

or, in the two-component form,

$$\begin{aligned} \phi'\ (\vec{r},t) \ &= \Big\{1 - \Big[(\vec{p}\cdot\vec{\beta})/2m\Big]\ \ (\tau_3 + i\,\tau_2)\Big\}\ \ \phi\ (\vec{r} + \vec{\beta}t, t + \vec{\beta}\cdot\vec{r}) \\ &\approx \Big\{1 - i\,\vec{\beta}\cdot\Big[\vec{r}\ H - t\,\vec{P} - i\ (\vec{P}/2m)\ (\tau_3 + i\,\tau_2)\Big]\Big\}\ \ \phi\ (\vec{r},t)\ . \end{aligned}$$

From this, we conclude that

$$\vec{K} \ = \ \vec{r}\,H - i\,(\vec{P}/2m)\,(\tau_3 + i\ \tau_2) - t\vec{P} = H\ \vec{r} - t\ \vec{P} + i\ (\vec{P}/2m)\ (\tau_3 + i\ \tau_2) \quad ,$$

and the "mean position" operator for the spinless particles takes the form [note that $H^{-1} = H/(\vec{p}^{\ 2} + m^2)$]

$$\begin{aligned} \vec{q} \ &= \ \vec{r} + i\ \Big[\vec{P}/2\ (\vec{p}^{\ 2} + m^2)\Big]\ \ \{(H/m)\ (\tau_3 + i\ \tau_2) - 1\} \\ &= \ \vec{r} + i\ \Big[\vec{P}/2\ (\vec{p}^{\ 2} + m^2)\Big]\ \ \tau_1 \quad . \end{aligned}$$

3. – A spinless particle in a square well potential

For a scalar particle in a spherically symmetric square well potential, the Klein–Gordon equation is

$$[E - V(\vec{r})]\ \phi \ = \ -(\tau_3 + i\ \tau_2)\ (\nabla^2/2m)\ \phi + m\ \tau_3\ \phi \quad ,$$

where

$$V\ (\vec{r}) \ = \ -V_0\ \Theta\ (r_0 - r) \ = \ \begin{cases} -V_0\ , & \text{if } r \leq r_0 \\ 0\ , & \text{if } r > r_0 \end{cases} \quad .$$

This problem can easily be solved for the spherically symmetric ground state

$$\phi\ (\vec{r}) = \frac{1}{r} \begin{pmatrix} U_1\ (r) \\ U_2\ (r) \end{pmatrix} \quad ,$$

giving (hereafter we shall usually take $m = 1$)

$$\phi\ (\vec{r}) = \begin{pmatrix} 1 + E - V\ (r) \\ 1 - E + V\ (r) \end{pmatrix}\ [U\ (r)/r] \quad , \tag{17}$$

where .0

$$U\ (r) = \begin{cases} A\ \sin\ (K_1 r)\ , & \text{if } r \leq r_0 \\ A\ \sin\ (K_1 r_0)\ \exp\{-K_2\ (r - r_0)\}\ , & \text{if } r > r_0\ , \end{cases}$$

and

$$K_1 = \sqrt{(E + V_0)^2 - 1}\ , \qquad K_2 = \sqrt{1 - E^2} \quad .$$

A is defined from the normalization condition

$$\int d\ \vec{r}\ \phi^+\ (\vec{r})\ \tau_3\ \phi\ (\vec{r}) = 1 \quad ,$$

and an evaluation of integrals gives

$$(18)\quad |A|^2 = (1/8\pi)\ \left\{(E+V_0)\ \left[r_0 - (1/K_1)\ \sin\,(K_1 r_0) \times \cos\,(K_1 r_0)\right] + (E/K_2)\ \sin^2\,(K_1 r_0)\right\}^{-1} \quad .$$

The smoothness of the solution shown in Eq. (17) demands the following relation

$$K_1\ \mathrm{ctg}\ (K_1 r_0) = -K_2 \quad ,$$

which defines energy eigenvalue E.

A mean value of r is

$$\langle r \rangle = \int d\,\vec{z}\ \phi^+\ (\vec{z})\ \tau_3\ z\ \phi\ (\vec{z}) \quad ,$$

and substituting Eq. (17), we get

$$\langle r \rangle \ = \ 8\,\pi\,|A|^2\ \Big\{(E+V_0)\ \Big[(r_0^2/2) + \{[\sin^2\,(K_1 r_0)]\,/2K_1^2\} - \{[r_0\ \sin\,(2K_1 r_0)]\,/2K_1\}\Big] + E\ \sin^2\,(K_1 r_0)\ \left[(r_0/K_2) + \{1/2K_2^2\}\right]\Big\} \quad .$$

To calculate an analogous quantity for the Newton–Wigner position operator

$$\vec{Q} = \Lambda_+\ (\vec{p})\ \left\{\vec{r} + i\ \left[\vec{P}/2(\vec{P}^{\,2} + m^2)\right]\ \tau_1\right\}\ \Lambda_+\ (\vec{p}) \quad ,$$

it is convenient to work in the Foldy–Wouthuysen representation:

$$(19)\qquad \varphi = U\ \phi\,, \qquad \vec{R} = U\ \vec{Q}\ U^{-1} \quad ,$$

where [14,8]

$$U = \sqrt{m/p_0}\ [(p_0 + \tau_3\ H_0)/(p_0 + m)]$$

$$= [1/(2\ \sqrt{m\ p_0})]\ [(p_0 + m) + \tau_1\ (p_0 - m)] \quad ,$$

(20)

$$H_0 = (\tau_3 + i\ \tau_2)\ (\vec{p}^{\ 2}/2m) + m\ \tau_3\ ,$$

and

$$p_0 = \sqrt{\vec{p}^{\ 2} + m^2} \quad .$$

The Newton–Wigner position operator has a particularly simple form in this representation

$$\vec{R} = (1/2)\ (1 + \tau_3)\ \vec{r} \equiv (1/2)\ (1 + \tau_3)\ [i\,(\partial/\partial\ \vec{p})]\ .$$

So for the mean value of $\vec{R}^{\ 2}$, we have

(21) $$\langle \vec{R}^{\ 2} \rangle = \int d\ \vec{p}\ \varphi^{+}\ (\vec{p})\ \tau_3\ \vec{R}^{\ 2}\ \varphi\ (\vec{p}) = -\int d\ \vec{p}\ \varphi_1^{+}\ (\vec{p})\ \left[\triangle_{\vec{p}}\ \varphi_1\ (\vec{p})\right]\ ,$$

and from Eqs. (19–20) (we have set $m = 1$),

(22) $$\varphi_1\ (\vec{p}) = [1/(2\sqrt{p_0})]\ [(p_0 + 1)\ \phi_1\ (\vec{p}) + (p_0 - 1)\ \phi_2\ (\vec{p})]\ .$$

The momentum space wave function $\phi(\vec{p}) = \begin{pmatrix} \phi_1\ (\vec{p}) \\ \phi_2\ (\vec{p}) \end{pmatrix}$ can be evaluated using Eq. (17):

$$\phi\ (\vec{p}) = [1/(2\pi)^{3/2}] \int d\,\vec{x}\ \phi\,(\vec{x}) \exp\{-i\vec{p}\cdot\vec{x}\}$$

$$= \sqrt{2/\pi}\ (A/p) \left\{ \begin{pmatrix} 1 + E + V_0 \\ 1 - E - V_0 \end{pmatrix} \right.$$

$$\times\ [\{K_1 \sin(r_0\ p) \cos(r_0\ K_1) - p \cos(r_0\ p) \sin(r_0\ K_1)\}\ /\ (p^2 - K_1^2)]$$

$$\left. + \begin{pmatrix} 1 + E \\ 1 - E \end{pmatrix} \left[\{\sin(K_1\ r_0)\ [K_2 \sin(r_0\ p) + p \cos(r_0\ p)]\}\ /\ (p^2 + K_2^2) \right] \right\} .$$

(23)

We see that $\varphi_1(\vec{p})$ is spherically symmetric and real, and (21) can be rewritten as

$$(24) \qquad \langle \vec{R}^{\,2} \rangle = -4\pi \int_0^\infty d\,p \cdot p\,\varphi_1\,(p)\,(d^2/dp^2)\,(p\,\varphi_1\,(p)) = 4\pi \int_0^\infty d\,p\,\left[(d/dp)\,(p\,\varphi_1\,(p))\right]^2 \quad .$$

$R = \sqrt{\langle \vec{R}^{\,2} \rangle}$ can be calculated numerically using Eqs. (24), (22–23) and (18).

The summary of the numerical results, for the case $r_0 = m = 1$, is presented in Figure 1. The solid line gives ground state energy E versus potential strength. It decreases as V_0 increases. For sufficiently large V_0, a second root appears near the lower continuum, which can be interpreted [15] as an antiparticle bound state. As V_0 increases further, the gap between particle and antiparticle bound states diminishes, and for some critical value $V_{cr} \approx 2.97$ of a potential strength, they coincide. Near the critical value of V_0, however, the mean size $\langle r \rangle$ of the bound system (dashed line) becomes negative, clearly indicating that a single particle interpretation of the Klein–Gordon equation is no longer possible in this region. Actually, a single particle interpretation in this case becomes questionable even for much smaller values of V_0, because the Newton–Wigner mean radius (dotted line) indicates that for $V_0 \approx 1$, we already have almost minimal localization.

4. – The Dirac particle in a square well Potential

As a second example, consider a spin-1/2 particle in a square well potential, $V(\vec{r}) = -V_0\Phi(r_0 - r)$. Although a comprehensive study of this, as well as the previous scalar particle case, can be found in the literature [16], for the sake of completeness, we give a brief self-contained treatment of the problem.

First of all, let us rewrite the Dirac equation

$$[\vec{\alpha}\cdot\vec{p} + \beta\, m + V\,(\vec{r})]\;\Psi = E\;\Psi$$

in a form which is convenient for spherically symmetric potentials [17]. Using $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$ with $\vec{A} = \vec{B} = \hat{\vec{r}} \equiv \vec{r}/r$ and $\vec{C} = \nabla$, we obtain

$$\nabla \;=\; \hat{\vec{r}}\,(\partial/\partial r) - \hat{\vec{r}}\times(\vec{r}\times\nabla) \;=\; \hat{\vec{r}}\,(\partial/\partial r) - i\,\hat{\vec{r}}\times\vec{L} \quad ,$$

where $\vec{L} = \vec{r}\times\vec{p} = -i\vec{r}\times\nabla$ is the orbital angular momentum operator. So

$$(25) \qquad \vec{\alpha}\cdot\vec{p} = -i\,\alpha_r\;(\partial/\partial r) - (\vec{\alpha}/r)\cdot\left(\hat{\vec{r}}\times\vec{L}\right)\;, \qquad \alpha_r = [(\vec{\alpha}\cdot\vec{r})/r]\;.$$

From $(\vec{\alpha}\cdot\vec{A})(\vec{\alpha}\cdot\vec{B}) = \vec{A}\cdot\vec{B} + i\vec{\sigma}\cdot(\vec{A}\times\vec{B})$ with $\vec{A} = \hat{\vec{r}}$ and $\vec{B} = \vec{L}$, we get, using $\vec{\sigma} = \gamma_5\vec{\alpha}$, $\hat{\vec{r}}\cdot\vec{L} = 0$,

$$\vec{\alpha}\cdot\left(\hat{\vec{r}}\times\vec{L}\right) \;=\; -i\,\alpha_r\;\vec{\sigma}\cdot\vec{L} \quad .$$

Substituting this into Eq. (25),

$$\vec{\alpha}\cdot\vec{p} \;=\; -i\,\alpha_r\;(\partial/\partial r) + i\,(\alpha_r/r)\;\vec{\sigma}\cdot\vec{L} \quad .$$

Therefore, the Dirac equation can be rewritten as [17]

$$(26) \quad \left[-i\,\gamma_5\,\sigma_r\;\{(\partial/\partial r) - [(\beta/r)\;K] + (1/r)\} + \beta\,m + V\right]\Psi = E\;\Psi \quad ,$$

where $\sigma_r = (\vec{\sigma} \cdot \vec{r})/r$ and the operator $K = \beta(\vec{\sigma} \cdot \vec{L} + 1)$ commutes with the Hamiltonian for a central potential.

For a ground state, we have

$$K\,\Psi = \Psi\,, \qquad \vec{J}^{\,2}\,\Psi = (3/4)\,\Psi\,, \tag{27}$$

and from $\vec{J}^{\,2} = (\vec{L} + (1/2)\vec{\sigma})^2$, we get

$$\vec{J}^{\,2}\,\Psi = -\vec{\sigma}\cdot\vec{L}\,\Psi\ . \tag{28}$$

Let us take

$$\Psi = \begin{pmatrix} g\,(r)\,\chi_1\,(\hat{\vec{r}}) \\ i\,f\,(r)\,\chi_2\,(\hat{\vec{r}}) \end{pmatrix}\ .$$

The imaginary unit i is introduced to make the radial functions g and f real. We then get, using Eqs. (27–28)

$$\vec{L}^{\,2}\,\chi_1\,(\hat{\vec{r}}) = 0\,, \qquad \vec{L}^{\,2}\,\chi_2\,(\hat{\vec{r}}) = 2\chi_2\,(\hat{\vec{r}})\ .$$

Therefore, χ_1 and χ_2 are spin-orbital functions with $\ell = 0$ and $\ell = 1$, respectively. Choosing, for definiteness, the up spin projection case, $J_z = 1/2$, the explicit expressions for them are [17]

$$\begin{aligned} \chi_1\,(\hat{\vec{r}}) &= \begin{pmatrix} Y_0^0 \\ 0 \end{pmatrix} = (1/\sqrt{4\pi})\begin{pmatrix} 1 \\ 0 \end{pmatrix}\,, \\ \chi_2\,(\hat{\vec{r}}) &= \begin{pmatrix} -(1/\sqrt{3})\ Y_1^0 \\ \sqrt{2/3}\ Y_1^1 \end{pmatrix} = -(1/\sqrt{4\pi})\begin{pmatrix} \cos\theta \\ e^{i\varphi}\sin\theta \end{pmatrix}\ . \end{aligned} \tag{29}$$

Radial equations for the ground state can be easily obtained now from Eq. (26). Using $K\Psi = \Psi$, and the relations

$$\sigma_r\,\chi_1\,(\hat{\vec{r}}) = -\chi_2\,(\hat{\vec{r}}) \quad \text{and} \quad \sigma_r\,\chi_2(\hat{\vec{r}}) = -\chi_1\,(\hat{\vec{r}})\,, \tag{30}$$

which readily follow from Eq. (29), and

$$\sigma_r = [(\vec{\sigma} \cdot \vec{r})/r] = \begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix} \quad ,$$

these radial equations are

$$(d\,g/d\,r) = (E - V + m) \cdot f \quad \text{and} \quad (d\,f/d\,r) = -(2/r)\,f - (E - V - m) \cdot g \,.$$

Solutions for a square well potential look like $g(r) = (U_1(r))/r$, $f(r) = (U_2(r))/r$, where (we take $m = 1$)

$$U_1\,(r) = \begin{cases} A \sin\,(K_1 r)\ , & \text{if } r \leq r_0 \\ A\ \sin\,(K_1\, r_0)\ \exp\{-K_2(r - r_0)\}\ , & \text{if } r > r_0 \end{cases}$$

$$U_2\,(r) = [1/(E - V + 1)]\ \left[(d/dr) - \frac{1}{r}\right]\ U_1\,(r)$$

$$= \begin{cases} [A/(E + V_0 + 1)]\ [K_1 \cos(K_1\, r) - (1/2)\sin(K_1\, r)]\ , & \text{if } r \leq r_0\ , \\ -\left\{[A \sin\,(K_1\, r_0)]\ /(1 + E)\right\} & \\ \qquad \times\ [K_2 + (1/r)]\ \exp\{-K_2(r - r_0)\}\,, & \text{if } r > r_0\ , \end{cases}$$

and $K_1 = \sqrt{(E + V_0)^2 - 1}$, $K_2 = \sqrt{1 - E^2}$.

The normalization constant A is defined by

$$\int d\,\vec{r}\,\psi^+\,(\vec{r})\ \psi\,(\vec{r}) = 1 \quad ,$$

which gives (for $r_0 = m = 1$)

$$|A|^2 = (1/2) - [\sin(2K_1)/4K_1] + [(\sin^2 K_1)/2K_2]$$

$$+ [(\sin^2 K_1)/(1+E)^2][(1+1/2K_2)]$$

$$+ [1/(1+V_0+E)^2][(K_1^2/2) - \sin^2 K_1 + \{[K_1 \sin(2K_1)]/4\}]\}^{-1} \ .$$

The energy eigenvalue equation follows from continuity of $U_2(r)$ at $r = r_0$

$$K_1 \text{ ctg } (K_1 r_0) = -K_2 - [V_0/(1+E)] \ [K_2 + (1/r_0)] \ .$$

It is straightforward to calculate a mean radius of the system $\langle r \rangle = \int d\vec{z}\ \psi^+(\vec{z})\ z\psi(\vec{z}) = \int_0^\infty dz \ \cdot z \cdot (U_1^2 + U_2^2)$. The solution has the form (for $r_0 = m = 1$):

$$\langle r \rangle = |A|^2 \Big\{ (1/4) + [(\sin^2 K_1)/4K_1^2] - [\sin(2K_1)/4K_1]$$

$$+ [(\sin^2 K_1)/2K_2^2] \ [(1/2) + K_2]$$

$$+ [(\sin^2 K_1)/(1+E)^2] \ [(5/4) + (K_2/2) - \exp\{2K_2\} \cdot Ei(-2K_2)]$$

$$+ [1/(1+V_0+E)^2] \ [(K_1^2/4) - (5/4) \ \sin^2 K_1 + \{[K_1 \ \sin(2K_1)]/4\}$$

$$+ (1/2) \ \{\gamma + \ell n \ (2K_1) - Ci(2K_1)\}] \Big\} \ ,$$

where the exponential and cosine integrals are defined by Euler's constant ($\gamma \approx 0.57721$):

$$Ei(x) = \int_{-\infty}^{x} (e^t/t)\, dt \ ,$$

$$Ci(x) = \gamma + \ell n \ |x| + \int_0^x [(\cos t - 1)/t]\, dt \ .$$

To evaluate the Newton–Wigner mean size of the system, it is convenient, as in the scalar case, to work with momentum space wave functions in the Foldy–Wouthuysen representation

$$\phi\ (\vec{p}) = \exp\{iW\}\ \Psi\ (\vec{p})\ ,$$

where

$$\begin{aligned}\exp\{iW\} &= [(p_0 + \beta\, H_0)/\sqrt{2p_0\ (p_0 + m)}] \\ &= \sqrt{[(1 + p_0)/2p_0} + [(\beta\ \vec{\alpha}\cdot\vec{p})/\sqrt{2p_0\ (p_0 + m)}]\ ,\end{aligned}$$

and

$$\Psi\ (\vec{p}) = \frac{1}{(2\pi)^{3/2}} \int d\ \vec{x}\ \exp\{-i\vec{p}\cdot\vec{x}\}\ \Psi\ (\vec{x})\ .$$

Angular integrals can be calculated by the use of a plane wave expression

$$\exp\{-i\vec{p}\cdot\vec{x}\} = 4\pi \sum\nolimits_{e,m} (-i)^{\ell}\ j_e\ (p\ r)\ Y_e^{*m}\ (\hat{\vec{x}})\ Y_e^{m}\ (\hat{\vec{p}})\ ,$$

and we get

$$\Psi\ (\vec{p}) = \begin{pmatrix} \tilde{U}_1\ (p)\ \chi_1\ (\hat{\vec{p}}) \\ \tilde{U}_2\ (p)\ \chi_2\ (\hat{\vec{p}}) \end{pmatrix}\ ,$$

with

$$\tilde{U}_1\,(p) \;=\; \sqrt{2/\pi}\,\int_0^\infty dz\cdot z\cdot j_0\,(p\,z)\,U_1\,(z) \;=\; \sqrt{2/\pi}\,(A/p)$$

$$\times\left[\left\{[K_1\sin\,(p)\cos\,(K_1)-p\sin\;K_1\cos\;p]/(p^2-K_1^2)\right\}\right.$$

$$\left.+\left\{[\sin\,(K_1)\;(p\cos\;p+K_2\sin\;p)]/(p^2+K_2^2)\right\}\right]\;,$$

$$\tilde{U}_2\,(p) \;=\; \sqrt{\frac{2}{\pi}}\int_0^\infty d\;z\cdot z\cdot j_1\,(p\,z)\,U_2\,(z)$$

$$=\;\sqrt{2/\pi}\,(A/p)\;\left\{1/(1+V_0+E)\,[(\sin\;p/p)\sin\;K_1\right.$$

$$-\;[K_1/(p^2-K_1^2)]\,(p\sin\;p\cos\;K_1-K_1\sin\;K_1\cos\;p)]$$

$$-\;[\sin\;K_1/(1+E)]$$

$$\left.\times\;[(\sin\;p/p)-[K_2/(p^2+K_2^2)\,(K_2\cos\;p-p\sin\;p)]\right\}\;,$$

j_0 and j_1 being spherical Bessel functions:

$$j_0\,(z)=(\sin\;z)/z \qquad\qquad j_1\,(z)=(1/z)\;[(\sin\;z/z)-\cos\;z]\;.$$

Using $\beta\vec{\alpha}\cdot\vec{p}=p\begin{pmatrix}0 & \sigma_p\\ -\sigma_p & 0\end{pmatrix}$ and Eq. (30), we get for the momentum space FW wave function

$$\phi\,,(\vec{p})\;=\;\begin{bmatrix}\left\{\sqrt{[(1+p_0)/2p_0]}\,\tilde{U}_1(p)-\left[p/\sqrt{2\;p_0\,(1+p_0)}\right]\,\tilde{U}_2(p)\right\}\chi_1(\hat{\vec{p}})\\[2ex] \left\{\sqrt{[(1+p_0)/2p_0]}\,\tilde{U}_2(p)+\left[p/\sqrt{2\;p_0\,(1+p_0)}\right]\,\tilde{U}_1(p)\right\}\chi_2(\hat{\vec{p}})\end{bmatrix}.$$

Therefore, the Newton–Wigner mean radius is $R = \sqrt{\langle \vec{R}^{\,2} \rangle}$, where

$$\begin{aligned} \langle \vec{R}^{\,2} \rangle &= \int d\vec{p}\, \phi^{+}(\vec{p})\, (1/2)\, (1+\gamma_0) \left[-\Delta_{\vec{p}}\, \phi(\vec{p}) \right] \\ &= \int_0^{\infty} \left[d\, G(p)/d\, p \right]^2 \, d\, p \ , \end{aligned}$$

and

$$G(p) \;=\; p \sqrt{[(1+p_0)/2p_0]}\, \tilde{U}_1\, (\vec{p}) - \left[p^2 / \sqrt{2\, p_0\, (1+p_0)} \right] \, \tilde{U}_2\, (p) \ .$$

Figure 2 presents the numerical results. Unlike the scalar case, the Newton–Wigner mean size remains close to the ordinary one up to the very vicinity of the critical point, and it seems as if minimal localization is not reached until this point.

5. Concluding remarks

Although only the square well potential was considered, we may speculate that minimal localization doesn't occur until the critical point for Dirac particles, and so the one-particle picture remains reasonable. This is no longer valid for scalar particles [18] because vacuum polarization effects are much more important in this case.

Near the critical point, the fact that virtual particles may be created and destroyed can be accounted for by effective potential. Our final remark provides an argument that this potential is expected to be nonlocal and energy dependent.

A formally exact relativistic equation for the bound-state problem is the Bethe–Salpeter (B-S) equation [19]. In the "center-of-mass" frame, with the fermion propagators replaced by the free ones with effective constituent masses m_1 and m_2, this equation looks like [20]

$$\left[p_0^{(1)} - H^{(1)}(\vec{p})\right]\left[p_0^{(2)} - H^{(2)}(-\vec{p})\right]\Psi(p) = (i/2\pi)\int d\,K\,G(K)\,\Psi(p+K)\,, \tag{31}$$

where $p_0^{(1)} = p_0 + \mu_1 E$, $p_0^{(2)} = \mu_2 E - p_0$, $\mu_1 = m_1/(m_1+m_2)$, $\mu_1 = m_2/(m_1+m_2)$, E is the bound-state energy eigenvalue, $H(\vec{p}) = \vec{\alpha}\cdot\vec{p} + \beta m$ is one particle Dirac Hamiltonian and $\Psi(p) \equiv \Psi(\vec{p}, p_0)$ is a 16–component spinor.

For the very localized systems that we are interested in, the instantaneous approximation looks reasonable. Suppose the B-S interaction kernel doesn't depend on relative energy (or on relative time in configuration space), then

$$G(K) \equiv G(\vec{K})\,;$$

so a three-dimensional equation can be derived [20] from Eq. (31) for the equal-time Schrödinger-type amplitude

$$\phi(\vec{p}) = \int_{-\infty}^{\infty}\Psi(\vec{p}, p_0)\,d\,p_0\,.$$

Using ($\Lambda_\pm$ being positive and negative energy projection operators)

.

$$\Lambda_\pm(\vec{p})\,H(\vec{p}) = \pm\sqrt{\vec{p}^{\,2}+m^2}\,\Lambda_\pm(\vec{p})\,, \tag{32}$$

Equation (31) can be written as a system of four equations

$$\Lambda_{++}\,\Psi\,(p) = (i/2\pi)\,\Bigg[\left(\mu_1\,E + p_0 - \sqrt{m_1^2 + \vec{p}^{\,2}} + i\,\epsilon\right) \times \left(\mu_2\,E - p_0 - \sqrt{m_2^2 + \vec{p}^{\,2}} + i\,\epsilon\right)\Bigg]^{-1} \times \Lambda_{++}\int d\,\vec{K}\;G(\vec{K})\;\phi\,(\vec{p} + \vec{K})$$

$$\Lambda_{+-}\,\Psi\,(p) = (i/2\pi)\,\Bigg[\left(\mu_1\,E + p_0 - \sqrt{m_1^2 + \vec{p}^{\,2}} + i\,\epsilon\right) \times \left(\mu_2\,E - p_0 + \sqrt{m_2^2 + \vec{p}^{\,2}} - i\,\epsilon\right)\Bigg]^{-1} \times \Lambda_{+-}\int d\,\vec{K}\;G(\vec{K})\;\phi\,(\vec{p} + \vec{K})$$

(33)

$$\Lambda_{-+}\,\Psi\,(p) = (i/2\pi)\,\Bigg[\left(\mu_1\,E + p_0 + \sqrt{m_1^2 + \vec{p}^{\,2}} - i\,\epsilon\right) \times \left(\mu_2\,E - p_0 - \sqrt{m_2^2 + \vec{p}^{\,2}} + i\,\epsilon\right)\Bigg]^{-1} \times \Lambda_{-+}\int d\,\vec{K}\;G(\vec{K})\;\phi\,(\vec{p} + \vec{K})$$

$$\Lambda_{--}\,\Psi\,(p) = (i/2\pi)\,\Bigg[\left(\mu_1\,E + p_0 + \sqrt{m_1^2 + \vec{p}^{\,2}} - i\,\epsilon\right) \times \left(\mu_2\,E - p_0 + \sqrt{m_2^2 + \vec{p}^{\,2}} - i\,\epsilon\right)\Bigg]^{-1} \times \Lambda_{--}\int d\,\vec{K}\;G(\vec{K})\;\phi\,(\vec{p} + \vec{K}) \quad ,$$

where

$$\Lambda_{++} = \Lambda_+^{(1)}(\vec{p})\,\Lambda_+^{(2)}(-\vec{p}) \quad , \qquad \Lambda_{+-} = \Lambda_+^{(1)}(\vec{p})\,\Lambda_-^{(2)}(-\vec{p})$$

$$\Lambda_{-+} = \Lambda_-^{(1)}(\vec{p})\,\Lambda_+^{(2)}(-\vec{p}) \quad , \qquad \Lambda_{--} = \Lambda_-^{(1)}(\vec{p})\,\Lambda_-^{(2)}(-\vec{p}) \quad .$$

Now we can integrate both sides of each equation in Eq. (33) over p_0 using

$$\int_{-\infty}^{\infty} [d\,p_0/((p_0 - a \pm i\,\epsilon)\,(p_0 - b \pm i\,\epsilon))] \;=\; 0\;,$$

and

$$\int_{-\infty}^{\infty} [d\,p_0/(p_0 - a \pm i\,\epsilon)\,(p_0 - b \mp i\,\epsilon)] \;=\; [\mp 2\pi\; i/(a-b)] \quad ,$$

and derive

$$\Lambda_{++}\,\phi(\vec{p}) = (\Lambda_{++}/E - \sqrt{m_1^2 + \vec{p}^{\,2}} - \sqrt{m_2^2 + \vec{p}^{\,2}})$$

$$\int d\vec{K}\,G(\vec{K})\,\phi(\vec{p} + \vec{K})\;,$$

$$\Lambda_{+-}\,\phi(\vec{p}) = \Lambda_{-+}\,\phi(\vec{p}) = 0$$

and

$$\Lambda_{--}\,\phi(\vec{p}) = (\Lambda_{--}/E + \sqrt{m_1^2 + \vec{p}^{\,2}} + \sqrt{m_2^2 + \vec{p}^{\,2}})$$

$$\int d\vec{K}\,G(\vec{K})\,\phi(\vec{p} + \vec{K})\;.$$

Taking into account Eq. (32), these four equations can be written as one single Salpeter equation [20]

$$(34) \quad \left[E - H^{(1)}(\vec{p}) - H^{(2)}(-\vec{p})\right]\;\phi(\vec{p}) \;=$$

$$= \left[\Lambda_+^{(1)}(\vec{p})\;\Lambda_+^{(2)}(-\vec{p}) \;-\; \Lambda_-^{(1)}(\vec{p})\;\Lambda_-^{(2)}(-\vec{p})\right] \int d\vec{K}\,G(\vec{K})\;\phi(\vec{p} + \vec{K})\;.$$

The reason we reproduced its derivation is as follows. In contrast to the Schrödinger wavefunctions, the B-S amplitude depends on the relative time variable (in configuration space). Clearly, one source of its appearance is a retardation of an interaction. The Salpeter equation shows that even in instantaneous approximation the effective interaction is nonlocal and energy dependent because of the presence of projective operators in the right hand side of Eq. (34). In nonrelativistic quantum mechanics, we can also derive the BS equation [21], but its Salpeter-like reduction gives exactly the Schrödinger equation with local potential. The crucial difference between the two cases is that in the nonrelativistic theory, we have retarded Green's functions instead of Feynman propagators. If we change the $m \rightarrow m - i\epsilon$ Feynman prescription to $p_0^{(1,2)} \rightarrow p_0^{(1,2)} + i\epsilon$, which corresponds to a retarded Green's function, and repeat the Salpeter reduction we end up with the Breit equation [22]

$$\left[E - H^{(1)}(\vec{p}) - H^{(2)}(-\vec{p})\right] \phi(\vec{p}) = \int d\vec{K}\, G(\vec{K})\, \phi(\vec{p} + \vec{K}) \quad ,$$

having local interaction.

Therefore, the second source for the essential relative time dependence of the B-S amplitude is the possibility for particles to turn backward in time. In the three-dimensional effective theory, this results in nonlocal and energy dependent interaction.

* * *

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Figure Captions

1. A summary of the numerical results (as described in the text) for a spinless particle in a square well potential versus potential strength. The solid line denotes the ground state energy (righthand axis); the dashed line shows the mean size of the bound system (lefthand axis); and the dotted line shows the Newton-Wigner mean size (lefthand axis).

2. A summary of the numerical results (as described in the text) for the Dirac particle in a square well potential versus potential strength. The solid line denotes the ground state energy (righthand axis); the dashed line shows the mean size of the bound system (lefthand axis); and the dotted line shows the Newton-Wigner mean size (lefthand axis).

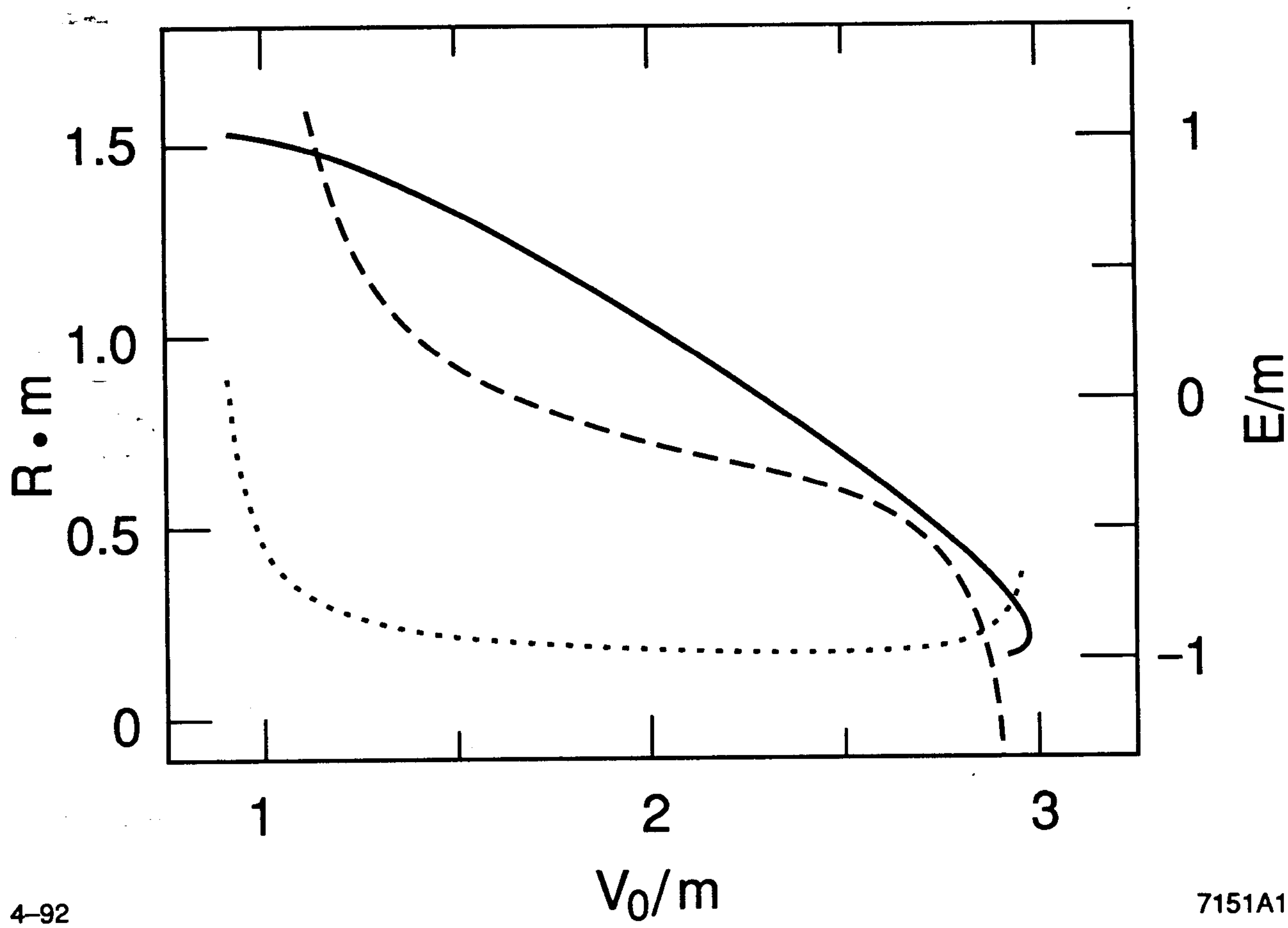

Fig. 1

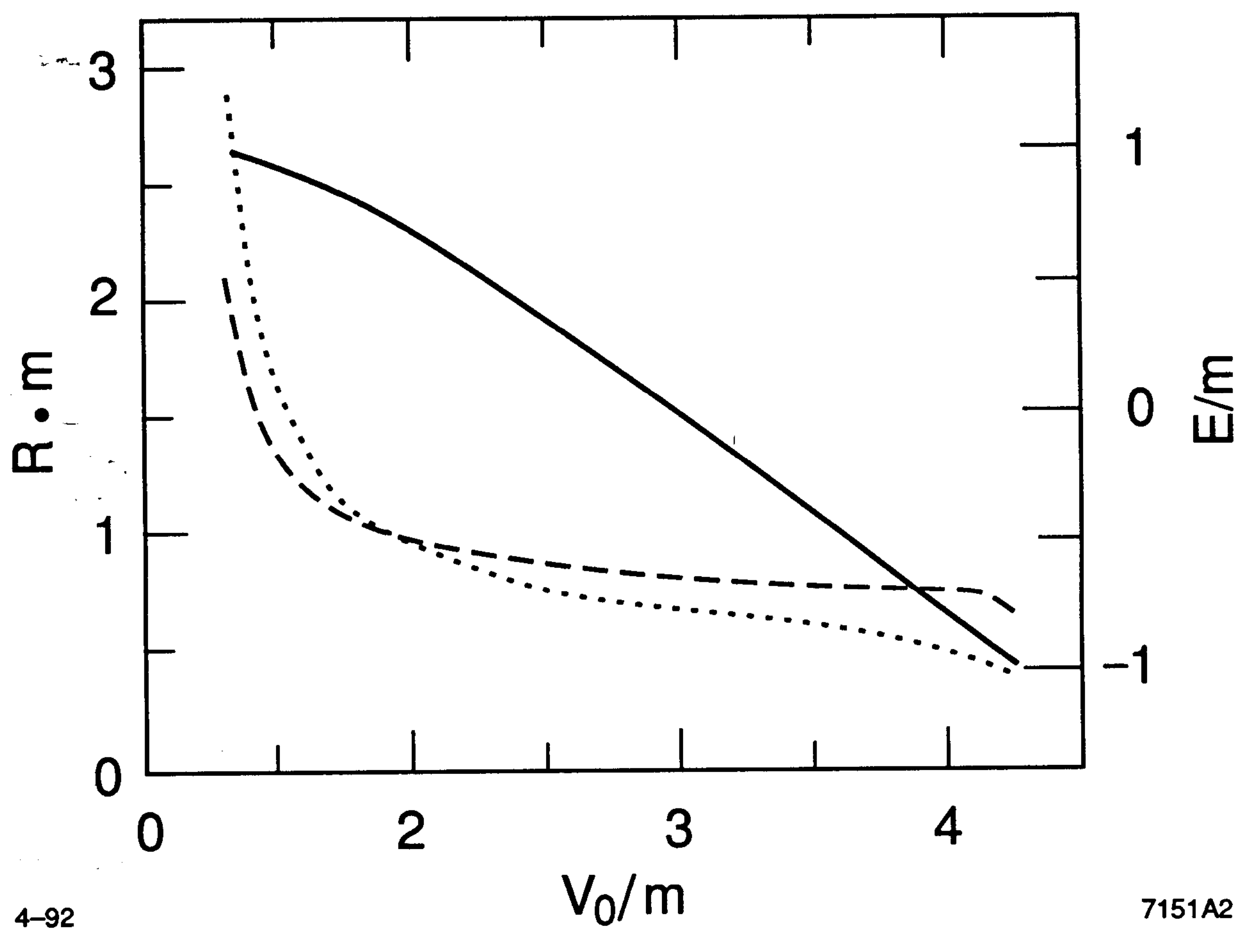

Fig. 2