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THE RQM TRIANGLE:

A Paradigm for Relativistic Quantum Mechanics*

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ABSTRACT

A simple way to relate Lorentz transformations to a finite and discrete model of counter firings is presented. We first abstract from counter firings with finite resolution to rational fraction velocities and a finite step-length. Discrete Lorentz transformations and quantized rotations follow. These are encoded by three integers and a triangle with integer sides: the *RQM Triangle.* The double slit experiment allows us to observe quantized space steps and measure invariant step lengths for any particle in rational ratio to one such length for any convenient reference particle. A discrete version of the Mandelstam analysis of elastic and anelastic scattering is implied. The results fit naturally into the bit-string formalism used in earlier work.

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1. Counter Physics

1.1. Particles and Events

A **PARTICLE** *is a conceptual carrier of conserved quantum numbers* Q^i *between events.*

An **EVENT** *is a region with incoming and outgoing particles through which quantum numbers are conserved:* $\Sigma Q^i_{in} - \Sigma Q^i_{out} = 0$.

According to Clive Kilmister my definition of particle is basically a paraphrase of Eddington. I do not know how Eddington thought of events. They cannot be, for us, the point events of Einstein's special relativity. Whitehead thought of events as the overlap between *space-time* regions; in his view, this overlap could be progressively narrowed as more and more information was supplied. I am fishing for a similar idea, but one which encompasses the multiple connectivities that preclude the "point limit" from arising. We must not allow the concept of a *pre-existent* space-time to creep in by the back door. Perhaps the word "region" used in my definition already carries too much conceptual baggage with it. Suggestions for a better definition of "event" would be welcome.

1.2. The Counter Paradigm

We start from considerations as close to contemporary experimental practice as possible. We try to pick concepts that allow a simple mathematical model to be abstracted. Gazing back toward early papers by Bastin and Kilmister on the interpretation of tracks in a cloud chamber, I now realize that my construction has interesting areas of contact with their thinking in the 50's. For me a "counter" is any device occupying some reasonably well defined spacial region in the laboratory, which can be activated ("fired") by either a massive particle or a gamma ray with a reasonably well defined time resolution. It can be thought of as containing an internal "clock," and a "memory" which records the time of firing and may record whether the event can be attributed to a particle or a gamma ray.

In practice, much of the apparatus which accomplishes this space-time localization of the counter and the event within it is actually external to the space-time region within which the event is presumed to have occurred; the details are unimportant in our abstract treatment. Much of the hard work in experimental particle physics is devoted to modeling this region, the probability distributions of particles and gamma rays which pass through it and the probability distributions which should (and should not) be attributed to what we summarize as "counter firings." The end product is a model of the apparatus reduced to bits on tape in some computer (or some equivalent form of discrete memory storage). Events are also recorded as bits in this context. They can then be used to test various theories of what is going on, also expressed as bits on tape. This makes a bit-string model for the theory particularly appropriate; whatever mathematics is used in formulating the theory, the actual comparison with experiment is always carried out at the bit-string level.

This practice assigns a spacial resolution $\Delta x, \Delta y, \Delta z$ and a time resolution Δt to the counter. To give more precision to the model, we need two counters a distance $L = N_L \Delta x$ apart in the x direction, and counter firings a time interval $T = N_T \Delta t$ apart. In order to measure T, we must first synchronize the clocks by sending a gamma ray to one and returning a gamma ray back. The time at the distant counter is, by the Einstein convention, taken to be half way between these two events in the reference counter. Empirically we find that no signal travels faster than a gamma ray; in the absence of further information any signal which travels close to this limiting velocity, which we call c if we need to express it in dimensional units, will serve for our clock synchronization. We reduce our results to dimensionless form by defining the rational fraction velocity $\beta = N_L \Delta x / N_T c \Delta t = n_\beta / d_\beta$ if the distant counter fires after the reference counter, and the negative of this rational fraction if the distant counter fires first. Equivalently, we can specify two other integers by $n_\beta = n_1 - n_0$, $d_\beta = n_1 + n_0$. The simplest bit-string

model for these two events is then a string with n_1 1's and n_0 0's. Note that if these two numbers have a common factor T_β, this velocity specifies a *periodic* phenomenon which might repeat T_β times between the two counter firings. This simple fact, together with the observation that in the absence of further information this periodicity is not directly observable, is the basis on which we will build the "wave-like" phenomena of relativistic quantum mechanics. Note that T_β/β is the distance a light signal will go during the time that the particle moves a distance βT_β. We connect these spacial and temporal periodicities by the definitions

$$\beta\lambda = 1; \quad 2\pi r = j\lambda \ . \tag{1.1}$$

We will see in due course that j is either integral or half-integral and in appropriate circumstances can be unambiguously identified as the angular momentum quantum number in units of $\hbar = h/2\pi$.

1.3. Three Counters

We now extend our counter paradigm to three (or more) counters. Because of our eventual interest in a bit-string model, we reserve the counter labeled "0" and the integer n_0 for special treatment. For 1+1 dimensional problems it will represent the origin of coordinates $(0,0)$. For the moment we take counter "1" to be the first counter of interest with respect to counter "0". We assign coordinates (x,t) to the firing of counter "1". We have seen that for a fixed space and time resolution we have a unit of length $\Delta x = c\Delta t$, and hence can express all coordinates as integers, and all velocities as rational fractions. This is, for the moment, an artifact of our technological competence. We can generalize the situation by assuming that, at least for some class of problems, there is a Lorentz invariant length $\ell_0 = ct_0$ which has physical significance. In current experience ℓ_0 is always much smaller than Δx. Our quest for a finite and discrete relativistic quantum mechanics could be viewed as a quest for a precise operational meaning to give to ℓ_0. We intend to show

that all other meanings ℓ_0 acquires in the practice of physics can be connected by finite, integer operations to our initial abstraction. For the moment, we will simply assume that we *can* give meaning to integer coordinates.

We now introduce a notation which it will be easy to generalize. We concentrate on two counters and two integers n_0 and n_1 assumed to have no common factor. Then we can define

$$x_0 := n_0 - n_1 = -x_{10}; \quad t_{01} := n_0 + n_1 = t_{10}$$

$$\beta_{01} := \frac{n_0 - n_1}{n_0 + n_1}; \quad \tau_{01}^2 := t_{01}^2 - x_{01}^2 = 4n_0n_1 = \tau_{10}^2 \ . \tag{1.2}$$

We also find that

$$\frac{t_{01}^2}{\tau_{01}^2} := \gamma_{01}^2 = \frac{(n_0 + n_1)^2}{4n_0n_1} = \frac{1}{1 - \beta_{01}^2} \ . \tag{1.3}$$

Because of our relation between space and time units, $t_{01}\ell_0/c = (n_0 + n_1)t_0$ is simply the time it takes a particle with velocity β_{01} to travel from counter "0" at $(0,0)$ and cause an event in counter "1" at (x_{01}, t_{01}). We introduce a third counter "2", with an associated integer n_2 and define coordinates relative to $(0,0)$ in the same way. But now we also have x_{12}, t_{12} and quantities derived from them and/or from n_1, n_2 which no longer refer to counter "0" or n_0. This is our starting point for constructing finite and discrete Lorentz transformations.

Boosts

When we consider the connection between (x_{01}, t_{01}) and (x_{20}, t_{20}) encoded in β_{12}, we have the usual two options. In the realistic language we have supplied so far, these two coordinate pairs can be interpreted as two counter firings in counters "1" and "2" which are a distance $|x_{12}|$ apart, for which the "obvious" causal explanation is the passage of a particle with velocity β_{12}. But, we can also take β_{12} to encode the velocity in the Lorentz transformation which connects *one*

event described in two different coordinate systems. Either interpretation rests on the same algebraic facts:

$$x_{01} + x_{12} + x_{20} = n_0 - n_1 + n_1 - n_2 + n_2 - n_0 = 0 \tag{1.4}$$

$$\begin{aligned}
\frac{\beta_{01} + \beta_{12}}{1 + \beta_{01}\beta_{12}} &= \frac{\frac{n_0 - n_1}{n_0 + n_1} + \frac{n_1 - n_2}{n_1 + n_2}}{1 + \frac{(n_0 - n_1)(n_1 - n_2)}{(n_0 + n_1)(n_1 + n_2)}} \\
&= \frac{(n_0 + n_2)(n_0 - n_1) + (n_0 + n_1)(n_1 - n_2)}{(n_0 + n_1)(n_1 + n_2) + (n_0 - n_1)(n_1 - n_2)} \qquad (1.5) \\
&= \frac{2n_0n_1 - 2n_1n_2}{2n_0n_1 + 2n_1n_2} = \frac{n_0 - n_2}{n_0 + n_2} = \beta_{02} \ .
\end{aligned}$$

Or, succinctly,

$$\beta_{02} = \frac{\beta_{01} + \beta_{12}}{1 + \beta_{01}\beta_{12}} \tag{1.6}$$

A little algebra then suffices to show that

$$x_{20} = \gamma_{12}(x_{01} + \beta_{12}t_{01}); \quad t_{20} = \gamma_{12}(t_{01} + \beta_{12}x_{01}) \tag{1.7}$$

$$\rho = \frac{1 + \beta}{1 - \beta}; \quad \gamma^2 = \frac{1}{2}(\rho + \rho^{-1}) \ . \tag{1.8}$$

i.e., the usual Lorentz boosts along the line connecting counters "1" and "2".

<u>Rotations; Kepler's Second Law</u> For rotations consider two counters "1" and "2" equidistant from counter "0" and a distance Δr apart, where the equal distances are measured by light signals and Δr is measured by a particulate velocity, i.e.

$$n_0 + n_1 = r = n_0 + n_2; \quad \Delta r = \lambda = \frac{n_1 + n_2}{n_1 - n_2} \ . \tag{1.9}$$

Kepler found that the distance to an isolated object moving around a center sweeps out equal areas in equal times, which is known as Kepler's Second Law. We now know that this is also true, in the absence of further information for an object

moving with constant velocity in a fixed direction past a center. Since we are considering constant velocity, we assume that Δr is the same for equal time intervals. Using the fact that the square of the area of a triangle with sides a, b, c , called A^2, is given by

$$16A^2 = (a+b+c)(a+b-c)(b+c-a)(c+a-b) \tag{1.10}$$

and hence in our case given by

$$16A^2 = (2r+\Delta r)(2r-\Delta r)(2\Delta r)^2 \tag{1.11}$$

or in dimensionless form by

$$(\frac{A}{\Delta r^2})^2 = (j-\frac{1}{2})(j+\frac{1}{2})(\frac{1}{2\pi})^2 \tag{1.12}$$

where $j\Delta r = 2\pi r$. Since the distance of closest approach ("impact parameter") is given by $b^2 = r^2 - (\Delta r/2)^2$, we can also write the square of the impact parameter as $b^2 = \ell(\ell+1)\Delta r^2$ where $\ell = j - \frac{1}{2}$. We can then easily convert Kepler's *kinematic* law into a recursion relation based on constant Δr by the initial condition $r = r_0 = r_1$ and the fact that a triangle with sides $r_n, r_{n+1}, \Delta r$ has area A_n given by

$$(\frac{A_n}{\Delta r^2})^2 = ((\frac{r_{n+1}+r_n}{2\Delta r})^2 - \frac{1}{4})(\frac{1}{4} - (\frac{r_{n+1}+r_n}{2\Delta r})^2) \tag{1.13}$$

which is conserved, or by noting that $b^2 = r_n^2 - (n-\frac{1}{2})^2\Delta r^2$ is conserved.

For straight line motion at constant velocity, we have already noted that our rational fraction velocities imply a periodicity in time. This implies, in our relativistic theory, a periodicity in space, as summarized in Eq. 2.1. To connect the periodicity for straight line motion to the periodicity for circular motion we note that r is the radius of a circle about the point which cuts the straight line at two symmetric positions a distance Δr apart. Measuring r in these units, we find that $2\pi r = j\lambda$ is the distance traveled in a circular orbit which returns to its starting point with period T_β. [DeBroglie used precisely this relation to derive

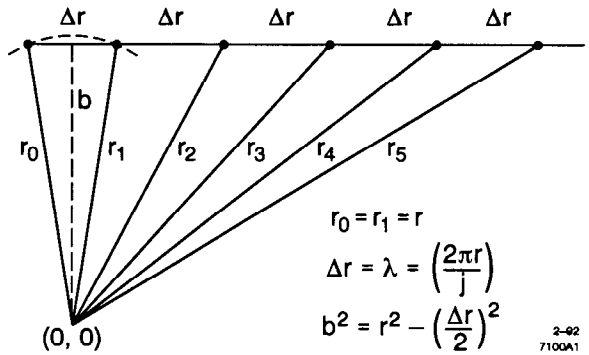

Figure 1. Kepler's Second Law for straight line motion.

the Bohr energy levels of the hydrogen atom.] Identifying Δr with λ, we find that $(A/\lambda^2) = \ell(\ell+1)(1/2\pi)^2$. Thus the unit for linear periodicity in space and for area conservation in motion relative to a point are related by $1/2\pi$ independent of the system of units. For more details see my discussion of Galileo's measurement [1] of π and my discussion of dimensional quantization [2].

1.4. The RQM Triangle

We can now generalize our treatment of integer lengths and times to a purely relative treatment in which any one of three counters can be the referent counter and any one of three directions can be the referent direction. Given three positive-*definite*, finite integers n_i, n_j, n_k with the three indices i, j, k finite, distinct, cyclic, positive-*definite* integers, i.e.,

$$n_i, n_j, n_k, i, j, k, \in 1, 2, 3, \ldots, N; \; N \text{ fixed}; \quad i \neq j \neq k \neq i \text{ cyclic} \tag{1.14}$$

we can define

$$t_{ij} := n_i + n_j; \; t_{ij}\beta_{ij} := n_i - n_j := x_{ij} = -x_{ji} \tag{1.15}$$

$$\tau_{ij}^2 := t_{ij}^2 - x_{ij}^2 = 4n_i n_j = t_{ij}^2(1 - \beta_{ij}^2) := t_{ij}^2\gamma_{ij}^2 \tag{1.16}$$

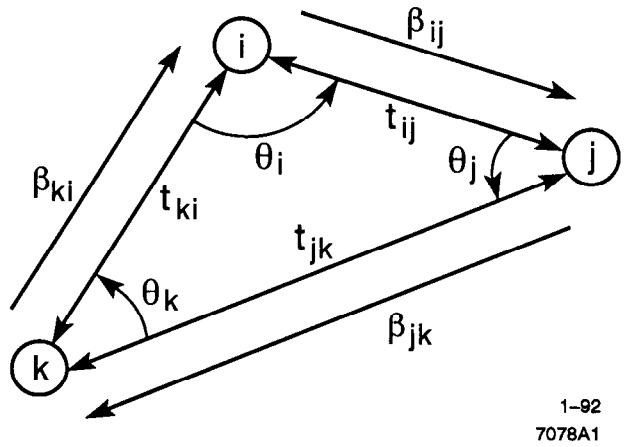

Figure 2. Kinematical interpretation of the three integers n_i, n_j, n_k.

with the consequences that

$$t_{ij}\beta_{ij} + t_{jk}\beta_{jk} + t_{ki}\beta_{ki} = 0 \tag{1.17}$$

and

$$-\beta_{ij} = \frac{\beta_{jk} + \beta_{ki}}{1 + \beta_{jk}\beta_{ki}} \ . \tag{1.18}$$

Further, since

$$|t_{ij} - t_{jk}| \leq t_{ki} \leq t_{ij} + t_{ki} \tag{1.19}$$

we can define

$$\odot(t_{ij}, t_{jk}; t_{ki}) = \odot(t_{jk}, t_{ij}; t_{ki}) := \frac{1}{2}[t_{ij}^2 + t_{jk}^2 - t_{ki}^2] \tag{1.20}$$

and draw a triangle (see Figure 2) with sides t_{ij}, t_{jk}, t_{ki} and angles

$$cos\ \theta_k := \frac{\odot(t_{ij}, t_{jk}; t_{ki})}{t_{ij}t_{jk}} = \frac{t_{ij}^2 + t_{jk}^2 - t_{ki}^2}{t_{ij}t_{jk}} \ . \tag{1.21}$$

Any one side can be interpreted as a combined rotation and boost taking the position and velocity of one event to another event with respect to a third event, as we will now show.

The figure can be thought of as three counters with associated clocks—synchronized using the Einstein convention—which keep a record of the time of arrival or departure of a signal, and whether it was a particle or indistinguishable locally from a gamma-ray (see Section 2.2). Using units with c=1, the *distances* between the counters i, j, k are simply t_{ij}, t_{jk}, t_{ki}. If we launch a signal with velocity β_{ij} from counter i toward counter j and simultaneously launch a signal with $-\beta_{ki}$ from i toward k which, on arrival at k, triggers a signal from k to j with velocity $-\beta_{jk}$, the signals from i to j and from k to j will arrive simultaneously at j. This explains why, if we pay proper attention to signs, we obtain the usual Lorentz velocity addition law independent of how far away counter k is from the ij path.

Note also that our cyclic convention can be used to define a direction out of the plane of the triangle whose sign reverses either if we change our convention from cyclic to anti-cyclic or if we interchange two of the indices. Clearly this is the "parity" transformation P. In contrast to classical relativistic kinematics, our finite assumption forces us to consider transformations which do *not* conserve parity. Further if we reverse *all* velocities—which corresponds to time reversal T—this discrete transformation produces the same result as the (cyclic $\leftrightarrow$ anti-cyclic) parity operation. Consequently the physical paradigm we use to interpret the formalism automatically guarantees that at this stage the theory is invariant under P^2, T^2, PT *and* TP. Full CPT invariance will have to wait until we define conserved quantum numbers analagous to and including electric charge. However, if we include forward or backward "motion in time" in order to define a conserved difference between the number of particles and the number of antiparticles, or left-right motion in a single direction to conserve helicity, we can immediately invoke these conservation laws to construct finite and discrete solutions to the Dirac equation in 1+1 dimensions [3].

Thanks to the velocity addition law derived from the conventional clock synchronization convention, the paradigm obviously has an Lorentz-invariant significance. We have established formal Lorentz invariance for boosts along a line

and rotations in a plane in Section 2.3. The RQM triangle gives a succinct description of a combined rotation and boost using only *relative* coordinates specified by three integers. We can use this integer construction to replace matrix representations of Lorentz transformations by simple bit-string discriminations, simplifying the analysis of relativistic particle kinematics, once we accept integer length or bit-string quantization.

Momentum space

We have used a space-time paradigm for our RQM triangle because space-time thinking ("kinematics") is so imbedded in the history and language of physics. But as abstract mathematics, we could just as well take $\beta = p/E$ as $\beta = x/t$ to describe the particulate connection between two events. The translation into operational terms would connect our theory to S-Matrix theory rather than to space-time relativistic quantum mechanics. Since no one has yet given *empirical* evidence to show that either language is false to facts in its application to the same agreed body of data, we assume that this translation will be easy in our theory.

3+1 dimensions

Our discrete physics allows only 3 macroscopic space dimensions and one universal ordering ("time") [4]. Nevertheless, to extend our RQM triangle to RQM tetrahedra which can be described by three *orthogonal* dimensions turned out to be more difficult than might be thought at first sight.

In a plane, there are many Pythagorean triples $a^2 + b^2 = d^2$ and corresponding inner products that vanish. Integer coordinates are easy to define relative to any set of them. Finding three simultaneous Pythagorean triples turned out to be more difficult. The problem is pointed up in our context by a theorem and corollary proved by Michael Gryk [5].

Theorem:

Given a Pythagorean triad $a^2 + b^2 = d^2$ where a, b, d have no common factor other than unity, then a is even and b is odd, or *visa versa.*

Proof:

The hypothesis excludes the case when both are even, so we need only show that the case when both a and b are odd is impossible. Since both are odd, their squares are odd, the sum of their squares is even, and hence both d^2 and d must be even. Writing

$$a = 2n_a + 1;\ b = 2n_b + 1; d = 2n_d$$

we have that

$$4n_a^2 + 4n_a + 4n_b^2 + 4n_b + 2 = 4n_d^2 \ .$$

Hence

$$2n_a^2 + 2n_a + 2n_b^2 + 2n_b + 1 = 2n_d^2 \ .$$

But then the left hand side is odd, and the right hand side is even, which is impossible. QED

This is obviously a modern version of an old Greek proof. But the Corollary, is—to my knowledge—novel.

Corollary:

Given six integers a, b, c, d, e, f with no common factor other than unity, then the three equations

$$a^2 + b^2 = d^2; \quad b^2 + c^2 = e^2; \quad c^2 + a^2 = g^2$$

cannot simultaneously be satisfied.

Proof:

Suppose a is even; then by the Theorem, b is odd. Then by the Theorem, if the second equation holds, c must be even. But then both a and c are even, contradicting the hypothesis that they have no common factor. QED

However, there is no need—other than parsimony—to assume that the problem *has* to be solved by three Pythagorean triples with no common factor, as Clive Kilmister was quick to point out [6]. He eventually tracked down the minimal example, which was found by Euler:

$$a = 44, \quad b = 117, \quad c = 240$$

with hypoteneuses 125, 244 and 267 as you can easily verify. Rather than work out an explicitly coordinate representation here, we will wait till the bit-string model and its simplifications are in hand.

1.5. The double slit experiment; ℓ_0

We have reduced our "counter paradigm" to the specification of three arbitrary, finite integers associated by our rules of correspondence to three laboratory counters and their firings under specified circumstances. But our units of length and time, although connected ($\ell_0 = ct_0$), remain arbitrary. We can reduce the arbitrariness by showing that for a beam of particles whose measured β is sufficiently well defined incident on two slits a distance w apart, there is a sequence of maximum counts a distance r away with a spacing s. This gives us a characteristic length $\ell = sw/r$ which we can identify with the *spacial* periodicity in our theory $\lambda = 1/\beta$ and check that the spacing s does vary inversely between them an even integer $w = 2n$. The problem is to give with β. Our Lorentz invariance then allows us to calculate the invariant length $\ell_0 = \beta\ell$ for this class of particles. Since I have given a detailed analysis of this experiment elsewhere [7], I will not take space here to discuss it in the language of this paper.

Clearly by taking any one type of particle as a referent, we can establish a particulate length scale in terms of rational fractions of this unit, and use it rather than momentum conservation to describe the kinematics of particle scattering processes. We will work all this out, including the reduction to Mandelstam variables,

on another occasion. As we discuss in our contribution to ANPA 13 already mentioned [1], we cannot give mass a universal significance until we construct gravitation. Even then it takes either baryon number conservation or lepton number conservation to give an absolute significance to the particulate mass scale.

2. Bit-Strings

We collect here recent formal and informal results that will allow a formal structure congruent with the above physical arguments to be constructed.

2.1. Discrimination and Concatenation

We specify a *bit-string*

$$\mathbf{a}(S) = (e_1^a e_2^a \ldots e_s^a \ldots e_S^a) \tag{2.1}$$

by its S ordered elements

$$e_s^a \in 0, 1; \qquad s \in 1, 2, \ldots S; \qquad 0, 1, \ldots, S \in \text{ ordinal integers} \tag{2.2}$$

and its norm by

$$|\mathbf{a}(S)| = \Sigma_{s=1}^{S} e_s^a = a(S) \ . \tag{2.3}$$

This is the usual Hamming measure for bit-strings.

Define the *null string* by $\mathbf{0}(S)$, $e_s^0 := 0$ for all s and the *anti-null* string by $\mathbf{1}(S)$, $e_s^1 := 1$ for all s. If we take the null string as the *reference ensemble*, the number of "1" 's in the string (i.e., the Hamming measure) as the *attribute*, and changing the Hamming measure by one unit as the *computational step*, then this

norm satisfies McGoveran's definition (FDP, Sec. 3.2, p.28) of *attribute distance.* With this definition, *attribute velocity* is given by

$$v_a(S) := a(S)/S \ . \tag{2.4}$$

Define *discrimination* ($\oplus$) by

$$e_s^{a\oplus b} := (e_s^a - e_s^b)^2; \quad \mathbf{a} \oplus \mathbf{b} := (\ldots\ e_s^{a\oplus b}\ \ldots\ e_S^{a\oplus b}) = \mathbf{b} \oplus \mathbf{a} \tag{2.5}$$

from which it follows that

$$\mathbf{a} \oplus \mathbf{a} = \mathbf{0}; \ \mathbf{a} \oplus \mathbf{0} = \mathbf{a} \ . \tag{2.6}$$

Define $\bar{\mathbf{a}}(S)$ by

$$\bar{\mathbf{a}} := \mathbf{a} \oplus \mathbf{1}; \ \textit{hence}\ \mathbf{a} \oplus \bar{\mathbf{a}} \oplus \mathbf{1} = \mathbf{0} \ . \tag{2.7}$$

Distinct strings which are *discriminately independent*, or *d.i.*, are those which when combined by discrimination in all possible non-repetitive ways do not produce the null string. Discriminately and anti-discriminately independent strings, or *d.i.a.d.* strings are *d.i.* strings which also do not produce the anti-null string. Note that this definition implies that a *single* "diadic" string cannot be either the null or the anti-null string.

Since discrimination is only defined for bit-strings of the same length S, we can often omit reference to the string length, as we have done above. However, when the norm *and* the anti-null string are involved we need to know the string length. In particular

$$|\mathbf{1}(S)| = S; \quad |\bar{\mathbf{a}}(S)| = S - a(S) \ . \tag{2.8}$$

For two strings $\mathbf{a}(S_a), \mathbf{b}(S_b)$ we define *concatenation* ($\|$) by

$$e_k^{a\|b} := e_s^a, \ s \in 1, 2, \ldots, S_a; \quad e_k^{a\|b} = e_j^b, \ j \in 1, 2, \ldots, S_b, k = S_a + j$$

$$\mathbf{a}(S_a)\|\mathbf{b}(S_b) = (\ldots\ e_i^a\ \ldots\ e_{S_a}^a)\|(\ldots\ e_j^b\ \ldots\ e_{S_b}^b) \tag{2.9}$$

$$= (\ldots\ e_k^{a\|b}\ \ldots\ e_{S_a+S_b}^{a\|b})\ . \tag{2.10}$$

Hence

$$a(S_a) + b(S_b) := |\mathbf{a}(S_a)\|\mathbf{b}(S_b)| = |\mathbf{b}(S_b)\|\mathbf{a}(S_a)| \tag{2.11}$$

but note that in general $\mathbf{a}\|\mathbf{b} \neq \mathbf{b}\|\mathbf{a}$.

2.2. Inner Product, Triangles and Tetrahedra

THEOREM 1:

If $\mathbf{f}(S)$ and $\mathbf{g}(S)$ and $\mathbf{h}_{fg}(S) := \mathbf{f}(S) \oplus \mathbf{g}(S)$ are d.i.a.d.

then the three Hamming measures f, g, h_{fg} specify a triangle with sides of these integer lengths, angles

$$cos\ \theta_{fg} = \frac{f^2 + g^2 - h_{fg}^2}{2fg}$$

$$cos\ \theta_{gh_{fg}} = \frac{g^2 + h_{fg}^2 - f^2}{2gh_{fg}} \tag{2.12}$$

$$cos\ \theta_{h_{fg}f} = \frac{h_{fg}^2 + f^2 - g^2}{2h_{fg}f}$$

and area A whose square is given by

$$A^2 = \frac{1}{16}(f + g + h_{fg})(f + g - h_{fg})(g + h_{fg} - f)(h_{fg} + f - g)\ . \tag{2.13}$$

The proof is easier if we first use the following theorem to establish the triangle inequalities.

THEOREM 2:

If the conditions of Theorem 1 are met *and* we define

$$n_f := \Sigma_{s=1}^{S} e_s^f (1 - e_s^g)$$

$$n_g := \Sigma_{s=1}^{S} e_s^g (1 - e_s^f)$$

$$n_{fg} := \Sigma_{s=1}^{S} e_s^f e_s^g$$

$$n_0 := \Sigma_{s=1}^{S} (1 - e_s^f)(1 - e_s^g) \tag{2.14}$$

then

$$f = n_f + n_{fg};\ g = n_g + n_{fg};\ h_{fg} = n_f + n_g;\ S = n_f + n_g + n_{fg} + n_0 \tag{2.15}$$

and

$$A^2 = (S - n_0) n_f n_g n_{fg}\ . \tag{2.16}$$

Once we note that $\Sigma_{s=1}^{S} 1 = S$, that $n_f + n_g = \Sigma_{s=1}^{S} (b_s^f - b_s^g)^2$ is the definition of $h_{fg} = |\mathbf{f}(S) \oplus \mathbf{g}(S)|$ and that, by definition, n_f, n_g, n_{fg} are positive integers or zero, the triangle equalities

$$|f - g| \leq h_{fg} \leq f + g$$

$$|g - h_{fg}| \leq f \leq g + h_{fg} \tag{2.17}$$

$$|h_{fg} - f| \leq g \leq h_{fg} + f$$

follow. From these, the results of Theorem 1 follow by standard methods.

Comment

The triangle inequalities do not, by themselves, exclude the case of a "triangle with zero area". Zero length sides are excluded by the d.i.a.d constraint. The case of zero area occurs when the lengths of two sides add to the length of the third side. This occurs when one of the three integers n_f, n_g, n_{fg} vanishes. If none of the three vanish $f + g = n_f + n_g + 2n_{fg} > h_{fg}$ and the area of the triangle cannot vanish. This gives us a straightforward way to impose this restriction, if we so desire.

The triangle inequalities make it sensible to define the inner product between two strings.

THEOREM 3:

If the conditions of Theorem 1 are met *and* we define

$$\mathbf{x}(S) \odot \mathbf{y}(S) := 2xy \; cos \; \theta_{xy} := x^2 + y^2 - |\mathbf{x} \oplus \mathbf{y}|^2 \tag{2.18}$$

then

$$\mathbf{f}(S) \odot \mathbf{h}_{fg}(S) + \mathbf{g}(S) \odot \mathbf{h}_{fg}(S) = h_{fg}^2 = |\mathbf{f} \oplus \mathbf{g}|^2 = h_{fg}|\mathbf{f} \oplus \mathbf{g}| \; . \tag{2.19}$$

Proof:

$$2[\mathbf{f}(S) \odot \mathbf{h}_{fg}(S) + \mathbf{g}(S) \odot \mathbf{h}_{fg}(S)] = f^2 + h_{fg}^2 - g^2 + g^2 + h_{fg}^2 - f^2 = 2h_{fg}^2 \, . \tag{2.20}$$

CORROLARY:

If

$$\mathbf{a} \oplus \mathbf{b} \oplus \mathbf{c} \oplus \mathbf{d} = \mathbf{0}(S)$$

then

$$\mathbf{a} \odot \mathbf{h_{ab}} + \mathbf{b} \odot \mathbf{h_{ab}} = \mathbf{c} \odot \mathbf{h_{ab}} + \mathbf{d} \odot \mathbf{h_{ab}} \tag{2.21}$$

where $\mathbf{h_{ab}}(S) = \mathbf{a} \oplus \mathbf{b}$. The proof is trivial because $\mathbf{a} \oplus \mathbf{b} = \mathbf{c} \oplus \mathbf{d}$.

THEOREM 4: Three d.i.a.d. bit-strings specify two tetrahedra, which are chiral pairs. All four faces of the tetrahedra are specified by the same triangle, and opposite edges have the same length. The theorem follows from the definitions

$$\mathbf{h}_{ab}(S) := \mathbf{a}(S) \oplus \mathbf{b}(S);\ \mathbf{h}_{bc}(S) := \mathbf{b}(S) \oplus \mathbf{c}(S);\ \mathbf{h}_{ca}(S) := \mathbf{c}(S) \oplus \mathbf{a}(S)$$

$$\mathbf{1}(S) \neq \mathbf{h}_{abc}(S) := \mathbf{a} \oplus \mathbf{b} \oplus \mathbf{c} \neq \mathbf{0}(S) \tag{2.22}$$

and the immediate consequence that

$$\mathbf{h}_{ab}(S) \oplus \mathbf{h}_{bc}(S) \oplus \mathbf{h}_{ca}(S) = \mathbf{0}(S)\ . \tag{2.23}$$

The necessary algebra for the proof is supplied by the definitions

$$n_a := \Sigma_s e_s^a (1 - e_s^b)(1 - e_s^c)$$

$$n_b := \Sigma_s e_s^b (1 - e_s^c)(1 - e_s^a)$$

$$n_c := \Sigma_s e_s^c (1 - e_s^a)(1 - e_s^b)$$

$$n_{ab} := \Sigma_s e_s^a e_s^b (1 - e_s^c)$$

$$n_{bc} := \Sigma_s e_s^b e_s^c (1 - e_s^a)$$

$$n_{ca} := \Sigma_s e_s^c e_s^a (1 - e_s^b)$$

$$n_{abc} := \Sigma_s e_s^a e_s^b e_s^c$$

$$n_0 := \Sigma_s (1 - b_e^a)(1 - e_s^b)(1 - e_s^c)$$

$$S := \Sigma_s 1 \tag{2.24}$$

with the immediate consequences that

$$S = n_0 + n_a + n_b + n_c + n_{ab} + n_{bc} + n_{ca} + n_{abc}$$

$$a = n_a + n_{ab} + n_{ca} + n_{abc}$$

$$b = n_b + n_{ab} + n_{bc} + n_{abc}$$

$$c = n_c + n_{bc} + n_{ca} + n_{abc}$$

$$h_{ab} = n_a + n_b + n_{bc} + n_{ca}$$

$$h_{bc} = n_b + n_c + n_{ab} + n_{ca}$$

$$h_{ca} = n_a + n_c + n_{ab} + n_{bc}$$

$$h_{abc} = n_a + n_b + n_c + n_{abc} \ . \tag{2.25}$$

The geometrical consequences are indicated in Figures 3 and 4.

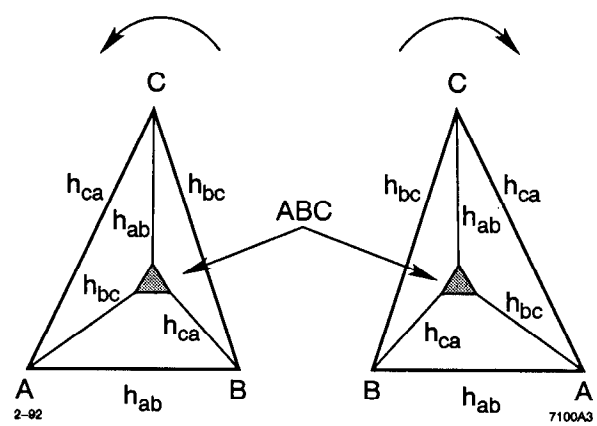

Figure 3. External chiral tetrahedra.

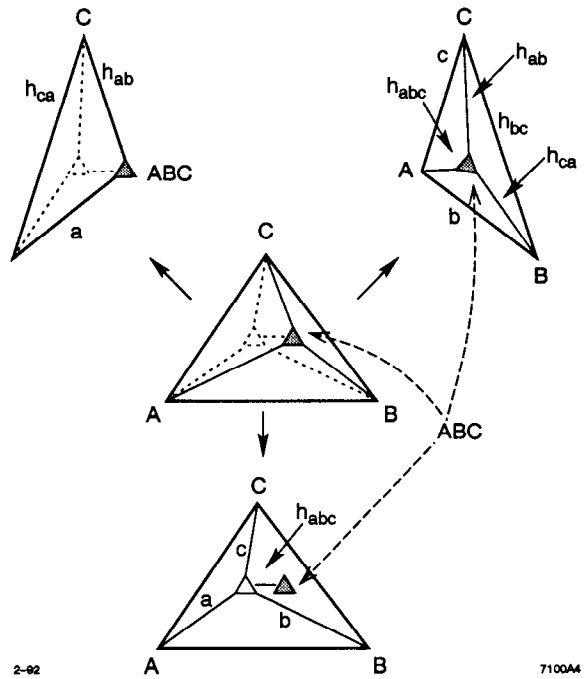

Figure 4. Decomposition of the invariant tetrahedron.

2.3. Standard representation

We need a standard notation. Consider I strings of Hamming measure h_i with

$$H = \Sigma_{i=1}^{I} h_i; i \in 1, 2, \ldots, I \ .$$

Pick an arbitrary parameter $h_0 \geq H + 1$ and define the string length to be $S = H + n_0$. Then a standard representation for $\mathbf{h}_i(S)$ is

$$\mathbf{h}_i(S) = \mathbf{0}(h_1)\|\mathbf{0}(h_2)\| \ldots \|\mathbf{0}(h_{i-1})\|\mathbf{1}(h_i)\|\mathbf{0}(h_{i+1})\| \ldots \|\mathbf{0}(h_I)\|\mathbf{0}(h_0) \ . \quad (2.26)$$

Further, define

$$\mathbf{h}_{\mathbf{ijk}\ldots} = \mathbf{h}_\mathbf{i} \oplus \mathbf{h}_\mathbf{j} \oplus \mathbf{h}_\mathbf{k} \oplus \ldots \ . \quad (2.27)$$

Note that $\mathbf{h_1}, \mathbf{h_2}, \mathbf{h_3}$ and $\mathbf{h_{123}}$ discriminate to the null string and define the basic tetrahedron in terms of three integers. $\mathbf{h_{12}}, \mathbf{h_{23}}, \mathbf{h_{31}}$ define the invariant triangle

in terms of the same three integers, and these seven strings give a specific way to describe level 2 of the *combinatorial hierarchy.* The relation to Kilmister's notation should be spelled out.

2.4. Bit-String Statistics

We can define a second measure on a bit-string by the number of "bends"—the number of times a sequence of 1's changes to a sequence of 0's or visa versa—which we call k. McGoveran [8] invented the following way to compute k given $\mathbf{a}(S)$. Form the string $\mathbf{a}'(S-1)$ defined by

$$b_s^{a'} := (b_s^a - b_{s+1}^a)^2; \; s \in 1, 2, \ldots, S-1 \; . \tag{2.28}$$

Then

$$k := |\mathbf{a}'(S-1)| = \Sigma_{s=1}^{S-1}(b_s^a - b_{s+1}^a)^2 \; . \tag{2.29}$$

The problem posed is, given a string with Hamming measure a of length $S \geq 2a+1$ characterized by k bends to show that the relative probability of having such a string is $a^k/k!$ independent of S. I call this *McGoveran's Transport Theorem* because he proved it in FDP and discussed it at subsequent ANPA meetings. Kilmister was not convinced until I developed a bit-string argument which I gave in various drafts of our paper on the Dirac Equation and related reports at ANPA and ANPA WEST meetings. Kilmister noted that this is similar to the counting of states in Bose-Einstein statistics, so I start the discussion above there rather than with bit-strings. I hope this will prove useful.

To prove the theorem for a string with Hamming measure a, lengths $S \geq 2a+1$ and k bends, note that, since k is the number of bits in a string of length $S-1$, the total number of such strings is

$$\binom{S-1}{k} = \frac{(S-1)!}{(S-1-k)!k!} \; . \tag{2.30}$$

Further the a 1's which occur in the k "boxes" (i.e., sequences of 1's between bends) can be assigned in a^k ways. Consequently the total number of ways we can have a string with the specified parameters S, a, k is

$$N(S, a, k) = \frac{(S-1)!a^k}{(S-1-k)!k!} \ . \tag{2.31}$$

However, as in Bose-Einstein statistics, the permutations among the arbitrary orderings of s are indistinguishable, so the factor $(S-1)!/(S-1-k)!$ can be divided out, leaving the desired result that the relative probability that a string with a 1's and k bends normalized to unity for $k = 0$ is simply

$$P(a, k; S \geq 2a+1) = \frac{a^k}{k!} \ . \tag{2.32}$$

For independent strings, which we can take to be of the same length so long as this lower limit on the length is satisfied for all of them, the number of cases is proportional to

$$\Pi_i \ \frac{a_i^{k_i}}{k_i!}$$

which is the starting point for the derivation of Bose-Einstein statistics [9]. Note that if we construct our standard bit-string representation by arbitrary, independent choices of the integers h_i, it automatically provides a precise representation for Bose-Einstein statistics.

But much more follows. Once we have McGoveran's transport theorem, we can construct the relativistic Schroedinger (Klein-Gordon) and Dirac equations. Going from the non-interacting to the interacting system simply requires the introduction of discrimination and working out the proper rules of correspondence with laboratory practice in particle physics. Working out the details will take a book, which I am writing [10]

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