SLAC–PUB–5732
February 1992
(T, Noyes)

On the Measurement of $\pi^\star$

H. Pierre Noyes

Stanford Linear Accelerator Center
Stanford University, Stanford, CA 94309

ABSTRACT

Inspired by Stillman Drake's definition of *Galilean Units* as those for which $\frac{L}{T^2} = (\frac{\pi^2}{8})g$ where g is any finite, constant acceleration measured in units of L and T, we construct a *kinematical* dimensional analysis based only on two universal, dimensionless constants. For the linear relation between L and T we use *Einsteinian Units* $\frac{L}{T} = (\mathbf{1})c$. For orbiting masses negligible compared to some mass unit M, we use *Keplerian Units* based on his second law $\frac{L^2}{T} = (\frac{1}{2\pi})\frac{h}{M}$. Then the unit for orbital angular momentum is $\hbar$, independent of the mass scale. This allows us to define dimensionless coupling constants $f^2 = \beta = \frac{v}{c}$ where v is the orbital velocity. We find that most of relativistic quantum mechanics requires only kinematical units. *Dynamical* units require a mass scale with universal significance, set by the orbital velocity $v = c$ (or $f^2 = 1$). In dimensional form this becomes $M = (\mathbf{1})[\frac{\hbar c}{G}]^{\frac{1}{2}}$. Assuming baryon number conservation, the fact that the proton is the lightest stable baryon allows us to calculate $\hbar c/Gm_p^2 \approx 1.7 \times 10^{38}$ as the Beckenstein number of the proton—the number of bits of information lost in its formation—and connects our units to the elementary particle mass scale.

Submitted to the Thirteenth Annual International Meeting of the
Alternative Natural Philosophy Association
Department of History and Philosophy of Science
Free School Lane, Cambridge, England; 12–15 September 1991

*Work supported by Department of Energy contract DE–AC03–76SF00515.

1. Introduction

1.1. Galileo's measurement of π

The idea for this paper came to me from Stillman Drake's discussion [1] of the actual *historical* route by which Galileo arrived at his "times squared law" for free fall. What Drake shows is that Galileo found, *by measurement*, that if the time t_ℓ it takes for a pendulum of a specific length ℓ to swing to the vertical through a small arc is 942 units, then the time t_d it takes a body to fall from rest through a distance equal to that length ($d = \ell$) is 850 units. Although Galileo had no way of knowing this, we now believe that this ratio must be given by

$$\frac{t_\ell}{t_d} = \frac{\pi}{2\sqrt{2}} = 1.1107 \ \ldots$$

"anywhere that bodies fall and pendulums oscillate" [2]. Consequently, we can now assert that Galileo's measurement of $942/850 = 1.108$ to four places was the first kinematical measurement of π. His measurement agrees with the currently predicted value to considerably better than 1 percent accuracy.

Drake's discovery of "Galileo's constant," which he symbolizes by $\sqrt{\mathrm{g}} = \pi/2\sqrt{2}$, is the result of a lifetime dedicated to painstaking research into the question of *when* and *how* and *to what accuracy* Galileo, in historical fact, arrived at his results. I do not have space here to do justice to his arguments. Drake spent many hours in "hands-on" examination of what remain of Galileo's working papers. His conclusions are supported by watermarks on the paper, when Galileo had arthritis, what it was prudent to destroy before the Inquisition seized his records, what was a fragment preserved by *literally* "cut and paste" from lost manuscripts, ... The methodology and many important conclusions were reported some time ago [3]. Some important conclusions rely on recent measurements with reconstructed apparatus. Equally important investigations of the records of Galileo's telescopic observations are relevant in demonstrating once again that the founder of *experimental* physics was a superb observer. I hope those of you who have not yet had the pleasure of reading this scientific "detective story" will be motivated to take a look at it.

1.2. Contexts for the measurement of π

I call Galileo's measurement *kinematical* because it involves the measurement of time in conjunction with length. The first *geometrical* measurement of π is lost in the mists of prehistory. The Bible quotes a value of 3, which sometimes makes trouble for fundamentalists, and even state legislators who are influenced by them.* The success of Euclidean Geometry established the presumption that π for perimeters, areas, and volumes is the same. So far as I know, David McGoveran was the first to suggest [4] that in Discrete Physics these three numbers are conceptually distinct and subject to empirical measurement; his argument is still implicitly geometrical. On the other side, once the concept of *mass* is introduced, the meaning of π changes again. I call measurements of π that involve length, time and mass *dynamical.* Following Einstein, Wheeler's geometrodynamics relates mass to the curvature of space, and freezes the universe into a static 4-space. I think of his theory as "geometrostatics" in contrast to our constructive, context-dependent approach which necessarily introduces multiple connectivities. These connections cannot be "flattened out" for the same reason that parallel processing computers of sufficient complexity cannot be reduced to a single Turing machine. Penrose [5] seems to be unaware of this latter fact.

Drake's analysis suggested to me that, just as our finite and discrete theory has a natural unit for velocity, it *also* has a natural unit for *acceleration.* This further suggests that, once physicists get used to the idea of accepting π as a context-dependent *empirical* number on the same footing as c, it will become easier to convince them that coupling constants and mass ratios can be computed from general structural requirements. This paper is a first effort in that direction.

* Toward the end of the nineteenth century my father, when he was Professor of Chemistry at Rose-Polytechnic Institute, once testified against legislation that had been proposed in the Indiana state legislature which would have required π to be exactly equal to 3 throughout the State!

2. Kinematical Units

2.1. Some Remarks on Dimensional Analysis

Dimensional analysis can start with the observation that, historically, the unit in which any physical quantity is measured is arbitrary. The units are chosen initially for convenience in measurement, and only as theory develops are comparisons made. Today these comparisons are customarily carried out in terms of theory-laden "fundamental constants." Physicists are used to the scale-invariant Newtonian system of units based on mass, length and time. But this, too, is arbitrary. In his excellent book on metrology, Petley notes that in different branches of physics and engineering more than three units may be useful and are in fact employed. In addition to mass, length and time, electrical engineers are accustomed to use charge, or some equivalent, as an independent dimensional concept with an independent unit. Petley finds that up to seven dimensional units [6] may be employed in standard contexts.

A fact that is often ignored in dimensional analysis is that measurement of zero or infinity is *impossible.* One way to build this fact into the methodology is to base measurement on *ratios* of finite quantities. Prior to Galileo, such ratios were always taken between quantities of the same logical type. In their rigorous mathematics, both Galileo and Newton used the Eudoxian theory of proportions, drawn from the paradigm of length ratios in Euclidean geometry. For instance, in the measurements mentioned in the first chapter, Galileo took the ratio between two times. It was only later in his work that he took the critical step of taking the ratio of a length to a time, and allowed this velocity to pass through all values starting from zero, or diminishing to and increasing from zero. Newton, following Galileo, allowed velocities to pass through zero without changing their direction. This is one way to extend the Euclidean concept of a *point* to space-time. Historically these continuously varying quantities which can include zero led to the conflict over "infinitesimals." Operationally, finite measurements in classical physics remained restricted to the comparison of finite ratios.

In a scale invariant theory based on the calculus, there was no conceptual difficulty in using continuous quantities represented empirically by measured finite ratios, once the calculus itself was given acceptable mathematical rigor. The situation in special relativity is usually represented as specifying a maximum or limiting velocity, but this is not the only way to talk about it. In a conventional relativistic wave theory in a dispersive medium, or for relativistic deBroglie waves, c is the geometric mean between the phase and the group velocity: $c^2 = v_{ph}v_{gp}$. One way of looking at the EPR "paradox" is to note that *causal* information transfer (forward light cone) is limited by the group velocity, while space-like *correlations* involve a supraluminal phase velocity. These distant coherent effects are no puzzle in the classical electromagnetic wave theory for dispersive media, and need be no puzzle in relativistic quantum mechanics if one accepts the deBroglie wave dispersion theory as a brute fact. In a sense, the absence of a material model for deBroglie wave dispersion need be no more puzzling than the absence of a material model for the electromagnetic ether. One is represented by a universal constant with dimensions L/T and the other by a universal constant with dimensions ML^2/T. Operationally, one can cut the Gordian knot there if one wants to.

The situation changes once there is a maximum or a minimum quantity in the theory, in addition to some convenient reference value which may or may not have deeper theoretical significance. In papers presented at this conference, both Constable and Reed have exploited this fact in different but related ways. I have thought a lot about how their different approaches work, and this manuscript has profited from these considerations. Prior to the development of quantum theory, there was no reason to believe that physics required the insertion of invariant maximum or minimum quantities. In our RQM theory [Relativistic Quantum Mechanics = RQM = Reconstruction of Quantum Mechanics] scale invariance is broken by the invariant length h/mc, rotational invariance is broken by the smallest quantized unit of angular momentum $\frac{1}{2}\hbar = \frac{h}{4\pi}$ and the mass scale is set by the largest coherent mass $[\frac{hc}{G}]^{\frac{1}{2}}$. Quantization based on h/mc being the length unit looks "dynamical" in that it involves a mass parameter, but careful operational analysis [7] at the kinematic level reveals that in practice the theory depends only on h/m, and mass ratios relative to any convenient reference mass, until gravitational phenomena are discussed. So long as this reference mass parameter is *arbitrary* we can still discuss measurement in terms of length and time units, and c as the geometric mean between two unknown upper and lower bounds. We will follow this approach before breaking scale invariance.

2.2. Velocity-Acceleration Units

We are accustomed in theoretical physics to use arbitrary units of length L and of time T. Theoretical expressions assume that quantities identified with "length" and "time" always use the same units. Otherwise no consistent way to compare theoretical predictions with experiment would be possible. Once the limiting velocity c appears, the same assumption applies. Using c in the theoretical expressions carries the implicit assumption that when a *numerical* value is required for comparison with experiment c will be given an appropriate numerical value in those units. The units themselves remain disconnected and arbitrary. Special relativity has given unique significance to the limiting velocity c which goes far beyond its connection to the Maxwell Equations and the "speed of light." For theoretical physicists it became customary to use "$c = 1$" in theoretical discussions. This is often confusing to the uninitiated, though considerably less dangerous than the "theorist's approximation" $\pi^2 \approx 10$ for order of magnitude calculations.

The route currently taken in SI units is to use the definition $c = 299\ 792\ 458$ meter sec^{-1}. This creates an unusual metrological situation, which is mentioned in Petley's book. Neither this convention nor $c = 1$ is quite general enough for my current purpose. Noting that Drake has introduced a pure number $\mathbf{g} = \pi^2/8$ to specify a connection between length and the square of a time—the square of "Galileo's constant"—we can also specify a pure number $\mathbf{c}$, which we can call "Einstein's constant." Since these are pure numbers, *theoretical equations*—which as mathematical expressions are themselves pure numbers—can contain $\mathbf{c}$ and $\mathbf{g}$ as *arbitrary* constants. Particular numerical choices, such as $\mathbf{c} = 1, \mathbf{g} = \pi^2/8$, simplify some expressions at the cost of complicating others. The choice is different from, but just as arbitrary (until the structure of the theory is taken into account) as the choice of the length of the king's foot, the weight of his head, and the time it takes to fall to the ground in the Place de la Republique as units of length, mass, and time. My proposal is to take the basic equations for velocity- acceleration units to be

$$\frac{L_c}{T_c} = c\mathbf{c};\ \frac{L_g}{T_g^2} = g\mathbf{g};\ T_{c,g} = \frac{c\mathbf{c}}{g\mathbf{g}};\ L_{c,g} = \frac{[c\mathbf{c}]^2}{g\mathbf{g}} \tag{2.1}$$

where $\mathbf{c}, \mathbf{g}$ are *pure numbers* picked for theoretical convenience, and c, g are physical parameters in units of L/T and L/T^2 respectively. As Drake has shown, practical experiments can be designed to *test* a specific theoretical value for $\mathbf{g}$ independent of the system of units L, T. We hope the systems of units presented below will show how this notation can be usefully employed to make new connections between theoretical ideas and physical parameters.

SI Velocity-Acceleration Units

The SI system uses L = meter; T = sec for length and time. After long discussion, it has now picked

$$c = 299\ 792\ 458 \text{ meter sec}^{-1} \tag{2.2}$$

as a *convention* which encapsules a host of empirical information. Of course, it is still possible to question, empirically, whether the "velocity of light" is indeed the same in different empirical situations. Only the metrological *language* relating metrology to laboratory practice has to be changed.

What is also much more obviously conventional is to define a "standard gravity" by the relation [8]

$$g = 9.806\ 65 \text{ meter sec}^{-2} \tag{2.3}$$

We can take this convention as defining a standard unit for acceleration. Take $\mathbf{g} = 1 = \mathbf{c}$. Then the units of time and length in this system, re-expressed in SI units, become

$$T_{c,g} = \frac{299\ 792\ 458}{9.806\ 65}\ \text{sec} = 30\ 507\ 323\ldots\ \text{sec} \approx 0.966\ 719\ \text{years}$$

$$L_{c,g} \approx 0.966\ 719\ \text{light years}\ . \tag{2.4}$$

In my verbal presentation at ANPA 13, I remarked that "This makes human interstellar travel within our galaxy feasible with current technology." Clive gently suggested that this remark needs elaboration. The argument assumes that (unless or until "anti-gravity" becomes a technological possibility, rendering g irrelevant) any interstellar drive will have to accelerate humans at something like g or less, making c/g the appropriate time scale. In fact, if one uses internal rocket power which delivers one g to the initial mass of the ship, and continues to deliver one g relative to the galactic frame, the remnant of the ship could reach almost anywhere in a ship-time close to c/g. But the acceleration *inside the ship* would squash the passengers flat well before that time was approached. However, using an *external* drive delivering g to the passenger compartment (interstellar hydrogen ram-jet, or the like), the ship could cross the galaxy in 20 years or so with the passengers experiencing only normal gravity. The conceptual design of ram-jets fueled by interstellar hydrogen has been discussed in terms of current technology; hence my remark.

A design of interstellar ships capable of reaching a few percent of the velocity of light is closer to current realization. Dyson has presented two designs, using deuterium bombs for the Orion-type propulsion system. The radiating design would move a community of 20,000 people at one parsec per century, and the ablating design would move a community of 2,000 people at 10 parsecs per century (1 parsec $\approx$ 3.3 light years). Since we have evolved under g, and are limited by that heritage, I find it amusing to note that if c/g were ten times smaller, interstellar travel would already be an interesting engineering topic; if it were ten times larger, most engineering schemes would almost inevitably have to wait for radically new technologies to be invented or discovered. Those who believe in the "anthropic principle", of whom I am not one, will undoubtedly take off from this fact, if they have not done so already.

Combined Galilean and Einsteinian Units

In modern notation, Galileo's law for falling bodies can be written

$$d = \frac{1}{2} g t_d^2 \tag{2.5}$$

where we use t_d rather than "t" to remind us of the experiment from which it comes. The related result for the time t_ℓ which it take a pendulum of length ℓ to swing through a small arc to the vertical, which came from Newton's dynamics, can be written

$$t_\ell = \frac{\pi}{2} [\frac{\ell}{g}]^{\frac{1}{2}} . \tag{2.6}$$

Consequently, when $d = \ell$, $\frac{t_\ell}{t_d} = \frac{\pi}{2\sqrt{2}}$ as already asserted. In order to utilize this combined collection of theoretical, experimental and historical facts, Drake proposes a system of "Galilean Units" based on Galileo's constant which can be defined, in the notation already established, by taking

$$\mathbf{g}_G = \pi^2/8 \tag{2.7}$$

which is *locally* valid in any region where g is some arbitrary acceleration which is constant over the relevant region within experimental error. This definition can obviously be extended to many more regions than the environments specified by Drake as "anywhere that bodies fall and pendulums oscillate."

In analogy to our definition of Galilean units, we define "Einsteinian units" by taking

$$\mathbf{c}_E = 1 \tag{2.8}$$

and c some conventional or empirical value based on laboratory experience. We combine these two conventions to obtain "Galileo-Einstein" units

$$T_{GE} = \frac{2^3}{\pi^2}[\frac{c}{g}]; \; L_{GE} = \frac{2^3}{\pi^2}[\frac{c^2}{g}] \,. \tag{2.9}$$

Since $8/\pi^2$ differs from unity by about 25 percent, the time unit is again close to a year, which makes the length unit close to a light year (by the same factor).

Centripetal Acceleration-Radius Units

Newton, starting from Galileo's parabolic law for projectile motion and the observation that the acceleration measured by Galileo is always directed toward the center of the earth, arrived at the conclusion that a projectile launched above the atmosphere parallel to the surface with a velocity $V_\oplus = \sqrt{gR_\oplus}$ would continue to move in a circle of radius $R_\oplus$ around the center of the earth with this constant velocity. This is obvious from the symmetry and the geometry of the situation once one accepts Galileo's "vector" addition of velocity and acceleration. Newton went on to draw *dynamical* conclusions from this kinematical calculation, but we need not follow his chain of thought.

Galileo, starting from laboratory measurements of space and time intervals determined what we now believe to be the dimensionless constant $\pi^2/8$ to reasonable accuracy. His methodology allowed him to do this using *arbitrary* units of length and time thanks to the Eudoxian theory of proportions. Newton's *theory* for g allowed him to connect local velocity and acceleration measurements to a "non-local" distance $R_\oplus$. I call this distance *non-local* because it can only be inferred from laboratory measurement and an *astronomical* theory based on Euclidean geometry, parallax, ...

The situation just described has three natural length and time parameters, the radius r, the circumference (length of the trajectory back to the starting point) $2\pi r$, and the time to return to the starting point (period) T. It also has the locally measurable acceleration g with which we are already familiar. We can (following Newton), define a centripetal acceleration equal to g and a corresponding circular velocity, $v_{gr}^2 = gr$. One way to define the time unit is to take $L_{gr} = r$, $T_{gr} = \frac{2\pi r}{v_{gr}}$. Compared to the Galilean Units defined by Drake, we find that

$$\frac{L_{gr}}{T_{gr}^2} = 4\pi^2 g; \ \frac{L_G}{T_G^2} = \frac{8g}{\pi^2} \ . \tag{2.10}$$

Since the length of the orbit used in the definition of v relies on the geometric "π for perimeters," and is not *locally* specified as noted above, comparison of these two kinematical measurements of π can be thought of as a test of whether the "plane of the orbit" is *flat*.

We can extend our analysis from satellites in circular orbit to planetary motion, using the velocity and distance at perihelion and the semi-major axis of the ellipse, as we discuss below. Drake tries to do this in a way that, he believes, gives him "non-Keplerian" results discussed in the last chapter of his book. In the light of the analysis which follows, we believe this claim should be treated with caution.

Velocity-Radius Units; coupling constants

Since the essential parameter in gr units is a velocity, once we introduce Einsteinian units, the situation is described by a *single* dimensionless parameter

$$\beta_{gr}^2 = \frac{gr}{c^2} \tag{2.11}$$

This allows us to relate this classical analysis directly to bit-string quantum mechanics [9]. For any rational fraction velocity $\beta = \frac{u-w}{u+w}$ any bit-string with n_0 0's and n_1 1's for which $n_1 = Nu$ and $n_0 = Nw$ will serve as a model. For any step-length interpret N as the number of times the dimensionless periodic boundary condition $\lambda = 1/\beta$ is repeated. If the bit-string is used to model a circular orbit where $\lambda = 2\pi r$, the periodicity represents the probability of an interaction which, on the average, delivers just enough centripetal acceleration to maintain the circular velocity β. For any rational fraction velocity (in units of c), the probability of an interaction occurring compared to λ steps in straight line motion at that velocity specifies a dimensionless coupling constant

$$f^2 = \beta = 1/\lambda \ . \tag{2.12}$$

The dividing line between "weak" and "strong" interactions defined by $f^2 = 1$ is just the interaction which will produce $v = c$ for a radius $r = \lambda/2\pi$. In this way our analysis achieves universal significance, independent of the unit of length. We still have an unknown parameter f^2 characterizing specific systems.

2.3. General Kinematical Units

Once one goes beyond the conceptual fusion between space and time symbolized by $\mathbf{c} = 1$ and allows time as an *independent* component of measured (and perceived?) experience, one can conceive of kinematical theories which have dimensionless constants other than $\mathbf{c} = 1$ for the linear relation between length and time and some dimensionless value for $\mathbf{g}$ relating length to the square of a time. The earliest such system is contemporary with Galileo, and can be ascribed to Kepler — in the apochrophal sense in which Drake defines units based on "Galileo's constant." It specifies an arbitrary constant which relates L^2/T to a *Keplerian* system of units based on his Second Law. In the spirit of our previous discussion we could, in any well specified observational context, pick a dimensionless constant $\mathbf{K_2}$ called "Kepler's Constant" and combine it with Einstein's constant. Or we could start from Kepler's Third Law, or from some definition appropriate to quantum mechanics, or .... Once we have done this, the assumption that there are only two fundamental kinematical constants will have non-trivial consequences.

Units based on Kepler's Second and Third Laws

Kepler discovered (using Tycho's data) that for the planets the line from the sun to the planet sweeps over equal areas in equal times. This defines the dimensional combination, $\frac{L^2}{T}$, rather than the linear ratio and the $\frac{L}{T^2}$ ratio we have so far considered. If we were geometrically motivated, we could introduce a unit of area based on Kepler's First Law (elliptical orbits with the sun at one focus) by using the area of the ellipse (πab, with a the semi-major and b the semi-minor axis). This would give us a kinematic way to measure π for areas and compare it with π for perimeters. Rather than take this route, I use recent work on foundations with Pat Suppes and Acacio DeBarros, parts of which will be reported elsewhere [10,11]. We can now construct finite and discrete Lorentz transformations from three integers (see Appendix).

Consider a circular orbit of radius r and a minimal step-length Δr. Any minimal step between two points on the orbit which (which keeps the radius constant) specifies an isosceles triangle with base Δr and sides r. The square of the area of the triangle [using the general formula for sides a, b, c that $16A^2 = (a+b+c)(a+b-c)(b+c-a)(c+a-b)$] and units of Δr^2 is

$$[\frac{\Delta A}{\Delta r^2}]^2 = [r/\Delta r - \frac{1}{2}][r/\Delta r + \frac{1}{2}] \ . \tag{2.13}$$

Since the time increment $\Delta t = \Delta r/v$ in units of the period $T = 2\pi r/v$ is $\frac{1}{2\pi}[\frac{\Delta r}{r}]$ we can define the half-integer $j = \frac{r}{\Delta r}$ and the integer $\ell = j - \frac{1}{2}$ with the consequence that in these units

$$[\frac{\Delta A}{\Delta t}]^2 = \ell(\ell+1)(\frac{1}{2\pi})^2 = (j^2 - \frac{1}{4})(\frac{1}{2\pi})^2 \ . \tag{2.14}$$

We conclude that the natural unit in which to express Kepler's second law is $1/2\pi$ for ℓ and $1/4\pi$ for j. We take $1/4\pi$ to be the minimal kinematic unit for angular momentum per unit mass. In the past we have quantized bit-string physics using the invariant step length $\lambda_0 = h/mc$. Consequently, if we know the mass scale, the minimal unit for angular momentum is $\frac{1}{2}\hbar$ This gives us an alternative, but consistent, route to quantization. Note that since we computed the area, we get only the *orbital* angular momentum with maximum projection $\pm\ell\hbar$ on some reference direction. To identify the spin contribution we can derive the Dirac equation in this framework, which we have done elsewhere [12]. Once again the π which comes in is the relation between linear and circular measures of periodicity.

Since h/m has dimensions of L^2/T, we can make our constant for Kepler's second law consistent with relativistic quantum mechanics simply by taking

$$\frac{L_{K_2}^2}{T_{K_2}} = K_2^{RQM} = \frac{1}{2\pi}\frac{h}{m} \tag{2.15}$$

or with Galileo-Einstein units by taking

$$K_2^{GE} = \frac{L_{GE}^2}{T_{GE}} = \frac{2^3}{\pi^2}\frac{c^3}{g} \ . \tag{2.16}$$

Once we have Kepler's First Law that the orbits are elliptical rather than circular, with the sun at one focus, and we generalize our version of velocity-radius units to perihelion velocity and distance, Kepler's Third Law is simply a consequence of (kinematic) dimensional analysis:

$$K_3^{GE} = \frac{L_{GE}^2}{T_{GE}^3} = \frac{\pi^2}{2^3} cg \ . \tag{2.17}$$

Thus, once one accepts my way of combining Galileo's and Einstein's kinematics, Kepler's Third Law—although dependent on reference to well defined geometries—is only a consequence of a choice of units which are easily related to the earlier definitions. Drake's final chapter entitled "Galilean Units Today" applies Galilean units to accepted astronomical data (relative to Mercury). At least from my point of view his numerical results are no surprise, and to call them "non-Keplerian" becomes more of a semantic than a physics issue.

Kinematical RQM Units

Although m appears in my quantum version of Kepler's second law, it remains arbitrary, as indeed it must in any kinematic system. Nevertheless, the essential parameter h/m can be given kinematic significance relative to some sufficiently stable reference particle, as I have already mentioned in earlier work [7]. Take $c = 1$, define velocity by a counter telescope, and measure the double slit interference pattern in arbitrary units of length. This defines the relativistic deBroglie wavelength for that particle as a function of velocity. Lengths characteristic of other particles can be determined in the same way. Length ratios are now operationally specified. For any choice of reference particle, the constant h/m_0 can be given universal significance using Kepler's second law, as we saw above. Mass ratios remain empirical. The choice of m_0 remains arbitrary.

3. Dynamical Units: Gravitation

In conventional parlance, as used for example by Drake, kinematics involves a *description* of motion while dynamics involves a causal, and in principle calculable, *explanation* based on the concept of *force*. Mach's *Science of Mechanics* tries to banish the concept of "force" from the subject because of its residual anthropomorphic connotations. His treatment remains Newtonian, however, in that his mass ratios are based on Newton's Third Law. In recent years, I have become increasingly aware of the desirability of separating kinematics from dynamics in the discussion of relativistic quantum mechanics. As noted above, I have replaced Newton's Third Law by operationally defined deBroglie wavelengths which specify the equivalent of mass ratios. Then I must *derive* relativistic three-momentum conservation. This is done in the Appendix. It remains to give mass an absolute significance.

Thanks to the work we have done above, this is straightforward. We have already defined the minimum unit of angular momentum in terms of h/m and the shortest possible length as that which gives orbital velocity c. Taking this minimum length as $\hbar/Mc = r$ and $gr/c^2 = 1 = GM/rc^2 = GM^2/\hbar c$ we have that the limiting mass for a *coherent* elementary system is $M = [\frac{\hbar c}{G}]^{\frac{1}{2}}$, which should come as no surprise. To get this result related to the proton mass takes a little more work.*

Zurek and Thorne [13] have shown that the number of bits of information lost in forming a rotating, charged black hole is equal to the area of the event horizon

* An earlier version of the next two paragraphs—SLAC–PUB–5588—was rejected by *Phys. Rev. Lett.* because it was too novel to be published as a "Comment."

in Planck areas, i.e., the*Beckenstein number* [14]. Wheeler [15] has suggested that this could be a significant clue in the search through the foundations of physics for links between information theory and quantum mechanics. If one accepts the conservation of baryon number, as attested by the *experimentally* unchallenged stability of the proton, one can argue that the proton is a stable, charged, rotating black hole with baryon number $+1$, charge $+e$, angular momentum $\frac{1}{2}\hbar$ and Beckenstein number $N = \hbar c/Gm_pc^2 \simeq 1.7 \times 10^{38}$.

Consider an assemblage of N proton-antiproton pairs with all quantum numbers zero which contains an additional proton; this system has baryon number $+1$, charge e, and angular momentum $\frac{1}{2}\hbar$. Suppose the average distance between each pair is $\hbar/m_pc$. Then the gravitostatic energy E is

$$E = \frac{NGm_p^2}{\hbar/m_pc} = N\frac{Gm_p^2}{\hbar c}(m_pc^2)$$

which is equal to the proton rest energy when $N = \hbar c/Gm_p^2$. This is analagous to the bound $N_e = 137 \simeq \hbar c/e^2$ on the number of charged particle-antiparticle pairs established by Dyson [16] when he showed that the renormalized QED perturbation series in α is not uniformly convergent. No particulate constituent of the gravitational system we envisage can escape; the escape velocity exceeds c. Yet proton-antiproton pairs can annihilate to produce Hawking radiation [17], which is not, necessarily, bound to the system. The predictable endpoint of this system, granted baryon number conservation, charge conservation and quantized angular momentum conservation is a system with mass and conserved quantum numbers indistinguishable from those of the proton. Since this system started from $N = \hbar c/Gm_p^2$ indistinguishable pairs, the number of bits of information *lost* in this way can reasonably be called "the Beckenstein number of the proton." Of course, any particulate mass can be gravitationally stabilized in this way, if it cannot decay to lighter particles. That the proton is the lightest (indeed, the only known) stable baryon makes the identification unique.

4. Appendix: Integer Lorentz Transformations

4.1. Basic Algebra

Given three positive-*definite*, finite integers n_i, n_j, n_k with the three indices i, j, k finite, distinct, cyclic, positive-*definite* integers, i.e.,

$$n_i, n_j, n_k, i, j, k, \in 1, 2, 3, \ldots, N; \quad N \text{ fixed}; \quad i \neq j \neq k \neq i \text{ cyclic} \tag{4.1}$$

we can define

$$t_{ij} := n_i + n_j; \quad t_{ij}\beta_{ij} := n_i - n_j := x_{ij} \tag{4.2}$$

$$\tau_{ij}^2 := t_{ij}^2 - x_{ij}^2 = 4n_in_j = t_{ij}^2(1 - \beta_{ij}^2) := t_{ij}^2\gamma_{ij}^2 \tag{4.3}$$

with the consequences that

$$t_{ij}\beta_{ij} + t_{jk}\beta_{jk} + t_{ki}\beta_{ki} = 0 \tag{4.4}$$

and

$$-\beta_{ij} = \frac{\beta_{jk} + \beta_{ki}}{1 + \beta_{jk}\beta_{ki}} \; . \tag{4.5}$$

Further, since

$$|t_{ij} - t_{jk}| \leq t_{ki} \leq t_{ij} + t_{ki} \tag{4.6}$$

we can define

$$\odot(t_{ij}, t_{jk}; t_{ki}) = \odot(t_{jk}, t_{ij}; t_{ki}) := \frac{1}{2}[t_{ij}^2 + t_{jk}^2 - t_{ki}^2] \tag{4.7}$$

and draw a triangle (see Figure 1) with sides t_{ij}, t_{jk}, t_{ki} and angles

$$cos\ \theta_k := \frac{\odot(t_{ij}, t_{jk}; t_{ki})}{t_{ij}t_{jk}} = \frac{t_{ij}^2 + t_{jk}^2 - t_{ki}^2}{t_{ij}t_{jk}} \; . \tag{4.8}$$

Any one side can be interpreted as a combined rotation and boost taking the position and velocity of one event to another event with respect to a third event, as we will now show.

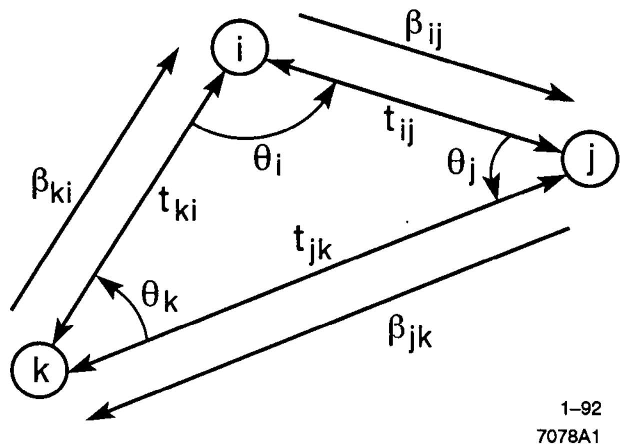

Figure 1. Kinematical interpretation of the three integers n_i, n_j, n_k.

4.2. The Three-Counter Paradigm; [The RQM Triangle]

The figure can be thought of as three counters with associated clocks—synchronized using the Einstein convention—which keep a record of the time of arrival or departure of a signal, and whether it was a particle or indistinguishable locally from a gamma-ray. Since we are using units with c=1, the *distances* between the counters i, j, k are simply t_{ij}, t_{jk}, t_{ki}. If we launch a signal with velocity β_{ij} from counter i toward counter j and simultaneously launch a signal with $-\beta_{ki}$ from i toward k which, on arrival at k, triggers a signal from k to j with velocity $-\beta_{jk}$, the signals from i to j and from k to j will arrive simultaneously at j. This explains why, if we pay proper attention to signs, we obtain the usual Lorentz velocity addition law independent of how far away counter k is from the ij path.

Note also that our cyclic convention can be used to define a direction out of the plane of the triangle whose sign reverses either if we change our convention from cyclic to anti-cyclic or if we interchange two of the indices. Clearly this is the "parity" transformation P. In contrast to classical relativistic kinematics, our finite assumption forces us to consider transformations which do *not* conserve parity. Further if we reverse *all* velocities—which corresponds to time reversal T—this discrete transformation produces the same result as the (cyclic $\leftrightarrow$ anti-cyclic) parity operation. Consequently the physical paradigm we use to interpret the formalism automatically guarantees that at this stage the theory is invariant under P^2, T^2, PT *and* TP. Full CPT invariance will have to wait until we define conserved quantum numbers analagous to and including electric charge. However, if we include forward or backward "motion in time" in order to define a conserved difference between the number of particles and the number of antiparticles, or left-right motion in a single direction to conserve helicity, we can immediately invoke these conservation laws to construct finite and discrete solutions to the Dirac equation in 1+1 dimensions [12].

Although, thanks to the velocity addition law derived from the usual clock synchronization convention, the paradigm obviously has an Lorentz-invariant significance, we have yet to establish formal Lorentz invariance.

4.3. Boosts; definition of c, λ_0

Although the last section interpreted the algebra of Section 4.1 as describing three synchronized counters fixed in the laboratory, it is also interpretable more abstractly as describing coordinate transformations. Consider first the connection between counter i and counter j. Note first that the separation $t = t_{ij} = n_j + n_k$ and the velocity $\beta_{ij} = (n_i - n_j)/t_{ij} := x_{ij}/t_{ij} = x/t$ only involve the two integers n_i and n_j. If we take as our referent the vanishing of these two integers, symbolized by $(0,0)$, for the 1+1 space-time integer coordinate (x,t), the square of the invariant interval between the two events at i and j is $t^2 - x^2 = 4n_in_j$ independent of the value of n_k or of the position of the counter k. If we take the counter k as the referent for both $(x',t') = (n_j - n_k, n_j + n_k)$ and for $(x",t") = (n_k - n_i, n_k + n_i)$, with invariant intervals $\tau^2_{ki;j} = 4n_jn_k$ and $\tau^2_{jk;i} = 4n_kn_i$ respectively, we see that

$$\tau^2_{jk} = [n_i/n_j]\tau^2_{ki} := \rho\tau^2_{ki} \ . \tag{4.9}$$

Note that this connection between these two invariant intervals is again independent of n_k and hence of the arbitrary reference system represented by counter k.

Clearly $\beta = \frac{\rho-1}{\rho+1}$ is simply the boost along the $i-j$ direction which brings the event at i and the event at j to a coordinate system in which the two events are at rest. Once this is understood, the Lorentz transformation taking (x', t') to (x'', t'') is easy to work out.

Our clock synchronization convention and resulting derivation of the Lorentz transformation establishes the fact that c has the customary physical significance. Note that the unit of length is arbitrary. If we take $\lambda_0 = h/m_0c$, this corresponds to our earlier quantization assumption. h/m and mass ratios measured by a double slit plus collimators follow.

4.4. ROTATIONS; DEFINITION OF $\hbar$

For rotations, instead of an invariant interval, we need to preserve an invariant length. Recall that the square of the area of the triangle is given by

$$16A^2 = (t_{ij} + t_{jk} + t_{ki})(t_{ij} + t_{jk} - t_{ki})(t_{ij} - t_{jk} + t_{ki})(-t_{ij} + t_{jk} + t_{ki})$$

$$= 16(n_i + n_j + n_k)n_i n_j n_k \ . \tag{4.10}$$

Take

$$t_{jk} = r = t_{ki}; \ \Delta r = t_{ij} \ . \tag{4.11}$$

Require that equal areas be swept out in equal times. Then, if the minimum step *for rotations* (including straight line periodicities) is $\hbar/mc = \Delta r$, the area swept out by this minimal step is, in these units,

$$[(mcr/\hbar)/(mc\Delta r/\hbar)]^2 = (j - \frac{1}{2})(j + \frac{1}{2}) = \ell(\ell + 1) \tag{4.12}$$

where we have defined $j = r/\Delta r$ in order to bring out the formal similarity between this quantal version of Kepler's second law and the usual quantization of angular momentum for particles with spin $\frac{1}{2}$. The details will be presented elsewhere.

Note that this route defines $\hbar$ independent of c. Then, relative to any stable mass, scale invariance is broken.

4.5. General Lorentz Transformations in a plane

Since we can now boost to a rest system, rotate, and boost to the final system, the basic problem of the Lorentz invariance of our theory has been solved. Given two arbitrary integers n_i, n_j representing events at $(0,0)$ and (x,t) connected by the velocity $\beta = x/t = p/E$ we can obviously always find a third event relative to which, in the rest system, the two distances satisfy Eq. 4.11. Taking $\Delta r = \frac{\hbar}{mc}$ gives us the unique quantum number j either for a free particle (impact parameter) or for a circular orbit. We already have the invariant interval $\tau^2 = 4n_in_j$ giving us two of the free particle quantum numbers. The third comes from using an integer 3-space coordinate system. The reference mass remains arbitrary until we introduce gravitation, which we can do via the combinatorial hierarchy. The self consistency between the *linear* step-length $\lambda = h/mc$, the unit for angular momentum of $\frac{1}{2}\hbar$ derived from Kepler's second law, and the deBroglie relation $p = h/\lambda = \hbar k$ in fully invariant form is what convinces me that, *finally*, I have the correct elementary starting point for relativistic quantum mechanics. Working out the details will take a book, which I am writing [18].

REFERENCES

1. Stillman Drake, *Galileo: Pioneer Scientist*, University of Toronto Press, 1990; see in particular the first and last chapters.
2. Ref. [1], pp. 8,237.
3. Stillman Drake, *Galileo at Work*, University of Chicago Press, 1978; see also the research papers referenced there and in Ref. [1].
4. David McGoveran, "Foundations for a Discrete Physics," in *DISCRETE AND COMBINATORIAL PHYSICS; Proc. ANPA 9*, H. P. Noyes, ed, ANPA WEST, 409 Lealand Ave., Palo Alto, CA 94306, p. 61 et seq.
5. R. Penrose, *The Emperor's New Mind,* Oxford University Press, 1989.
6. B. W. Petley, *The Fundamental Physical Constants and the Frontier of Measurement,* Adam Hilger Ltd., Bristol, 1985, p. 28.
7. H. P. Noyes and S. Starson, "Discrete Antigravity" in *Instant Proceedings, ANPA WEST 7*, F.Young, ed., ANPA WEST, 409 Lealand Ave., Palo Alto, CA 94306, 1991, and SLAC–PUB–5429 (March 1991).

8. *PARTICLE PROPERTIES DATA BOOKLET,* April 1990, North Holland, available from Berkeley and CERN, p. 3.
9. D. O. McGoveran and H. P. Noyes, *Physics Essays,* **4**, pp. 115–120 (1991).
10. H. P. Noyes, *The RQM Triangle: a Paradigm for Relativistic Quantum Mechanics,* to be presented at ANPA WEST 8, February 15–17, Stanford University.
11. H. P. Noyes, *Second Quantization in Bit-String Physics,* to be presented at the Workshop on Harmonic Oscillators, March 25–28, University of Maryland, March 25–28.
12. H. P. Noyes, *Lectures on Bit-String Physics,* in Philosophy 242a, Stanford University, Fall quarter, 1991.
13. W. H. Zurek and K. S. Thorne, *Phys. Rev. Lett.* **54**, 2171–2175 (1985).
14. J. D. Beckenstein, *Phys. Rev.* **D 7**, 2333 (1973).
15. J. A. Wheeler, "Information, Physics, Quantum: the Search for Links," *Proc. 3rd ISFQM*, Tokyo 1989, pp. 334–368.
16. F. J. Dyson, *Phys. Rev.* **51**, 631 (1952).
17. S. W. Hawking, *Phys. Rev.* **D13**, 191 (1976).
18. H. P. Noyes, *An Introduction to Bit-String Physics,* J. C. van den Berg, ed., (in preparation).