Heavy Meson Form Factors from QCD Sum Rules[*]

MATTHIAS NEUBERT[†]

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94309

ABSTRACT

A consistent framework is developed for studying hadronic form factors of heavy mesons using QCD sum rules in the heavy quark effective theory, including the next-to-leading order renormalization group improvement. Sum rules are derived for the asymptotic value of the meson decay constant f_P and for the universal Isgur-Wise form factor. It is shown that renormalization group effects considerably enhance the prediction for f_P and bring its asymptotic value in accordance with recent lattice results. Including finite mass corrections, the dependence of the physical decay constants on the meson mass is investigated. We obtain $f_D \simeq 170 \pm 30$ MeV and $f_B \simeq 190 \pm 50$ MeV. The origin of the break-down of the heavy quark expansion for f_D is analyzed. In the case of heavy meson transition form factors, both the QCD and $1/m_Q$ corrections are moderate and under control. A sum rule for the renormalized Isgur-Wise function is derived and evaluated. The theoretical result is compared to experimental data.

Submitted to *Physical Review D*

---

1. Introduction

There is recently intense interest in the hadronic form factors of particles containing a heavy quark. The reason is that, in the limit of infinite quark mass, QCD reveals a spin-flavour symmetry that is not explicit in its Lagrangian [1−6]. It implies that the spin and mass of a heavy quark decouple from the hadronic dynamics. This symmetry becomes manifest to lowest order of an effective field theory describing the strong interactions of heavy quarks [7−9]. In this effective theory, Green's functions are expanded in powers of $1/m_Q$, with m_Q being the renormalization-group invariant "physical" pole mass of the heavy quark.

The simplest type of a form factor is that describing the current-induced generation of a heavy meson out of the vacuum. Consider, for instance, the coupling of a pseudo scalar meson $P = (Q\,\bar{q})_{0^-}$ with momentum p to the axial vector current

$$\langle\, 0\, |\, \bar{q}\,\gamma_\mu\gamma_5\, Q\, |P(p)\rangle = i\, f_P\, p_\mu \,. \tag{1.1}$$

The decay constant f_P is a measure of the strength of the quark-antiquark attraction inside the bound state. It is, therefore, a hadronic quantity of primary theoretical interest. Governing the strength of leptonic and nonleptonic weak decays of heavy mesons as well as phenomena like $B - \overline{B}$ mixing, which provide information on the mass of the top quark and on the CKM-matrix, decay constants are also of considerable phenomenological importance. Based on the non-relativistic constituent quark model, it was known for a long time that, up to logarithmic corrections, f_P obeys the asymptotic scaling law

$$f_P \propto \frac{1}{\sqrt{m_P}} \tag{1.2}$$

as $m_P \to \infty$. In the framework of the heavy quark expansion, such a behaviour can be shown to be a general consequence of QCD. Furthermore, the leading and next-to-leading logarithmic corrections to (1.2) have been calculated and summed to all orders in perturbation theory [4, 10−12]. The asymptotic dependence of f_P on the

mass of the heavy meson is thus well-understood. A study of the mass-dependence of physical decay constants provides, therefore, an estimate of the corrections to the infinite quark mass limit. In the absence of experimental information, considerable attention has been devoted to the theoretical calculation of f_P and its dependence on m_P. Besides QCD-inspired potential models [13,14], the sum rule approach of Shifman, Vainshtein, and Zakharov [15] and lattice gauge theory are the tools that have been most extensively used for this purpose. While there is general agreement (within the intrinsic uncertainties of each method) on the value of f_D, recent lattice results indicate an unexpectedly large value $f_B > f_D$ in vast contradiction to the scaling law (1.2) [16−19]. This has been interpreted as a signal for a break-down of the $1/m_Q$ expansion for the case of charmed particles. QCD sum rule calculations, on the other hand, yield considerably smaller values for f_B [20−28]. One of the purposes of this paper is to combine the sum rule technique with the heavy quark expansion, and to understand and resolve the discrepancy between sum rule and lattice calculations.

Probably the most fruitful application of the heavy quark spin-flavour symmetry is encountered in weak decays of heavy hadrons. Isgur and Wise have worked out the symmetry relations imposed on the various hadronic form factors describing current matrix elements between two heavy mesons or baryons [6]. For instance, they have shown that as $m_b, m_c \to \infty$, all the many form factors describing transitions between any two of the mesons $\overline{B}, \overline{B}^*, D$, and D^* become related to a single universal function $\xi(v \cdot v')$. This so-called Isgur-Wise form factor [6,29] is a universal function of QCD, summarizing all long distance effects active in the weak transition. It only depends on the velocities of the heavy particles and is normalized at zero recoil, where $v \cdot v' = 1$. The reduction of form factors implies a significant simplification of the theoretical description of semileptonic weak decay processes, as has been extensively discussed in the recent literature [30−36].

In this paper, we develop a consistent framework for studying hadronic form factors of heavy mesons by combining QCD sum rule and renormalization group techniques with the effective theory for heavy quarks. The method is applied to

calculate the asymptotic value of $f_P\sqrt{m_P}$, and the Isgur-Wise form factor. In Sect. 2.1, we rewrite the standard Laplace sum rule for f_P in a form which is suitable for a $1/m_Q$ expansion. We then present, in Sect. 2.2, a rederivation of the asymptotic form of this sum rule (valid for infinitely heavy mesons) by using the effective field theory formalism of Georgi [7]. One advantage of this second approach is that the sum rule in the effective theory only depends on low energy parameters, which are independent of the heavy quark mass. These parameters are *a priori* not known in the standard approach. The most important advantage of the effective theory is, however, that a renormalization group improvement can be performed by summing the large logarithms $(\alpha_s \ln m_Q)^n$ and $\alpha_s(\alpha_s \ln m_Q)^n$ to all orders in perturbation theory. In particular, the scale ambiguity associated with the leading QCD correction is resolved. These renormalization group effects, which have not been taken into account in previous calculations of the asymptotic value of $f_P\sqrt{m_P}$ [21, 37], turn out to be very significant. They bring the sum rule result in accordance with lattice computations. In Sect. 2.3, the expressions derived in the effective theory are combined with the standard Laplace sum rule for f_P to obtain an improved calculation of the physical decay constant as a function of the heavy meson mass. Deviations from the $m_Q \to \infty$ limit and from the scaling law (1.2) are investigated in detail.

The second part of the paper is devoted to the calculation of the universal Isgur-Wise form factor. After a short introduction into the effective field theory formalism, we derive the Laplace sum rule for $\xi(v \cdot v')$ in Sect. 3.2, and discuss its renormalization group improvement in Sect. 3.3. This completes two recent calculations of the universal form factor, which have ignored renormalization effects [36, 38]. The Isgur-Wise function is given by the ratio of a sum rule for a three-point correlator and a sum rule for a two-point correlator. A Ward identity ensures its correct normalization at zero recoil. It is shown that, unlike the situation encountered for f_P, the sum rule calculation of $\xi(v \cdot v')$ is not affected by unusually large QCD corrections, nor does one expect large $1/m_Q$ corrections to the infinite quark mass limit. Our theoretical prediction for the Isgur-Wise form

4

factor compares well with a recent extraction of this function from experimental data on $\overline{B} \to D^* \ell \overline{\nu}_\ell$ decays [34]. Sect. 4 contains the conclusions.

2. Decay Constants of Heavy Mesons

2.1 LAPLACE SUM RULE FOR f_P

The QCD sum rules proposed by Shifman, Vainshtein, and Zakharov (SVZ) [15] have proved to be a powerful phenomenological tool in the study of low energy parameters of hadrons, like their masses or couplings to currents. The idea is that hadronic properties may be studied in a self-consistent way by equating an integral over a physical spectral function to an approximation of the operator product expansion of the time-ordered product of two (or more) local currents. QCD sum rules for the pseudo scalar decay constant f_P have been first considered in Refs. 20 and 23. One studies the two-current correlator

$$\Pi_5(q^2) = i \int \mathrm{d}^4x \, e^{iq \cdot x} \, \langle \, 0 \, | \, T\{A_5(x), A_5(0)^\dagger\} \, | \, 0 \, \rangle \tag{2.1}$$

with $A_5 = im_Q \, \overline{q}\gamma_5 Q = \partial^\mu(\overline{q}\gamma_\mu\gamma_5 Q)$ being the divergence of the axial vector current in the limit where the light quark is massless ($m_q = 0$). Since this current is partially conserved, $\Pi_5(q^2)$ is a renormalization-group invariant quantity.

According to the philosophy of SVZ, the correlator is evaluated in two ways. In the euclidean region $q^2 \ll 0$, it can be calculated perturbatively because of asymptotic freedom of QCD. Short distance effects are taken care off by Wilson coefficients, while long distance confinement effects are included as power corrections and are parametrized in terms of vacuum expectation values of local operators, the so-called condensates [15]. Hence

$$\Pi_5(q^2) = \Pi_5^{pert}(q^2) + \Pi_5^{cond}(q^2) \, . \quad (q^2 \ll 0) \tag{2.2}$$

On the other hand, the correlator can be expressed as an dispersion integral over a physical spectral function, which gets contributions from the ground state meson

5

P as well as from higher resonances. The residue of the pole is given by the decay constant f_P. The resonance contributions are usually approximated by a dual perturbative continuum above a threshold s_c. Using (1.1) one thus writes

$$\Pi_5(q^2) = \frac{f_P^2\, m_P^4}{m_P^2 - q^2 - i\,\epsilon} + \frac{1}{\pi} \int\limits_{s_c}^{\infty} \mathrm{d}s\, \frac{\mathcal{I}m\, \Pi_5^{pert}(s)}{s - q^2 - i\,\epsilon} + \text{ subtractions.} \qquad (2.3)$$

Equating the two expressions for $\Pi_5(q^2)$ yields the sum rule, from which f_P can be determined. It is necessary, however, to improve the convergence by suppressing the continuum contribution to the spectral function. This is achieved by applying the Borel operator

$$\widehat{B}_{M^2} = \lim_{\substack{n \to \infty \\ -q^2 \to \infty \\ M^2 = \frac{-q^2}{n}\ \text{fixed}}} \frac{\left(q^2\right)^n}{\Gamma(n)} \left(-\frac{\mathrm{d}}{\mathrm{d}q^2}\right)^n \qquad (2.4)$$

to both sides of the sum rule. In particular, this also eliminates possible subtractions required for the convergence of the dispersion integral in (2.3). The result is the so-called Laplace sum rule

$$\frac{f_P^2\, m_P^4}{M^2}\, e^{-m_P^2/M^2} = \frac{1}{\pi M^2} \int\limits_{m_Q^2}^{s_c} \mathrm{d}s\, \mathcal{I}m\, \Pi_5^{pert}(s)\, e^{-s/M^2} + \widehat{B}_{M^2}\, \Pi_5^{cond}\ , \qquad (2.5)$$

where the perturbatively calculated spectral function has a cut starting at $s = m_Q^2$. Explicit expressions for the functions appearing on the right-hand side of (2.5) can be found in Ref. 23. In order to determine f_P in a self-consistent way, one tries to optimize the value of the continuum threshold s_c in such a way that the computed value of the decay constant is stable with respect to variation of the Borel parameter M^2 in a region where the theoretical calculation of Π_5^{pert} and Π_5^{cond} is reliable. For too small values of M^2 the power corrections blow up, while the

continuum contribution becomes dominant at large M^2. One thus aims for stability in an intermediate region, where both the power and continuum contributions stay reasonably small.

In its above form, the Laplace sum rule is not suited for an expansion in powers of $1/m_Q$, since the dependence of the parameters M^2 and s_c on the heavy quark mass is *a priori* not determined. It is convenient to introduce a set of new parameters by

$$m_Q T \equiv M^2 \, ,$$
$$m_Q \, \omega_c \equiv s_c - m_Q^2 \, , \tag{2.6}$$
$$m_Q \, \widetilde{\Lambda} \equiv m_P^2 - m_Q^2 \, .$$

In the following section we will show that the new variables T, ω_c, and $\widetilde{\Lambda}$ become constant low energy parameters in the $m_Q \to \infty$ limit. Substituting them into (2.5), the Laplace sum rule takes the following form

$$f_P^2 \, m_P \left(\frac{m_P}{m_Q} \right)^3 e^{-\widetilde{\Lambda}/T}$$

$$= \frac{3T^3}{8\pi^2} \int_0^{\omega_c/T} dz \, \frac{z^2 \, e^{-z}}{1 + z\frac{T}{m_Q}} \left\{ 1 + \frac{2\alpha_s}{\pi} \left[\ln \frac{m_Q}{T} + \frac{13}{6} + \frac{2\pi^2}{9} - \ln z + \frac{2}{3} K\left(z \frac{T}{m_Q} \right) \right] \right\}$$

$$- \langle \bar{q}q \rangle (m_Q) \left\{ 1 + \frac{2\alpha_s}{3\pi} \left[1 - 3\frac{T}{m_Q} \int_0^\infty dz \, \frac{e^{-z}}{1 + z\frac{T}{m_Q}} \right] \right\} - \frac{\langle \alpha_s \, GG \rangle}{12\pi m_Q}$$

$$+ \frac{m_0^2 \langle \bar{q}q \rangle (m_Q)}{4T^2} \left(1 - \frac{2T}{m_Q} \right) + \frac{4\pi}{81} \frac{\alpha_s \langle \bar{q}q \rangle^2}{T^3} \left(1 + \frac{3T}{m_Q} - \frac{12T^2}{m_Q^2} \right) + \cdots \, ,$$

$$\tag{2.7}$$

where m_0^2 is defined as ratio of the so-called mixed condensate and the quark condensate, $g_s \langle \bar{q} \, \sigma_{\mu\nu} G^{\mu\nu} \, q \rangle \equiv m_0^2 \langle \bar{q}q \rangle$. The ellipses stand for power corrections from higher-dimensional condensates $(d > 6)$. These are expected to be very small. Already the four-quark condensate is suppressed by a factor of 10^{-3} as compared to the quark condensate.

In (2.7) we have included the radiative corrections to the dispersion integral and to the contribution of the quark condensate. The first one has been calculated

in Ref. 39. In our notation, the function $K(x)$ is given by

$$K(x) = 2\operatorname{Li_2}(-x) + \ln x \, \ln(1+x) - \frac{x \ln x}{1+x} + \frac{1+x}{x}\ln(1+x) - 1$$
$$= -\frac{3}{2}x + \left(\frac{1}{3} + \frac{\ln x}{2}\right)x^2 + \mathcal{O}(x^3) \tag{2.8}$$

with $\operatorname{Li_2}(z)$ being the dilogarithm. The correction to the quark condensate contribution is new. In (2.7), m_Q is the physical heavy quark mass defined as the pole of the renormalized propagator. This renormalization-group invariant quantity is related to the running mass of the $\overline{\text{MS}}$ subtraction scheme by [40, 41]

$$\overline{m}_Q(\mu) = m_Q \left(\frac{\alpha_s(\mu)}{\alpha_s(m_Q)}\right)^{d_m} \left\{1 - \frac{4\alpha_s(m_Q)}{3\pi}\right\} \; ; \quad d_m = \frac{12}{(33 - 2n_f)} \; . \tag{2.9}$$

The running of the quark condensate is given by

$$\frac{\langle \bar{q}q \rangle(\mu)}{\langle \bar{q}q \rangle(m_Q)} = \left(\frac{\alpha_s(\mu)}{\alpha_s(m_Q)}\right)^{-d_m} \tag{2.10}$$

and is such that the product $m_Q \langle \bar{q}q \rangle$ is renormalization-group invariant. In the sum rule (2.7), the condensate is to be evaluated at the scale of the heavy quark.

Eq. (2.7) is completely equivalent to the more familiar form (2.5) of the sum rule, which has been first investigated in Ref. 23. For its evaluation we use the standard values of the vacuum condensates [15, 42]

$$\langle \bar{q}q \rangle(1 \text{ GeV}) = -(230 \text{ MeV})^3 \; ,$$
$$\langle \alpha_s GG \rangle = 0.038 \text{ GeV}^4 \; ,$$
$$m_0^2 = 0.8 \text{ GeV}^2 \; , \tag{2.11}$$
$$\alpha_s \langle \bar{q}q \rangle^2 = 6 \times 10^{-5} \text{ GeV}^6 \; ,$$

as well as the flavour-independent value $\widetilde{\Lambda} = 1$ GeV, corresponding to the pole masses $m_c \simeq 1.44$ GeV and $m_b \simeq 4.80$ GeV. The effect of a variation of $\widetilde{\Lambda}$ is

discussed below. The only free parameter in (2.7) is the threshold energy ω_c. Evaluating the radiative corrections at the scale of the heavy quark, one finds good stability in the wide region 0.5 GeV $< T <$ 2.0 GeV for $\omega_c \simeq$ 2.4 GeV for f_D and $\omega_c \simeq$ 2.0 GeV for f_B, corresponding to $s_c^D \simeq$ 5.5 GeV and $s_c^B \simeq$ 33 GeV, respectively. Unlike the continuum thresholds, the optimal ω_c-values are rather insensitive to the mass of the heavy quark. In the stability region, the values of the decay constants are $f_D \simeq 170 \pm 30$ MeV and $f_B \simeq 140 \pm 30$ MeV. They agree with those obtained in Ref. 23. Variation of the QCD parameters within the standard limits does not significantly change these results.

Under the premises that T, ω_c, and $\widetilde{\Lambda}$ are indeed low energy parameters, one can immediately perform the limit $m_Q \to \infty$ in (2.7). This non-relativistic form of the sum rule has been investigated as early as 1982 in a paper by Shuryak [21]. Recently, it has been rederived and discussed in Ref. 37. At this point, it is worth noting that the right-hand side of (2.7) shows a logarithmic dependence on the heavy quark mass both in the radiative correction to the dispersion integral and in the running of the quark condensate. Eventually, one would like to separate this dependence from the sum rule, and sum the large logarithms $\left(\alpha_s \ln m_Q \right)^n$ to all orders in perturbation theory. It is also obvious that the radiative correction to the dispersion integral is dangerously large, even if the strong coupling is evaluated at the scale of the heavy quark. However, it is not obvious that $\mu = m_Q$ really is the appropriate scale to use. For this reason, a next-to-leading order calculation, which resolves the scale ambiguity problem, is most desireable. In the following section, we will derive the correct asymptotic form of the sum rule and perform the complete next-to-leading order renormalization group improvement. To this end, it is necessary to employ an effective theory for heavy quarks.

2.2 SUM RULE IN THE EFFECTIVE THEORY

A convenient framework for systematically analyzing both the QCD and the $1/m_Q$ corrections to the infinite quark mass limit is provided by the so-called heavy-quark effective theory developed by Georgi [7]. The basic observation is that as $m_Q \rightarrow \infty$, the velocity v of a heavy quark becomes a conserved quantity with respect to soft processes. It is then possible to remove the mass-dependent piece of the momentum operator by the velocity-dependent field redefinition

$$h_Q(v, x) = e^{i m_Q \slashed{v} \cdot x} \, \psi_Q(x) \, , \qquad (2.12)$$

such that

$$i \, \slashed{\partial} \, h_Q(v, x) = \slashed{k} \, h_Q(v, x) \, ; \quad k_\mu = P_\mu - m_Q \, v_\mu \, , \qquad (2.13)$$

where P is the total momentum of the heavy quark, m_Q is its physical pole mass, and k denotes the residual "off-shell" momentum which is typically of order Λ_{QCD}. In terms of the new fields $h_Q(v, x)$, the effective Lagrangian consists of an infinite series of operators with increasing canonical dimension, multiplied by increasing powers of $1/m_Q$. To lowest order in the $1/m_Q$ expansion, this Lagrangian explicitly exhibits the spin-flavour symmetry for heavy quarks [7−9].

The axial vector current $A_\mu = \overline{q} \, \gamma_\mu \gamma_5 \, Q$ can be expanded in terms of operators of the effective theory as follows

$$
\begin{aligned}
A_\mu \simeq \; & D_1 \left(\frac{m_Q}{\mu} \right) \overline{q} \, \gamma_\mu \gamma_5 \, h_Q(v) - D_2 \left(\frac{m_Q}{\mu} \right) \overline{q} \, v_\mu \gamma_5 \, h_Q(v) \\
& + \sum_{n=1}^{\infty} \left(\frac{1}{m_Q} \right)^n \sum_i D_i^{(n)} \left(\frac{m_Q}{\mu} \right) \mathcal{A}_\mu^{(i,n)} \, ,
\end{aligned}
\qquad (2.14)
$$

where the index i labels operators of the same canonical dimension, and we have explicitly written down the two lowest-dimensional operators. The symbol "$\simeq$" means that (2.14) is an equality for matrix elements only. In the full theory, matrix elements of the axial vector current depend on the mass of the heavy quark, but

are independent of the renormalization scale since the anomalous dimension of A_μ vanishes. Matrix elements of operators in the effective theory, on the other hand, are independent of the heavy quark mass. All reference to m_Q is either in form of powers of $1/m_Q$ that multiply the higher-dimensional operators $\mathcal{A}_\mu^{(i,n)}$, or in the short-distance coefficients $D_i^{(n)}$. These functions appear since the properties of the effective theory under renormalization differ from QCD. In particular, the effective current operators have non-zero anomalous dimensions, such that matrix elements depend on the renormalization scale μ. From the fact that the scale-dependence of matrix elements must exactly cancel against that of the short-distance coefficients, one can derive the renormalization group equation for the functions $D_i^{(n)}$. In order to solve this equation to first order in the physical coupling $\alpha_s(m_Q)$, one needs the two-loop anomalous dimensions of the current operators in the effective theory. For the lowest-dimensional operators in (2.14) the anomalous dimension is the same. It has recently been calculated in Refs. 11 and 12. For our purpose, we only need the sum of the coefficients D_1 and D_2, the complete next-to-leading order expression for which is [43]

$$
\begin{aligned}
C_F\left(\frac{m_Q}{\mu}\right) &\equiv D_1\left(\frac{m_Q}{\mu}\right) + D_2\left(\frac{m_Q}{\mu}\right) \\
&= \left(\frac{\alpha_s(\mu)}{\alpha_s(m_Q)}\right)^{d_m/2} \left\{ 1 + \frac{\alpha_s(m_Q)}{\pi} Z_{n_f} - \frac{\alpha_s(\mu)}{\pi}\left(Z_{n_f} + \delta\right) \right\},
\end{aligned}
\tag{2.15}
$$

where δ is a scheme-dependent constant. In the $\overline{\text{MS}}$ subtraction scheme, $\delta_{\overline{\text{MS}}} = 2/3$. The constant Z_{n_f} is scheme-independent and reads

$$
Z_{n_f} = 3\frac{153 - 19n_f}{(33 - 2n_f)^2} - \frac{381 - 30n_f + 28\pi^2}{36\left(33 - 2n_f\right)} - \frac{2}{3},
\tag{2.16}
$$

where n_f is the number of light quark flavours. (The value for Z_{n_f} quoted in Ref. 11 differs from our result by $-2/3$. The correct next-to-leading order correction is only half as large as stated in that paper.)

Matrix elements of operators in the effective theory respect the heavy quark symmetries and can be most concisely computed employing a compact trace formalism [29, 30]. For the relevant matrix elements of the operators appearing in (2.14), one writes

$$\langle 0 | \, \bar{q} \, \Gamma \, h_Q(v) \, | P(v) \rangle = \frac{F(\mu)}{2} \, \text{Tr} \left\{ \Gamma \, \mathcal{P}(v) \right\}$$

(2.17)

with Γ being an arbitrary Dirac matrix. $F(\mu)$ is a scale-dependent low energy parameter independent of m_Q, and

$$\mathcal{P}(v) = -i \, \sqrt{m_P} \, \frac{(1 + \not{v})}{2} \, \gamma_5$$

(2.18)

denotes the spin wave function of the pseudo scalar meson P.

Substituting the expansion (2.14) into (1.1) and carrying out the traces in (2.17), we obtain the following relation between the physical decay constant f_P and the low energy parameter $F(\mu)$

$$f_P \, m_P^{1/2} = C_F \left(\frac{m_Q}{\mu} \right) F(\mu) + \mathcal{O}\left(\frac{1}{m_Q} \right) .$$

(2.19)

Since f_P is a physical quantity, the right-hand side of this equation must be independent of the subtraction scale and of the renormalization scheme adopted for the calculation of C_F. It is thus convenient to define the μ-independent short-distance coefficient $\hat{C}_F(m_Q)$ and the renormalized constant F_{ren} by

$$\begin{aligned}
\hat{C}_F(m_Q) &= \left[\alpha_s(m_Q) \right]^{-d_m/2} \left\{ 1 + \frac{\alpha_s(m_Q)}{\pi} \, Z_{n_f} \right\} , \\
F_{ren} &= \left[\alpha_s(\mu) \right]^{d_m/2} \left\{ 1 - \frac{\alpha_s(\mu)}{\pi} \, (Z_{n_f} + \delta) \right\} F(\mu) ,
\end{aligned}$$

(2.20)

such that

$$C_F \left(\frac{m_Q}{\mu} \right) F(\mu) = \hat{C}_F(m_Q) \, F_{ren} .$$

(2.21)

These new quantities are renormalization-group invariant. F_{ren} is a universal low energy parameter of QCD. It can only be estimated using non-perturbative techniques. In (2.20), it is to be understood that the number of active flavours, i.e. the

value of n_f in Z_{n_f} and d_m, changes as one scales down from m_Q to μ. Explicitly, one has

$$\hat{C}_F(m_c) = \left[\alpha_s(m_c)\right]^{-2/9} \left\{ 1 + \frac{\alpha_s(m_c)}{\pi} Z_3 \right\} \simeq 1.22 ,$$

$$\hat{C}_F(m_b) = \left(\frac{\alpha_s(m_b)}{\alpha_s(m_c)}\right)^{-6/25} \left[\alpha_s(m_c)\right]^{-2/9} \qquad (2.22)$$

$$\times \left\{ 1 + \frac{\alpha_s(m_b) - \alpha_s(m_c)}{\pi} Z_4 + \frac{\alpha_s(m_c)}{\pi} Z_3 \right\} \simeq 1.39$$

for the D and B meson, respectively. For the numerical estimate, we have used $m_c = 1.45$ GeV and $m_b = 4.67$ GeV [41]. The running coupling is accurate to second order. In the $\overline{\text{MS}}$ scheme,

$$\alpha_s(m_Q) = \frac{12\pi}{(33 - 2n_f) \ln \nu} \left[1 - 6 \frac{(153 - 19n_f)}{(33 - 2n_f)^2} \frac{\ln \ln \nu}{\ln \nu} \right] ; \quad \nu = \left(\frac{m_Q}{\Lambda_{n_f}^{\overline{\text{MS}}}}\right)^2 . \quad (2.23)$$

We use $\Lambda_4^{\overline{\text{MS}}} = 0.2$ GeV and adjust the n_f-dependence of this parameter such that the running coupling is a continuous function of m_Q.

In order to compute the renormalized parameter F_{ren} using QCD sum rules in the effective theory, we investigate the correlator

$$\Gamma_5(\omega) = i \int d^4x \, e^{ik \cdot x} \, \langle 0 | \, \mathcal{T}\{\mathcal{A}_5^{(v)}(x), \mathcal{A}_5^{(v)}(0)^\dagger\} \, | 0 \rangle \; ; \quad \omega \equiv 2k \cdot v , \qquad (2.24)$$

with $\mathcal{A}_5^{(v)} = \bar{q} \, \gamma_5 \, h_Q(v)$ being the effective pseudo scalar current. In the rest frame of the heavy quark, ω is twice the external "off-shell energy". The perturbative contribution to Γ_5 is obtained by evaluating the diagrams shown in Fig. 1a. The Feynman rules of the effective theory are given in the appendix. Using dimensional regularization and the $\overline{\text{MS}}$ subtraction scheme, we find

$$\frac{1}{\pi} \mathcal{I}m \, \Gamma_5^{pert}(\omega) = \frac{3\omega^2}{8\pi^2} \Theta(\omega) \left\{ 1 + \frac{2\alpha_s}{\pi} \left[\ln \frac{\mu}{\omega} + \frac{13}{6} + \frac{2\pi^2}{9} + \delta \right] \right\} , \qquad (2.25)$$

where δ is the same scheme-dependent constant that appears in (2.15), i.e. $\delta_{\overline{\text{MS}}} = 2/3$. Next we compute the nonperturbative power corrections to Γ_5. The contributions involving the quark and gluon condensate are depicted in Fig. 1b and c,

respectively. The calculation is most easily performed in the coordinate gauge $x_\mu A^\mu(x) = 0$. Including all condensates with dimension $d \leq 6$, the result is

$$\Gamma_5^{\langle \overline{q}q \rangle}(\omega) = \frac{\langle \overline{q}q \rangle(\mu)}{\omega} \left\{ 1 + \frac{2\alpha_s}{\pi} \left(\frac{1}{3} + \delta \right) \right\} ,$$

$$\Gamma_5^{\langle GG \rangle}(\omega) = 0 ,$$

$$\Gamma_5^{\langle \overline{q}\sigma G q \rangle}(\omega) = -\frac{g_s}{2} \frac{\langle \overline{q}\,\sigma_{\mu\nu} G^{\mu\nu}\, q \rangle}{\omega^3} = -\frac{m_0^2 \langle \overline{q}q \rangle}{2\omega^3} ,$$

$$\Gamma_5^{\langle \overline{q}q \rangle^2}(\omega) = \frac{8\pi}{27} \frac{\alpha_s \langle \overline{q}q \rangle^2}{\omega^4} ,$$

(2.26)

where the factorization approximation has been used to reduce the four-quark condensates to $\langle \overline{q}q \rangle^2$ [15]. Note, in particular, the vanishing of $\Gamma_5^{\langle GG \rangle}$ resulting from the explicit calculation of the diagrams shown in Fig. 1c. This is in accordance with the fact that the contribution of the gluon condensate in (2.7) is associated with a factor $1/m_Q$ instead of $1/T$.

Concerning the calculation of the pole contribution to the phenomenological side of the sum rule, we note that according to (2.12) the total external momentum in (2.24) is $P = m_Q v + k$, such that the propagator of the heavy meson becomes $(P^2 - m_P^2)^{-1} \rightarrow [m_Q(\omega - \widetilde{\Lambda})]^{-1}$ with $\widetilde{\Lambda}$ as defined in (2.6). The hadronic matrix elements in the effective theory are readily evaluated using (2.17). Approximating higher resonance contributions by the perturbative continuum above a "threshold energy" ω_c, which is the analog of the continuum threshold s_c, the phenomenological expression for the spectral function becomes

$$\Gamma_5^{phen}(\omega) = \frac{F^2(\mu)}{\widetilde{\Lambda} - \omega - i\epsilon} + \frac{1}{\pi} \int\limits_{\omega_c}^{\infty} d\omega' \frac{\mathcal{I}m\,\Gamma_5^{pert}(\omega')}{\omega' - \omega - i\epsilon} + \text{ subtractions.}$$

(2.27)

The Laplace sum rule for $F^2(\mu)$ is obtained by applying the Borel operator with respect to ω

$$\widehat{B}_T = \lim_{\substack{n \to \infty \\ -\omega \to \infty \\ T = \frac{-\omega}{n} \text{ fixed}}} \frac{\omega^n}{\Gamma(n)} \left(-\frac{d}{d\omega} \right)^n$$

(2.28)

14

and equating the different expressions for the correlator. The result is

$$
F^2(\mu)\, e^{-\tilde{\Lambda}/T} = \frac{3T^3}{8\pi^2} \int\limits_0^{\omega_c/T} dz\, z^2\, e^{-z} \left\{ 1 + \frac{2\alpha_s}{\pi} \left[\ln\frac{\mu}{T} + \frac{13}{6} + \frac{2\pi^2}{9} + \delta - \ln z \right] \right\}
$$

$$
- \langle \bar{q}q \rangle(\mu) \left\{ 1 + \frac{2\alpha_s}{\pi}\left(\frac{1}{3} + \delta \right) - \frac{m_0^2}{4T^2} - \frac{4\pi}{81}\frac{\alpha_s\langle \bar{q}q \rangle}{T^3} \right\}.
$$

$$(2.29)$$

We observe that the sum rule in the effective theory precisely corresponds to the leading term in the $1/m_Q$ expansion of (2.7), showing that the parameters T, ω_c, and $\tilde{\Lambda}$ introduced in (2.6) are in fact properly chosen low energy parameters. The only difference is the replacement of the heavy quark mass m_Q by the subtraction scale μ and the appearance of the scheme-dependent terms proportional to δ. These terms cancel if one computes the renormalized quantity F_{ren}^2.

With the help of (2.20), one can readily perform the renormalization group improvement of (2.29). The logarithmic dependence on μ can be summed to all orders to produce a factor $\left(\frac{\alpha_s(\mu)}{\alpha_s(T)} \right)^{-d_m}$. For the quark condensate this is already known from (2.10). The next-to-leading order corrections split into a contribution $2\frac{\alpha_s(\mu)}{\pi}\left(Z_{n_f} + \delta \right)$ and a μ-independent correction. For this latter, the running coupling is to be evaluated at a characteristic low energy scale of the effective theory. We choose $\tilde{\Lambda}$ for this scale, since it provides a measure of the average off-shell energy ω of the heavy quark in the meson. An alternative choice would be the Borel parameter T. The differences are formally of order α_s^2 and hence beyond the accuracy of the present calculation. Because of the size of the order-α_s correction, the numerical results are not insensitive to this choice, however. We shall comment on this below.

Putting everything together, we obtain the following renormalization-group improved sum rule for the renormalized parameter F_{ren}^2 defined in (2.20)

$$F_{ren}^2 \, e^{-\widetilde{\Lambda}/T} = \left[\alpha_s(T)\right]^{4/9} \left\{ 1 - \frac{2\alpha_s(\widetilde{\Lambda})}{\pi} Z_3 \right\}$$

$$\times \left\{ \frac{3T^3}{8\pi^2} \int_0^{\omega_c/T} \mathrm{d}z \, z^2 \, e^{-z} \left[1 + \frac{2\alpha_s(\widetilde{\Lambda})}{\pi} \left(\frac{13}{6} + \frac{2\pi^2}{9} - \ln z \right) \right] \right. \tag{2.30}$$

$$\left. - \langle \bar{q}q \rangle(T) \left[1 + \frac{2\alpha_s(\widetilde{\Lambda})}{3\pi} - \frac{m_0^2}{4T^2} - \frac{4\pi}{81} \frac{\alpha_s \langle \bar{q}q \rangle}{T^3} \right] \right\} ,$$

where we have used that the number of light quark flavours in the effective low energy theory is $n_f = 3$. In Fig. 2, the function $F_{ren}(T)$ is shown for $\widetilde{\Lambda} = 1.25$ GeV and various values of the threshold energy ω_c. We find good stability over a wide range of the Borel parameter T. For other values of $\widetilde{\Lambda}$, the resulting curves look very similar. The values of ω_c and T providing optimal stability approximately scale with $\widetilde{\Lambda}$. We obtain

$$F_{ren} \simeq \begin{cases} 0.34 \pm 0.03 \ \mathrm{GeV}^{3/2} & ; \ \widetilde{\Lambda} = 1.0 \ \mathrm{GeV} \\ 0.42 \pm 0.04 \ \mathrm{GeV}^{3/2} & ; \ \widetilde{\Lambda} = 1.25 \ \mathrm{GeV} \\ 0.47 \pm 0.05 \ \mathrm{GeV}^{3/2} & ; \ \widetilde{\Lambda} = 1.5 \ \mathrm{GeV} \end{cases} \tag{2.31}$$

for the renormalized low energy parameter. The quoted errors reflect the variation of the results with respect to changes in ω_c and in the values of the vacuum condensates. The intrinsic uncertainty of the sum rule calculation might be considerable larger, however. This can be inferred from the following observation. If the scale $\widetilde{\Lambda}$ in the next-to-leading logarithmic corrections is replaced by T, the running of $\alpha_s(T)$ leads to a suppression of $F_{ren}(T)$ at large values of the Borel parameter. This effect, which is formally of order α_s^2, is so significant that it changes the continuum threshold providing best stability from $\omega_c \simeq 2.3$ GeV to $\omega_c \simeq 3.0$ GeV (for $\widetilde{\Lambda} = 1.25$ GeV). As a consequence, the result for F_{ren} increases by almost 20%. This is in conflict with the SVZ philosophy that the stability of the sum rule should result from a balance between its perturbative and nonperturbative parts. In order to avoid this effect, we evaluate the next-to-leading order corrections at the fixed scale $\widetilde{\Lambda}$.

Eq. (2.31) exhibits a rather strong dependence of F_{ren} on the mass difference $\widetilde{\Lambda}$, which is a low energy parameter that can again in principle be determined using nonperturbative techniques like lattice gauge theory or QCD sum rules. While no lattice results are available today, an estimate of the pole mass of the b-quark can be obtained from a QCD sum rule analysis of the bottonium spectrum. The extracted values range from $m_b = 4.80 \pm 0.03$ GeV [44] to $m_b = 4.67 \pm 0.10$ GeV [41] and $m_b = 4.55 \pm 0.05$ GeV [26], corresponding to $\widetilde{\Lambda}_{(b)} = 1.01 \pm 0.06$ GeV, 1.30 ± 0.20 GeV, and 1.57 ± 0.10 GeV, respectively. This is the range of values covered in (2.31), and we shall assume that $\widetilde{\Lambda}_{(b)}$ is not very different from the asymptotic value that $\widetilde{\Lambda}$ acquires in the infinite quark mass limit.

From (2.31) and (2.22) one can compute the so-called static limit of the decay constant of the B meson, which is given by the first term on the right-hand side of (2.19). We find

$$f_B^{stat} = \frac{\hat{C}_F(m_b)}{\sqrt{m_B}} F_{ren} \simeq 200 - 300 \text{ MeV}. \tag{2.32}$$

These sum rule results can be compared to recent lattice calculations of the decay constants of heavy mesons, which use the static approximation of the heavy quark propagator [1]. In these computations, one determines the parameter $F(\mu = a^{-1})$ in (2.19) in units of the inverse lattice spacing a^{-1}. The numerical result can be converted into physical units by normalizing to the decay constant of a light pseudo scalar meson (f_π or f_K). Furthermore, a renormalization factor is required to relate the lattice result to the renormalized parameter F_{ren} defined in (2.20). In next-to-leading order, the relation is

$$F_{ren} = \left[\alpha_s(a^{-1})\right]^{d_m/2} \left\{ 1 - \frac{\alpha_s(a^{-1})}{\pi} \left(Z_{n_f} + \delta_{\overline{MS}} + \delta_{latt} \right) \right\} F_{latt}(a^{-1}), \tag{2.33}$$

where $\delta_{\overline{MS}} = 2/3$ accounts for the matching between QCD and the effective theory for heavy quarks, while δ_{latt} provides the matching between the effective theory in the continuum and on the lattice. For Wilson fermions (with $r = 1$), the value

of this constant is $\delta_{latt} \simeq 4.38$ [45, 46]. For a lattice spacing corresponding to $a^{-1} \simeq 2$ GeV, the relation (2.33) thus reads $F_{ren} \simeq 0.48\,F_{latt}(a^{-1})$. We note that in next-to-leading order of renormalization-group improved perturbation theory, the correction factor in (2.33) is 20% smaller than that used in three recent lattice computations of f_B^{stat} [16−18]. Rescaling the values obtained there, we find (The result of Ref. 16 increases to 0.43 ± 0.07 if the lattice spacing is determined without reference to f_π and f_K.)

$$F_{ren}^{latt} \simeq \begin{cases} 0.36 \pm 0.06 \text{ GeV}^{3/2} & ; \text{ Ref. 16} \\ 0.50 \pm 0.08 \text{ GeV}^{3/2} & ; \text{ Ref. 17} \\ 0.44 \pm 0.07 \text{ GeV}^{3/2} & ; \text{ Ref. 18} \end{cases} \qquad (2.34)$$

corresponding to $f_B^{stat} = 218 \pm 36$ MeV, 302 ± 48 MeV, and 266 ± 42 MeV, respectively. Comparison with (2.31) and (2.32) shows that the values obtained from our improved sum rule is completely consistent with the lattice results.

Before proceeding, we would like to make a comment on the size of the QCD correction in (2.30). After the renormalization group improvement, the scale ambiguity associated with the next-to-leading corrections has been resolved. The radiative correction to the dispersion integral is proportional to the physical coupling $\alpha_s(\widetilde{\Lambda})$ in the low energy theory and is accompanied by the large coefficient

$$\frac{13}{3} + \frac{4\pi^2}{9} - 2\,Z_3 = \frac{887}{162} + \frac{122}{243}\,\pi^2 \simeq 10.43 \,, \qquad (2.35)$$

which (in Feynman gauge) is mainly due to the gluon exchange between the heavy and the light quark. With $\frac{\alpha_s(\widetilde{\Lambda})}{\pi} \simeq 0.1$, the radiative correction amounts to a 100% enhancement of F_{ren}^2, corresponding to a 50% enhancement of f_P. We would like to stress that this is a purely perturbative result not specific for QCD sum rules. It is rather a general property of the two-current correlator Γ_5. The physical origin of this effect is likely to be the Coulomb interaction between the quarks. We believe that the size of the correction might indicate an interesting nonperturbative enhancement of the decay constants, which could ultimately limit any perturbative approach to calculate f_P. This should be kept in mind when considering the significance of the sum rule results (2.31) and (2.32).

2.3 Improved Sum Rule for f_P

After this *caveat*, we proceed by deriving an improved sum rule for f_P, which allows for a quantitative estimate of finite-mass corrections to the infinite quark mass limit. To this end, we combine the renormalization-group improved sum rule (2.30) derived in the effective theory with the standard sum rule (2.7) by adding back those term that vanish in the $m_Q \to \infty$ limit. Let us define the quantity $G(m_Q)$ by

$$f_P \sqrt{m_P} \left(\frac{m_P}{m_Q} \right)^{3/2} \equiv \hat{C}_F(m_Q) \, G(m_Q) \, ,$$

$$\lim_{m_Q \to \infty} G(m_Q) = F_{ren} \, . \tag{2.36}$$

This function is free of large logarithms and has as well-defined behaviour in the infinite quark mass limit. From (2.30) and (2.7), we obtain the sum rule

$$G^2(m_Q) \, e^{-\tilde{\Lambda}/T} = \left[\alpha_s(T) \right]^{4/9} \left\{ 1 - \frac{2\alpha_s(\tilde{\Lambda})}{\pi} Z_3 \right\}$$

$$\times \left\{ \frac{3T^3}{8\pi^2} \int_0^{\omega_c/T} dz \, \frac{z^2 \, e^{-z}}{1 + z \frac{T}{m_Q}} \left[1 + \frac{2\alpha_s(\tilde{\Lambda})}{\pi} \left(\frac{13}{6} + \frac{2\pi^2}{9} - \ln z + \frac{2}{3} K\left(z \frac{T}{m_Q} \right) \right) \right] \right.$$

$$- \langle \bar{q}q \rangle(T) \left[1 + \frac{2\alpha_s(\tilde{\Lambda})}{3\pi} \left(1 - 3 \frac{T}{m_Q} \int_0^\infty dz \, \frac{e^{-z}}{1 + z \frac{T}{m_Q}} \right) - \frac{m_0^2}{4T^2} \left(1 - \frac{2T}{m_Q} \right) \right]$$

$$\left. - \frac{\langle \alpha_s GG \rangle}{12\pi m_Q} + \dots \right\} \, , \tag{2.37}$$

where we have neglected the tiny contribution of the four-quark condensate. To first order in α_s, Eqs. (2.36) and (2.37) are equivalent to (2.7). In the improved sum rule, however, the large logarithms $(\alpha_s \ln m_Q)^n$ and $\alpha_s (\alpha_s \ln m_Q)^n$ are correctly summed to all orders in perturbation theory and factorized into the short distance coefficient $\hat{C}_F(m_Q)$. The leading corrections not taken into account are of order α_s^2 or $\frac{1}{m_Q} (\alpha_s \ln m_Q)^2$, neither one of which becomes large as $m_Q \to \infty$.

In Table 1, we investigate the m_Q-dependence of $G(m_Q)$ and related quantities for $\widetilde{\Lambda} = 1$ GeV and 1.25 GeV. Shown are the values providing best stability only. The optimal value of ω_c is found to slightly increase as the heavy quark mass becomes smaller. This effect stabilizes the mass-dependence of $G(m_Q)$. As a consequence, we find a moderate mass-dependence of this function even in the region of the charm quark. Typically, $G(m_b)$ and $G(m_c)$ are 5–10% and 15–20% smaller than the asymptotic value $G(\infty) = F_{ren}$, respectively. The $1/m_Q$ corrections being of the naively expected order of magnitude ($\sim \Lambda_{QCD}/m_Q$), we conclude that the heavy quark expansion works well for the quantity $G(m_Q)$.

Table 1a: Mass-dependence of the sum rule results for $\widetilde{\Lambda} = 1$ GeV

m_Q [GeV]	$m_c = 1.44$	3.0	$m_b = 4.80$	20.0	∞
m_P [GeV]	$m_D = 1.87$	3.46	$m_B = 5.28$	20.5	∞
ω_c [GeV]	2.4	2.2	2.1	2.0	1.9
s_c [GeV^2]	5.5	15.6	33.1	440	∞
$G(m_Q)$ [GeV$^{3/2}$]	0.286	0.310	0.317	0.335	0.340
$\left(\frac{m_Q}{m_P}\right)^{3/2} G(m_Q)$ [GeV$^{3/2}$]	0.193	0.251	0.275	0.323	0.340
f_P [MeV]	172	180	166	109	0

Table 1b: Mass-dependence of the sum rule results for $\widetilde{\Lambda} = 1.25$ GeV

m_Q [GeV]	$m_c = 1.35$	3.0	$m_b = 4.69$	20.0	∞
m_P [GeV]	$m_D = 1.87$	3.57	$m_B = 5.28$	20.6	∞
ω_c [GeV]	3.0	2.6	2.5	2.4	2.3
s_c [GeV^2]	5.9	16.8	33.7	448	∞
$G(m_Q)$ [GeV$^{3/2}$]	0.330	0.361	0.373	0.404	0.411
$\left(\frac{m_Q}{m_P}\right)^{3/2} G(m_Q)$ [GeV$^{3/2}$]	0.203	0.278	0.312	0.386	0.411
f_P [MeV]	179	196	189	130	0

The situation changes if one considers the quantity $\left(\frac{m_Q}{m_P}\right)^{3/2} G(m_Q)$, which up

to the short-distance correction determines the size of $f_P\sqrt{m_P}$. The additional mass ratio amounts to a further suppression of f_B by $\sim 15\%$ and f_D by $\sim 40\%$. Therefore, the decay constants themselves are subject to very large finite-mass corrections. It is important to remember, however, that the explicit appearance of quark masses is a specific feature of the pseudo scalar decay constants. Such an effect does not occur, $e.g.$, in the case of heavy quark transition form factors. The fact that the decay constants are subject to very large scaling violations should therefore not be considered as an indication of a general failure of the heavy quark expansion for the case of the charm quark.

Using the pole masses as given in Ref. 41, $m_b = 4.67 \pm 0.10$ GeV and $m_c = 1.45 \pm 0.05$ GeV, our final results are $f_D \simeq 170 \pm 30$ MeV and $f_B \simeq 190 \pm 50$ MeV, $i.e.$, the decay constant of the B meson is indeed found to be larger than that of the D meson. The quoted errors mainly reflect the uncertainty in the value of $\widetilde{\Lambda}$ and in the scale used in the next-to-leading logarithmic corrections. Two effects are responsible for the large value of f_B (as compared to the standard sum rule estimate at the end of Sect. 2.1), the most important one being that after renormalization group improvement the radiative corrections have to be evaluated at a low energy scale instead of at the scale of the heavy quark mass. A second increase is due to the different quark masses used and the strong sensitivity of f_P to $\widetilde{\Lambda} \simeq 2(m_P - m_Q)$. The decay constant of the D meson, on the other hand, agrees with our earlier estimate.

3. Sum Rule Calculation of the Isgur-Wise Form Factor

3.1 CURRENT MATRIX ELEMENTS IN THE EFFECTIVE THEORY

We now turn to the study of the form factors describing current-induced transitions between two heavy mesons P_1 and P_2. Specifically, consider transitions between any two of the ground-state pseudo scalar and vetor mesons $D, D^*, \overline{B}$, and $\overline{B}^*$. In general, the corresponding matrix elements involve a large set of $\grave{a}$ $priori$ unrelated form factors. Exploiting the consequences of the spin-flavour symmetry for the heavy quarks, however, Isgur and Wise have shown that in the

limit $m_b, m_c \to \infty$ all these form factors become proportional to a single universal function $\xi(v \cdot v')$ [6]. This so-called Isgur-Wise form factor is independent of the masses of the heavy quarks. It only depends on the velocity-transfer and is normalized at zero recoil.

The relations among form factors that the heavy quark symmetries generate can again be most concisely worked out by using the trace formalism already discussed in Sect. 2.2. In the effective theory, the matrix element describing the transition of a heavy meson P_1 with velocity v to a heavy meson P_2 with velocity v' is given by [29]

$$\langle P_2(v') |\, \overline{h}_{Q_2}(v')\, \Gamma\, h_{Q_1}(v)\, |P_1(v)\rangle = -\xi(v \cdot v', \mu)\, \mathrm{Tr}\left\{\overline{\mathcal{P}}_2(v')\, \Gamma\, \mathcal{P}_1(v)\right\}, \qquad (3.1)$$

where Γ is an arbitrary Dirac matrix, and the scale-dependent function $\xi(v \cdot v', \mu)$ is the analog of the low energy parameter $F(\mu)$ in (2.17). By evaluating the current matrix element for equal mesons one can readily shown that this function satisfies the zero recoil normalization $\xi(1, \mu) = 1$. The spin wave function $\mathcal{P}(v)$ of a pseudo scalar meson has been given in (2.18). For a vector meson with polarization ϵ_μ, one has instead

$$\mathcal{P}(v) = \sqrt{m_P}\, \frac{(1 + \not{v})}{2}\, \not{\epsilon}. \qquad (3.2)$$

The effective heavy quark current in (3.1) is related to the corresponding current $\overline{Q}_2\, \Gamma\, Q_1$ of the full theory by an expansion similar to (2.14). In this case, however, the associated short-distance coefficients C_i are not only functions of the two heavy quark masses and the renormalization scale, but also of the velocity-transfer [29]. These coefficients have recently been calculated in next-to-leading order of renormalization-group improved perturbation theory [43]. As in (2.20), they can be factorized into scale-independent functions $\hat{C}_i(m_{Q_1}, m_{Q_2}, v \cdot v')$ and a μ-dependent factor, which is independent of the heavy quark masses and precisely cancels the scale-dependence of matrix elements in the effective theory. With the help of this factor, we define the renormalized Isgur-Wise function in analogy to

22

the definition of F_{ren} in (2.20). Putting everything together, one obtains for the full QCD matrix element [43]

$$
\begin{aligned}
\langle P_2(v')| \, &\overline{Q}_2 \, \Gamma \, Q_1 \, |P_1(v)\rangle \\
&= -\xi_{ren}(y) \sum_i \hat{C}_i(m_{Q_1}, m_{Q_2}, y) \, \mathrm{Tr}\left\{\overline{\mathcal{P}}_2(v') \, \Gamma_i \, \mathcal{P}_1(v)\right\} + \mathcal{O}\left(\frac{1}{m_{Q_i}}\right),
\end{aligned} \qquad (3.3)
$$

where for abbreviation $y = v \cdot v'$, and Γ_i are Dirac matrices with the same canonical dimension as Γ. For the vector current, e.g., it is convenient to choose $\Gamma_1 = \Gamma = \gamma_\mu, \Gamma_2 = -v_\mu$, and $\Gamma_3 = -v'_\mu$. The renormalized Isgur-Wise form factor is defined as

$$
\xi_{ren}(y) = \left[\alpha_s(\mu)\right]^{-a_L(y)} \left\{1 - \frac{\alpha_s(\mu)}{\pi} \delta_Z(y)\right\} \xi(y, \mu) . \qquad (3.4)
$$

The velocity-dependent anomalous dimension $a_L(y)$ is given by [47, 29]

$$
a_L(y) = \frac{8}{33 - 2n_f} \left[y \, r(y) - 1\right] ; \quad r(y) = \frac{1}{\sqrt{y^2 - 1}} \ln\left(y + \sqrt{y^2 - 1}\right) \qquad (3.5)
$$

with $n_f = 3$ in the low energy theory. The next-to-leading order corretion $\delta_Z(y)$ in (3.4) is a complicated, scheme-dependent function [47, 43]. In the $\overline{\mathrm{MS}}$ renormalization scheme, one has

$$
\delta_Z^{\overline{\mathrm{MS}}}(y) = \frac{8(109 - 5n_f - 9\pi^2)}{729} (y - 1) + \mathcal{O}\left[(y - 1)^2\right] , \qquad (3.6)
$$

which is a sufficient approximation for $y < 1.5$. Notice that both $a_L(y)$ and $\delta_Z(y)$ vanish at $y = 1$. As a consequence, the renormalization prescription (3.4) preserves the normalization of the Isgur-Wise form factor at zero recoil, i.e., $\xi_{ren}(1) = 1$. The renormalized form factor is an observable, universal function of QCD, which contains all long distance dynamics active in the hadronic transition.

The $1/m_Q$ corrections in (3.3) can be systematically classified using the effective field theory approach [48, 49, 35]. In particular, Luke has shown that there

are no leading $1/m_Q$ corrections to (3.3) at zero recoil [48]. This is very different from the case of the meson decay constants, where no such restriction holds. As a consequence, the $1/m_Q$ corrections to $b \to c$ transitions are expected to be small for basically all values of $v \cdot v'$ that are kinematically accessible. This is indeed confirmed by model estimates [35], and the heavy quark symmetries have proved to be a useful tool in the theoretical description of weak decays of heavy mesons [31, 34, 36].

3.2 LAPLACE SUM RULE FOR THE ISGUR-WISE FUNCTION

In order to calculate the universal function $\xi_{ren}(v \cdot v')$ using QCD sum rules, we study the following correlator of currents in the effective theory

$$
i^2 \int d^4x\, d^4y\, e^{i(k' \cdot x - k \cdot y)} \langle 0 | \mathcal{T} \left\{ \mathcal{A}_5^{(v')}(x), \mathcal{J}_\Gamma^{(v,v')}(0), \mathcal{A}_5^{(v)}(y)^\dagger \right\} | 0 \rangle
$$
$$
\equiv \Xi(\omega, \omega', v \cdot v')\, \mathrm{Tr} \left[\frac{\not{v}' + 1}{2} \Gamma \frac{\not{v} + 1}{2} \right] ; \quad \omega^{(')} \equiv 2\, k^{(')} \cdot v^{(')} , \tag{3.7}
$$

where $\mathcal{J}_\Gamma^{(v,v')} = \overline{h}_Q(v') \Gamma\, h_Q(v)$, and as previously $\mathcal{A}_5^{(v)} = \overline{q}\, \gamma_5\, h_Q(v)$. In defining the three-point function Ξ, we have factored out the trace which determines its Lorentz structure in the effective theory. Ignoring QCD corrections, the perturbative contribution to Ξ is obtained by evaluating the triangle diagram shown in Fig. 3a. The result can be written as a double disperion integral

$$
\Xi^{pert}(\omega, \omega', v \cdot v') = \int d\widetilde{\omega}\, d\widetilde{\omega}'\, \frac{\rho(\widetilde{\omega}, \widetilde{\omega}', v \cdot v')}{(\widetilde{\omega} - \omega - i\,\epsilon)(\widetilde{\omega}' - \omega' - i\,\epsilon)} + \text{subtractions.} \tag{3.8}
$$

Setting again $y = v \cdot v'$, the spectral density is given by [36, 38]

$$
\rho(\widetilde{\omega}, \widetilde{\omega}', y) = \frac{3}{16\pi^2}\, \frac{(\widetilde{\omega} + \widetilde{\omega}')}{(y + 1)\sqrt{y^2 - 1}}\, \Theta(\widetilde{\omega})\, \Theta(\widetilde{\omega}')\, \Theta(2y\, \widetilde{\omega}\widetilde{\omega}' - \widetilde{\omega}^2 - \widetilde{\omega}'^2) . \tag{3.9}
$$

For the nonperturbative power corrections to the correlator, we find

$$\Xi^{\langle \bar{q}q \rangle}(\omega, \omega', y) = -\frac{\langle \bar{q}q \rangle}{\omega \omega'} \,,$$

$$\Xi^{\langle GG \rangle}(\omega, \omega', y) = \frac{\langle \alpha_s \, GG \rangle}{12\pi} \, (y-1) \, I_G(\omega, \omega', y) \,, \qquad (3.10)$$

$$\Xi^{\langle \bar{q}\sigma Gq \rangle}(\omega, \omega', y) = \frac{m_0^2 \langle \bar{q}q \rangle}{2\omega\omega'} \left\{ \frac{1}{\omega^2} + \frac{1}{\omega'^2} + \frac{4y-1}{3\omega\omega'} \right\} \,.$$

The diagram involving the quark condensate is shown in Fig. 3b. In contrast to the sum rule for the decay constant f_P, we also find a non-vanishing contribution from the gluon condensate. It arises from the diagram depicted in Fig. 3c (in coordinate gauge) and is proportional to the parameter integral

$$I_G(\omega, \omega', y) = \int\limits_0^\infty du \, \frac{u}{1+u^2+2y\,u} \left\{ \frac{3u(\omega+\omega')}{(\omega+u\omega')^4} - \frac{2(1+u)}{(\omega+u\omega')^3} \right\} \,. \qquad (3.11)$$

The phenomenological side of the sum rule is as usually obtained by saturating the correlator with the lowest intermediate pseudo scalar meson states and approximating the contributions of higher resonances by the perturbative continuum. Using (2.17) and (3.1), this gives

$$\Xi^{phen}(\omega, \omega', y) = \frac{\xi(y, \mu) \, F^2(\mu)}{(\tilde{\Lambda} - \omega - i\,\epsilon)(\tilde{\Lambda} - \omega' - i\,\epsilon)}$$

$$+ \int\limits_{\omega_c}^\infty d\tilde{\omega} \int\limits_{\omega_c}^\infty d\tilde{\omega}' \, \frac{\rho(\tilde{\omega}, \tilde{\omega}', y)}{(\tilde{\omega} - \omega - i\,\epsilon)(\tilde{\omega}' - \omega' - i\,\epsilon)} + \text{subtractions.} \qquad (3.12)$$

To obtain the Laplace sum rule for the Isgur-Wise function, one equates the above expressions after applying the Borel operator with respect to both ω and ω'. In particular, this greatly simplifies the contribution proportional to the gluon condensate. Since the correlator is symmetric in ω and ω', we set the associated

Borel parameters τ and τ' equal,

$$\tau = \tau' \equiv 2T \ . \tag{3.13}$$

The factor 2 is chosen for later convenience. The resulting sum rule reads

$$\xi(y,\mu)\, F^2(\mu)\, e^{-\widetilde{\Lambda}/T} = \frac{3}{8\pi^2} \frac{T^3}{(y+1)\sqrt{y^2-1}}$$
$$\times \int_0^{\omega_c/T} \mathrm{d}z \int_0^{\omega_c/T} \mathrm{d}z' \, \frac{(z+z')}{2} \, e^{-\frac{1}{2}(z+z')} \, \Theta(2y\, zz' - z^2 - z'^2)$$
$$- \langle \bar{q}q \rangle \left\{ 1 - \frac{(2y+1)}{3} \frac{m_0^2}{4T^2} \right\} + \frac{\langle \alpha_s\, GG \rangle}{48\pi\, T} \left(\frac{y-1}{y+1} \right) \ . \tag{3.14}$$

Let us evaluate this equation in the zero recoil limit $y = v \cdot v' \to 1$. Because of the normalization of the Isgur-Wise function, it then becomes a sum rule for $F^2(\mu)$ which must agree with that derived in Sect. 2.2. As $y \to 1$, the Θ-function in the dispersion integral reduces to $\sqrt{y^2-1}\,(z+z')\,\delta(z-z')$, and it is readily seen that, apart from QCD corrections, one indeed recovers (2.29). This is a consequence of a Ward identity that relates the three-point function Ξ to the derivative of the two-point correlator Γ_5 [50]. It was for this reason that we introduced the factor 2 in (3.13). The empirical observation that the Borel parameter of a three-point sum rule should be chosen approximately twice as large as that of the corresponding two-point sum rule has been first made in Ref. 51. In the infinite quark mass limit and for $v \cdot v' = 1$, this relation becomes exact.

It is clear from the above discussion that if one divides the sum rule (3.14) by (2.29), one obtains an expression for the Isgur-Wise function which explicitly obeys the normalization condition $\xi(1,\mu) = 1$, independent of the value of T. The resulting equation is furthermore independent of $\widetilde{\Lambda}$ and $F^2(\mu)$. This is welcome since the variation of the decay constants with $\widetilde{\Lambda}$ was rather strong (see Table 1), and $F^2(\mu)$ was associated with the factor $\left(\frac{m_Q}{m_P} \right)^3$, which induced large $1/m_Q$ corrections. In addition, we shall see in the following section that also the uncomfortably large

QCD corrections to $F^2(\mu)$ do not affect the final expression for the Isgur-Wise function.

Before proceeding, however, it is necessary to modify the sum rule (3.14) in two respects. The first one concerns the simulation of higher resonance contributions. The integration domain for the dispersion integral is the "kite-like" area shown in Fig. 4. The separation between the pole and the continuum contribution appears to be rather crude and to a large extent arbitrary. Performing the integral over the region specified in (3.14), one can shown that the derivative of the Isgur-Wise form factor with respect to $v \cdot v'$ diverges at zero recoil, a result which is certainly unphysical. One should, therefore, change the integration domain. To this end, it is convenient to introduce new variables $x = (z + z')/2$ and $q = z - z'$ and integrate over a symmetric triangle (see Fig. 4), such that the q-integration becomes trivial. Instead of the double integral in (3.14), we then obtain

$$
\int_0^{\omega_c/T} \mathrm{d}z \int_0^{\omega_c/T} \mathrm{d}z' \ldots \rightarrow 4 \left(\frac{y-1}{y+1} \right)^{1/2} I\left(\sigma(y) \frac{\omega_c}{T} \right),
$$

$$
I(z) = \int_0^z \mathrm{d}x\, x^2\, e^{-x} = 2 - (2 + 2z + z^2)\, e^{-z}.
$$

(3.15)

According to Fig. 4, any choice for $\sigma(y)$ in the range

$$
\frac{1}{2} \left(y + 1 - \sqrt{y^2 - 1} \right) < \sigma(y) < 1
$$

(3.16)

is reasonable. Note, however, that $\sigma(1) = 1$ in any case. A finite slope of the Isgur-Wise function is obtained, e.g., for $\sigma(y) = \frac{y+1}{2y}$. The arbitrariness of $\sigma(y)$ yields to a significant uncertainty of the sum rule prediction for the universal form factor, as will be discussed in Sect. 3.3 below. Facing the lack of information on the structure of higher resonance contributions to the spectral function Ξ, this uncertainty is unavoidable.

A second modification is required to improve the large-recoil behaviour of the sum rule (3.14). For large values of $y = v \cdot v'$ (corresponding to large negative values of q^2) the Isgur-Wise function should tend to zero, whereas the nonperturbative contributions in (3.14) stay constant or even increase with y. Hence higher-order power corrections must compensate the contributions of the lowest-dimensional ones. In order to model this cancellation and to simulate the effect of higher-order corrections, one approximates the non-local quark condensate by a Gaussian distribution in Euclidean space-time [52−54]

$$\langle \overline{q}(x)q(0) \rangle \simeq \langle \overline{q}q \rangle \, e^{-x^2/\lambda^2} \; ; \quad \lambda^2 = \frac{16}{m_0^2} \, . \tag{3.17}$$

The damping length λ is chosen to be consistent with the short-distance expansion

$$\langle \overline{q}(x)q(0) \rangle = \langle \overline{q}q \rangle - \frac{x^2}{16} \, g_s \, \langle \overline{q} \, \sigma_{\mu\nu} G_{\mu\nu} \, q \rangle + \ldots \tag{3.18}$$

The standard value $m_0^2 \simeq 0.8 \text{ GeV}^2$ corresponds to $\lambda \simeq 1$ fm. For the simple Gaussian ansatz (3.17), the contribution of the diagram shown in Fig. 3b can be calculated in closed form. Denoting the "double-borelized" correlator by $\widehat{\Xi}$, the result is

$$4T^2 \, \widehat{\Xi}^{\langle \overline{q}q \rangle} = -\langle \overline{q}q \rangle \, \exp\left\{ -\frac{m_0^2}{4T^2} \left(\frac{y+1}{2} \right) \right\} \, . \tag{3.19}$$

In contrast to the old result, the contribution of the non-local condensate becomes exponentially small at large recoil. Comparison with (3.14) shows that part of the contribution of the mixed condensate is taken into account by (3.19). In the following, we shall assume that all nonperturbative contributions have the same exponential damping factor at large values of y. It must be noted that for physical values of $y = v \cdot v'$ relevant for $\overline{B} \to D^{(*)}$ decays, the numerical effect of the exponentiation is less than 3%.

Putting everything together, we obtain the following expression for the universal form factor [36, 38]

$$\xi(y,\mu) = \frac{\frac{3T^3}{8\pi^2}\left(\frac{2}{y+1}\right)^2 I\left(\sigma(y)\frac{\omega_c}{T}\right) + C(T,y)}{\frac{3T^3}{8\pi^2} I\left(\frac{\omega_c}{T}\right) + C(T,1)} , \qquad (3.20)$$

where the condensate contributions are given by (the scale in the quark condensate is chosen for later convenience)

$$C(T,y) = \left\{ -\langle \bar{q}q \rangle(T) \left[1 - \frac{(y-1)}{6}\frac{m_0^2}{4T^2} \right] + \left(\frac{y-1}{y+1}\right)\frac{\langle \alpha_s GG \rangle}{48\pi T} \right\}$$
$$\times \exp\left\{ -\frac{m_0^2}{4T^2}\frac{(y+1)}{2} \right\} . \qquad (3.21)$$

This result explicitly obeys the normalization condition $\xi(1,\mu) = 1$. Since, up to now, QCD corrections have been ignored, our theoretical expression does not yet reproduce the proper scale-dependence of $\xi(y,\mu)$. The calculation of the renormalized form factor defined in (3.4) is subject of the next section.

3.3 Calculation of the Renormalized Isgur-Wise Form Factor

In order to calculate the renormalized Isgur-Wise function, one has to include the QCD corrections to both the numerator and denominator of (3.20). The corrections to the denominator have been calculated in Sect. 2.2 and are given in (2.29). The corrections to the three-point correlator Ξ involve loop integrals in the effective theory with two heavy quarks, which depend on ω, ω', and $v \cdot v'$. As an example, we have calculated the radiative corrections to the contribution involving the quark condensate (see Fig. 3b). After applying the double Borel transformation, we find using the formulas given in the appendix

$$\widehat{\Xi}^{\langle \bar{q}q \rangle} = -\frac{\langle \bar{q}q \rangle(\mu)}{4T^2}\left\{ 1 + \frac{2\alpha_s}{3\pi}\left(1 + 3\delta + 2\left[y\,r(y) - 1\right]\ln\frac{T}{\mu} + c(y) \right) \right\} , \qquad (3.22)$$

where $\delta_{\overline{\text{MS}}} = 2/3$, and $c(y)$ is a scheme-dependent function, which vanishes at zero

recoil. In the $\overline{\text{MS}}$ scheme,

$$
\begin{aligned}
c_{\overline{\text{MS}}}(y) = &\left[y\,r(y) - 1\right]\left[\ln 8(y+1) - 2\gamma_E\right] - (y-1)\,r(y) + 2 \\
&- \frac{y}{\sqrt{y^2 - 1}}\left[L_2(1 - y_-^2) - L_2(1 - y_+^2) + L_2(1 - y_+) - L_2(1 - y_-)\right] \\
= &\frac{4}{3}\left(\ln 4 - \gamma_E - \frac{7}{3}\right)(y - 1) + \mathcal{O}\!\left[(y-1)^2\right],
\end{aligned}
$$

$$\tag{3.23}$$

where $y_\pm = y \pm \sqrt{y^2 - 1}$, $\gamma_E \simeq 0.5772$, and $r(y)$ has been defined in (3.5). The important observations are:

- At zero recoil, the radiative correction is precisely that encountered in (2.29) in the calculation of the correlator Γ_5. The equivalence of the sum rules at zero recoil is preserved by renormalization.

- The logarithmic dependence on μ in (3.22) can be summed to yield

$$
\langle \bar{q}q \rangle(\mu)\left\{1 + \frac{4\alpha_s}{3\pi}\left[y\,r(y) - 1\right]\ln\frac{T}{\mu}\right\} \to \langle \bar{q}q \rangle(T)\left(\frac{\alpha_s(\mu)}{\alpha_s(T)}\right)^{-4/9 + a_L(y)}.
$$

$$\tag{3.24}$$

The factor $\left(\frac{\alpha_s(\mu)}{\alpha_s(T)}\right)^{-4/9}$ also appears in the renormalization-group improved denominator of (3.20) and drops out of the ratio. The remaining μ-dependent anomalous scaling factor is precisely cancelled if one computes the renormalized Isgur-Wise form factor from (3.4).

- The functions $c(y)$ and $\delta_Z(y)$ from (3.4) combine to give a scheme-independent result

$$
\frac{2}{3}\,c(y) - \delta_Z(y) \simeq -1.42\,(y-1) + \mathcal{O}\!\left[(y-1)^2\right].
$$

$$\tag{3.25}$$

As a consequence of the restriction that it must vanish at $y = 1$, this velocity-dependent next-to-leading order correction stays small for all relevant values of $y = v \cdot v'$. We shall neglect it from now on.

The radiative corrections to the triangle diagram of Fig. 3a involve two-loop diagrams and are hard to calculate. However, they must exhibit the same structures

as encountered above. The logarithmic dependence on μ/T must be of the same form as in (3.24), and the correction must reduce to that given in (2.29) at zero recoil. There is thus no need to calculate the two-loop diagrams as long as one neglects next-to-leading order corrections which vanish at $y = 1$. In view of the uncertainties associated with the simulation of higher resonance contributions to the perturbative part of the sum rule, this is a safe approximation. In total, we obtain from (3.4) and (3.14) for the renormalized Isgur-Wise function

$$\xi_{ren}(y) = \left[\alpha_s(T)\right]^{-a_L(y)} \frac{\dfrac{3T^3}{8\pi^2} \left(\dfrac{2}{y+1}\right)^2 \widetilde{I}\!\left(\sigma(y)\,\dfrac{\omega_c}{T}\right) + \eta_{QCD}^{-1}\, C(T,y)}{\dfrac{3T^3}{8\pi^2}\, \widetilde{I}\!\left(\dfrac{\omega_c}{T}\right) + \eta_{QCD}^{-1}\, C(T,1)} \,, \qquad (3.26)$$

where

$$\eta_{QCD} = 1 + \frac{\alpha_s(\widetilde{\Lambda})}{\pi}\left(\frac{11}{3} + \frac{4\pi^2}{9}\right) \simeq 1.8 \qquad (3.27)$$

is essentially the difference of the QCD corrections to the dispersion integral and to the quark condensate, and we have introduced the new function

$$\widetilde{I}(z) = \int\limits_0^z \mathrm{d}x\, x^2\, e^{-x} \left\{1 - \frac{2\alpha_s(\widetilde{\Lambda})}{\pi}\ln x\right\}\,. \qquad (3.28)$$

Our final result (3.26) for the universal form factor has an appealing structure. The normalization condition $\xi_{ren}(1) = 1$ is explicitly fulfilled. In contrast to the sum rule for the decay constant f_P discussed in Sect. 2.2, the large QCD corrections to the perturbative contribution cancel in the ratio, since they are y-independent. The only remnant of this correction is the QCD factror $\eta_{QCD}^{-1} \simeq 0.6$, which leads to a significant suppression of the nonperturbative contributions. This is still a large correction, but it only affects the small power corrections to the leading perturbative contribution, which therefore determines the shape of the universal form factor. The corresponding y-dependence is approximately of the form $\left(\frac{2}{y+1}\right)^2$, corresponding to a double pole at $q^2 = (m_b + m_c)^2$. Corrections to a pure pole behaviour arise, however, from the function $\sigma(y)$.

31

As mentioned previously, the final result for the heavy quark form factor would be unaffected from mass factors of the type $\frac{m_Q}{m_P}$ even if $1/m_Q$ corrections were included. In the framework of QCD sum rules, we therefore do not expect these corrections to be unusually large. This is very different from the situation encountered for f_P.

For the numerical evaluation of (3.26), we vary the continuum threshold ω_c between 1.9 and 2.5 GeV, and the Borel parameter T between 1.0 and 2.5 GeV, corresponding to the range of values providing stability of the two-point sum rule investigated in Sect. 2.2. The radiative corrections in (3.27) and (3.28) are evaluated at the scale $\widetilde{\Lambda} = 1.25$ GeV. For the function $\sigma(y)$ in the dispersion integral we consider the two extreme choices $\sigma_{min}(y) = \frac{1}{2}(y + 1 - \sqrt{y^2 - 1})$ and $\sigma_{max} = 1$ [cf. (3.16)], as well as $\sigma_0(y) = \frac{y+1}{2y}$, which for all $y > 1$ lies in between these limits. The latter ansatz gives results very similar to those obtained evaluating the original form of the dispersion integral in (3.14) but, as discussed earlier, yields a finite slope of the universal form factor at zero recoil. The results are summarized in Fig. 5. The three bands correspond to the three choices for $\sigma(y)$. The width of the bands arises from the variation of the sum rule parameters ω_c and T in the limits specified above. Obviously the dependence on these parameters is very weak. Also a variation of $(\eta_{QCD} - 1)$ by $\pm 50\%$ changes the results by less than 4%, indicating that the QCD corrections are under control. The main uncertainty is due to the arbitrariness of the choice of $\sigma(y)$. We consider the band obtained using $\sigma_0(y)$ as a reasonable estimate, whereas the results obtained using $\sigma_{min}(y)$ and σ_{max} should be regarded as very conservative lower and upper limits, respectively.

Also shown in Fig. 5 are "data" for the universal form factor $\xi_{ren}(v \cdot v')$ that have been extracted from an experimental measurement of the differential branching ratio for the semileptonic decay $\overline{B}^0 \rightarrow D^{*+} \ell \overline{\nu}_\ell$ by the CLEO and ARGUS collaboration [55], accounting for the leading QCD and $1/m_Q$ corrections to the infinite quark mass limit [34]. The normalization of the data corresponds to $|V_{cb}| \tau_{B^0} = 0.044 \times 1.18$ ps. It is seen that the sum rule result obtained using $\sigma_0(y)$ nicely compares to the data. Defining the slope parameter ϱ by $\xi'_{ren}(1) = -\varrho^2$,

we obtain the prediction $\varrho_{s.r.} = 1.13 \pm 0.11$. This is in accordance with previous estimates of this parameter like $\varrho = 1.20 \pm 0.17$ in Ref. 31 and $\varrho = 1.13 \pm 0.23$ in Ref. 34. A simple parametrization of the sum rule result in terms of a pole-type function is

$$\xi_{ren}^{s.r.}(y) = \left(\frac{2}{y+1}\right)^{\beta(y)} \; ; \quad \beta(y) \simeq 2 + \frac{0.6}{y} \; . \tag{3.29}$$

It exhibits dipole behaviour at large recoil.

4. Conclusions

We have presented a consistent framework for the calculation of hadronic form factors of heavy mesons using QCD sum rules in the effective theory for heavy quarks. Since the effective currents have non-vanishing anomalous dimensions, a renormalization of the form factors is required. We have included the complete next-to-leading order renormalization group improvement, thereby summing the large logarithms $(\alpha_s \ln m_Q)^n$ and $\alpha_s(\alpha_s \ln m_Q)^n$ to all orders in perturbation theory. This procedure resolves the scale ambiguity problem associated with the order-α_s corrections. We have applied this technique to derive sum rules for the asymptotic value of the scaled meson decay constant $f_P\sqrt{m_P}$ and of the universal Isgur-Wise form factor, which describes current matrix elements between two heavy mesons in the infinite quark mass limit.

In the case of f_P, the renormalization group improvement turns out to be most important and resolves the discrepancy between previous sum rule and lattice calculations. Against naive expectation, the large radiative correction to the correlator of two axial currents has to be evaluated at a low energy scale rather than at the scale of the heavy quark. While this effect is unimportant for the decay constant of the D meson, it considerably enhances previous estimates of the value of f_B. In the static approximation (*i.e.* neglecting $1/m_b$ corrections), we find $f_B^{stat} \simeq 200 - 300$ MeV in agreement with recent lattice results.

Combining the renormalization-group improved sum rule derived in the effective theory with the standard Laplace sum rule for f_P, we have investigated the

effect of finite heavy quark masses. The appearance of an explicit factor $\left(\frac{m_Q}{m_P}\right)^3$ in front of the sum rule is found to be the origin of unnaturally large finite-mass corrections to the decay constant of the D meson. This effect is, however, specific for decay constants of pseudo scalar particles and should not be considered as an indication for a general break-down of the heavy quark expansion for charm. Taking into account the full dependence on m_Q, we find the physical decay constants $f_D \simeq 170 \pm 30$ MeV and $f_B \simeq 190 \pm 50$ MeV. For large values of m_Q, the decay constants depend rather strongly on the mass difference $\widetilde{\Lambda} \simeq 2(m_P - m_Q)$. The quoted value for f_B refers to $m_b = 4.67 \pm 0.10$ GeV.

The sum rule for the renormalized Isgur-Wise form factor $\xi_{ren}(v \cdot v')$ is obtained from the study of a three-current correlation function in the effective theory. At zero recoil, a Ward identity relates this function to the two-current correlator from which one derives the sum rule for f_P. As a consequence, the Isgur-Wise function can be expressed a the ratio of two sum rules in such a way that its normalization at zero recoil is explicitly obeyed. This ratio is independent of the parameter $\widetilde{\Lambda}$ and, after renormalization group improvement, is not affected by uncomfortably large QCD corrections. The main uncertainty arises from the way in which higher resonance contributions to the perturbative part of the sum rule are approximated. We have given a prescription that ensures a finite slope of the universal form factor at zero recoil, and discussed conservative lower and upper limits for $\xi_{ren}(v \cdot v')$. Our final result compares nicely to data on the form factor extracted from the differential decay rate for $\overline{B} \to D^* \ell \overline{\nu}_\ell$ decays.

APPENDIX

The Feynman rules of the heavy quark effective theory can be readily obtained from the effective Lagrangian derived by Georgi [7]. The momentum of the heavy quark is split into an on-shell and an off-shell piece, $P = m_Q v + k$, where k acts as an infrared cutoff and is of order Λ_{QCD}. In momentum space, the propagator of the heavy quark is then given by $\frac{i}{v \cdot k} \frac{\not{v}+1}{2}$. The heavy quark-gluon coupling is $ig_s v_\mu t_a$. Master integrals for the calculation of one and two-loop integrals involving only a single heavy quark are given in Ref. 12. The master one-loop integral for diagrams involving two heavy quarks with velocities v, v' and off-shell energies $\omega = 2v \cdot k, \omega' = 2v' \cdot k'$ is (in D space-time dimensions)

$$
\begin{aligned}
I_{\alpha\beta\gamma}(\omega, \omega', v \cdot v') &= \int d^D t \left(-\frac{1}{t^2}\right)^\alpha \left(\frac{1}{\omega + 2v \cdot t}\right)^\beta \left(\frac{1}{\omega' + 2v' \cdot t}\right)^\gamma \\
&= i\pi^{D/2} I(\alpha, \beta, \gamma) \int_0^\infty du \, \frac{u^{\gamma-1}}{[\Omega(u)]^{\beta+\gamma}} \left(-\frac{\Omega(u)}{V(u)}\right)^{D-2\alpha}
\end{aligned}
$$

with $(v \cdot v' \equiv \cosh\theta)$

$$
\Omega(u) = \omega + u\omega' \ ,
$$
$$
V(u) = \left(1 + 2u \, v \cdot v' + u^2\right)^{1/2} = \left(u + e^\theta\right)^{1/2}\left(u + e^{-\theta}\right)^{1/2} \ ,
$$

and coefficients

$$
I(\alpha, \beta, \gamma) = \frac{\Gamma(2\alpha + \beta + \gamma - D)\,\Gamma(D/2 - \alpha)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma)} \ .
$$

It is convenient to apply the double Borel transformation with respect to ω and ω' before performing the integral over the Feynman parameter u. Useful formulas are

$$
\widehat{B}_{\tau'}\widehat{B}_\tau \left[-\Omega(u)\right]^{-\alpha} = \frac{1}{\Gamma(\alpha)\,\tau^{\alpha-1}\,\tau'}\,\delta(u - \tau/\tau') \ ,
$$
$$
\widehat{B}_{\tau'}\widehat{B}_\tau \frac{1}{\omega\omega'} \left[-\Omega(u)\right]^{-\alpha} = \frac{1}{\Gamma(\alpha+1)\,\tau\tau'} \left\{\frac{\Theta(\tau/\tau' - u)}{\tau^\alpha} + \frac{\Theta(u - \tau/\tau')}{(u\tau')^\alpha}\right\} \ .
$$

Additional powers of ω' or $\omega = \Omega(u) - u\omega'$ in the numerator can be generated by derivatives with respect to the Feynman parameter u.
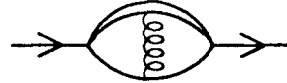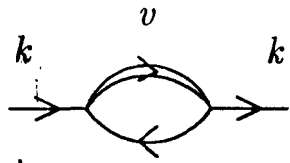
REFERENCES

[1] E. Eichten and F. Feinberg, Phys. Rev. **D23** (1981) 2724;

E. Eichten, Nucl. Phys. **B** (Proc. Suppl.) **4** (1988) 170;

E. Eichten and B. Hill, Phys. Lett. **B234** (1990) 511.

[2] G.P. Lepage and B. Thacker, Nucl. Phys. **B** (Proc. Suppl.) **4** (1988) 119.

[3] B. Grinstein, Nucl. Phys. **B339** (1990) 253.

[4] M.B. Voloshin and M.A. Shifman, Sov. J. Nucl. Phys. **45** (1987) 292.

[5] M.B. Voloshin and M.A. Shifman, Sov. J. Nucl. Phys. **47** (1988) 511.

[6] N. Isgur and M.B. Wise, Phys. Lett. **B232** (1989) 113, Phys. Lett. **B237** (1990) 527, Nucl. Phys. **B348** (1991) 276.

[7] H. Georgi, Phys. Lett. **B240** (1990) 447.

[8] J.G. Körner and G. Thompson, Phys. Lett. **B264** (1991) 185.

[9] T. Mannel, W. Roberts and Z. Ryzak, Harvard preprint HUTP–91/A017 (1991).

[10] H.D. Politzer and M.B. Wise, Phys. Lett. **B206** (1988) 681, Phys. Lett. **B208** (1988) 504.

[11] X. Ji and M.J. Musolf, Phys. Lett. **B257** (1991) 409.

[12] D.L. Broadhurst and A.G. Grozin, Phys. Lett. **B267** (1991) 105.

[13] For a review see: S. Capstick and S. Godfrey, Phys. Rev. **D41** (1990) 2856.

[14] P. Colangelo, G. Nardulli and M. Pietroni, Phys. Rev. **D43** (1991) 3002.

[15] M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Nucl. Phys. **B147** (1979) 385, 448.

[16] C.R. Allton *et al.*, Nucl. Phys. **B349** (1991) 598.

[17] C. Alexandrou *et al.*, Phys. Lett. **B256** (1991) 60.

[18] C. Bernard, J. Labrenz and A. Soni, Brookhaven report BNL–45581 (1990).

[19] For a review see: L. Maiani, Helv. Phys. Acta **64** (1991) 853.

[20] V.A. Novikov *et al.*, Phys. Rev. Lett. **38** (1977) 626.

[21] E.V. Shuryak, Nucl. Phys. **B198** (1982) 83.

[22] V.L. Chernyak, A.R. Zhitnitsky and I.R. Zhitnitsky, Sov. J. Nucl. Phys. **38** (1983) 773.

[23] T.M. Aliev and V.L. Eletskii, Sov. J. Nucl. Phys. **38** (1983) 936.

[24] V.S. Mathur and T. Yamawaki, Phys. Rev. **D29** (1984) 2057.

[25] L.J. Reinders, H. Rubinstein and S. Yazaki, Phys. Rep. **127** (1985) 1.

[26] L.J. Reinders, Phys. Rev. **D38** (1988) 947.

[27] C.A. Dominguez and N. Paver, Phys. Lett. **B197** (1987) 423 [Erratum **B199** (1987) 596], Phys. Lett. **B246** (1990) 493.

[28] S. Narison, Phys. Lett. **B218** (1989) 238.

[29] A.F. Falk, H. Georgi, B. Grinstein and M.B. Wise, Nucl. Phys. **B343** (1990) 1.

[30] J.D. Bjorken, invited talk given at Les Recontre de la Valle d'Aoste La Thuile, Aosta Valley, Italy, March 1990, SLAC report SLAC–PUB–5278 (1990), Lecture at the 1990 SLAC Summer Institute on Particle Physics, SLAC report SLAC–PUB–5389 (1990).

[31] J.L. Rosner, Phys. Rev. **D42** (1990) 3732, see also Chicago preprint EFI–90–80 (1990).

[32] N. Isgur and M.B. Wise, Phys. Rev. **D42** (1990) 2388.

[33] T. Mannel, W. Roberts and Z. Ryzak, Phys. Lett. **B248** (1990) 392, Phys. Lett. **B255** (1991) 593, Phys. Lett. **B259** (1991) 359, Nucl. Phys. **B355** (1991) 38.

[34] M. Neubert, Phys. Lett. **B264** (1991) 455.

[35] M. Neubert and V. Rieckert, Heidelberg preprint HD–THEP–91–6 (1991), submitted to Nucl. Phys. B.

[36] M. Neubert, V. Rieckert, B. Stech and Q.P. Xu, Heidelberg preprint HD–THEP–91–28 (1991).

[37] C.A. Dominguez and N. Paver, Trieste preprint 91–0393 (1991).

[38] A.V. Radyushkin, CEBAF preprint CEBAF–TH–91–09 (1991).

[39] D.J. Broadhurst, Phys. Lett. **B101** (1981) 423;

D.J. Broadhurst and S.C. Generalis, Open University preprint OUT–4102–8/R (1982).

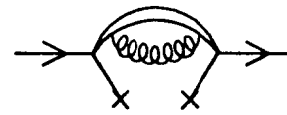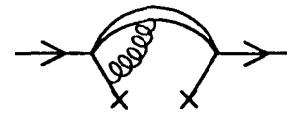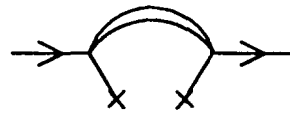[40] J. Gasser and H. Leutwyler, Phys. Rep. **87** (1982) 77.

[41] S. Narison, Phys. Lett. **B197** (1987) 405 and references therein, Phys. Lett. **B216** (1989) 191.

[42] S. Narison, *QCD Spectral Sum Rules*, World Scientific, Singapore, 1989.

[43] M. Neubert, Heidelberg preprint HD THEP 91-30 (1991), submitted to Phys. Rev. D.

[44] M.B. Voloshin and Y.M. Zaitsev, Sov. Phys. Usp. **30** (1987) 553.

[45] Ph. Boucaud, C.L. Lin and O. Pène, Phys. Rev. **D40** (1989) 1529 [Erratum **D41** (1990) 3541].

[46] E. Eichten and B. Hill, Phys. Lett. **B240** (1990) 193.

[47] G.P. Korchemsky and A.V. Radyushkin, Nucl. Phys. **B283** (1987) 342; G.P. Korchemsky, Mod. Phys. Lett. **A4** (1989) 1257.

[48] M.E. Luke, Phys. Lett. **B252** (1990) 447.

[49] A.F. Falk, B. Grinstein and M.E. Luke, Nucl. Phys. **B357** (1991) 185.

[50] V.N. Baier and A.G. Grozin, Z. Phys. **C47** (1990) 669.

[51] B.L. Ioffe and A.V. Smilga, Phys. Lett. **B114** (1982) 353, Nucl. Phys. **B216** (1983) 373.

[52] S.V. Mikhailov and A.V. Radyushkin, Sov. J. Nucl. Phys. **49** (1989) 494.

[53] E.V. Shuryak, Nucl. Phys. **B328** (1989) 85.

[54] P. Ball, V.M. Braun and H.G. Dosch, Heidelberg preprint HD–THEP–91–16 (1991).

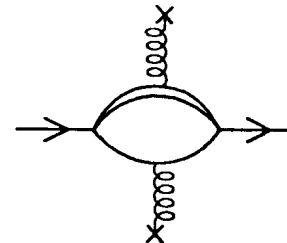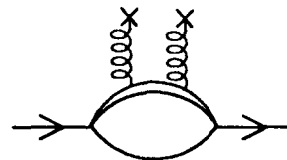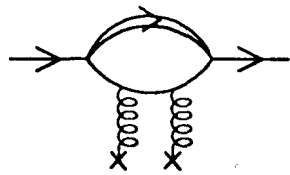[55] D. Bortoletto and S. Stone, Phys. Rev. Lett. **65** (1990) 2951.

FIGURE CAPTIONS

1) Feynman diagrams for the two-current correlator $\Gamma_5(\omega)$: (a) perturbative contributions, (b) power corrections involving the quark condensate, (c) corrections involving the gluon condensate. The heavy quark propagators are represented by double lines

2) Numerical evaluation of the sum rule (2.30) for $\widetilde{\Lambda} = 1.25$ GeV and various values of the threshold parameter ω_c

3) Lowest-order diagrams for the three-current correlator $\Xi(\omega, \omega', v \cdot v')$: (a) perturbative contribution, (b) correction involving the quark condensate, (c) correction involving the gluon condensate. In (c) only the diagram yielding a non-vanishing contribution (in coordinate gauge) is shown

4) Integration domain for the double integral in (3.14). The original "kite-like" region is replaced by a symmetric triangle. The small (shaded) triangle corresponds to the lower bound for $\sigma(y)$ in (3.16), the large (lined) one to $\sigma(y) = 1$

5) Sum rule result for the renormalized Isgur-Wise form factor. The three bands correspond, from top to bottom, to $\sigma(y) = 1, \sigma_0(y)$, and $\sigma_{min}(y)$, respectively. The data points are taken from Ref. 34
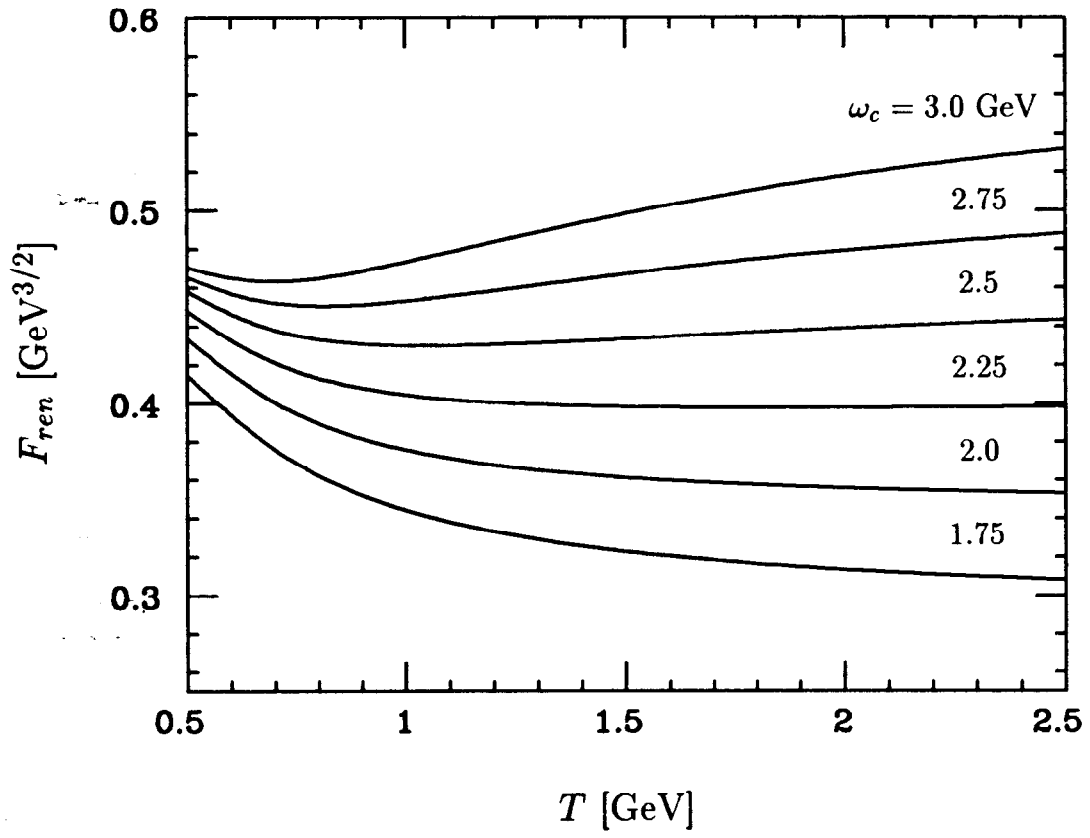
Fig. 1

Fig. 2

Fig 3

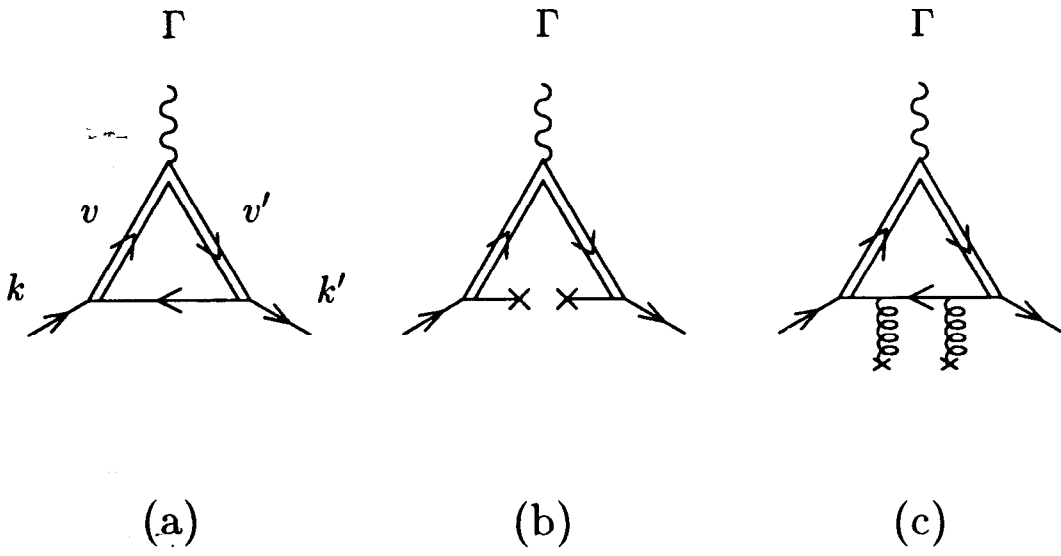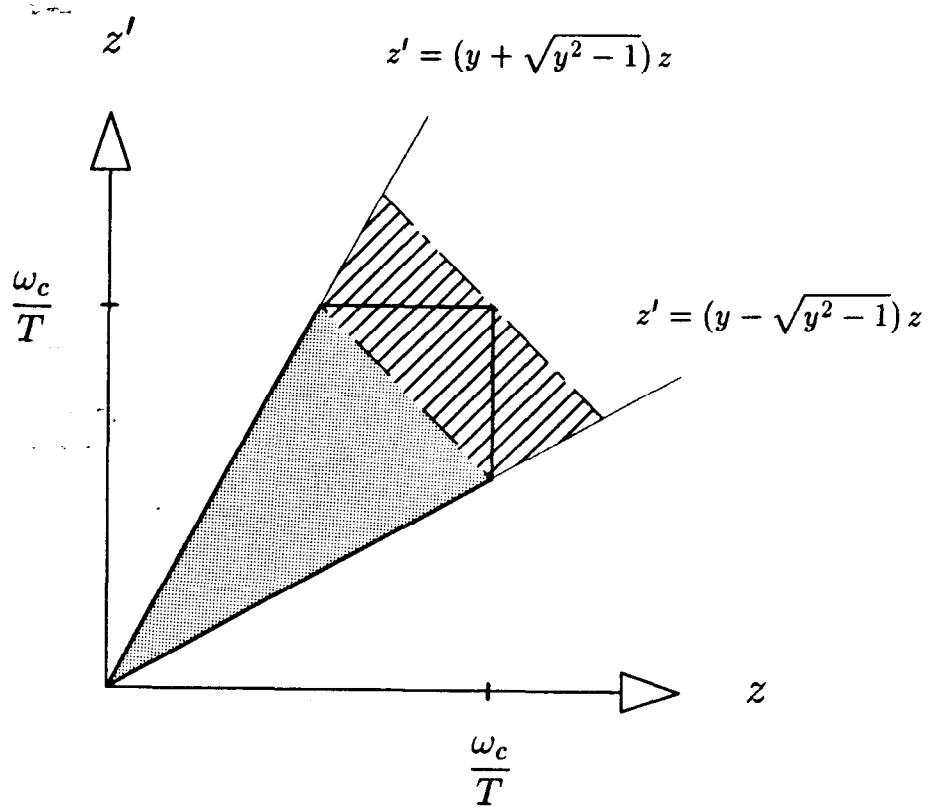$$z' = (y + \sqrt{y^2 - 1})\, z$$

$$z' = (y - \sqrt{y^2 - 1})\, z$$

$\dfrac{\omega_c}{T}$

$\dfrac{\omega_c}{T}$

Fig. 4

Fig. 5