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Tolerances for the Vertical Emittance in Damping Rings*

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Abstract

Future damping rings for linear colliders will need to have very small vertical equilibrium emittances. In the limit of low beam current, the vertical emittance is primarily determined by the vertical dispersion and the betatron coupling. In this paper, the contributions to these effects from random misalignments are calculated and tolerances are derived to limit the vertical emittance with a 95% confidence level.

1. Introduction

Future damping rings for linear colliders will need to achieve very small vertical equilibrium emittances. In the limit of low beam current, the vertical emittance is primarily determined by the vertical dispersion and the betatron coupling. It is standard to estimate tolerances to limit these effects from the expected value of the vertical emittance. But, the emittance due to any given set of errors can deviate substantially from this expected value. Thus, in this paper, we calculate the distribution density of the emittance assuming a gaussian distribution of errors. This will be used to determine the variation of the emittance about the expected value in an ensemble of machines having the same rms alignment tolerances. In particular, we will use the distribution density of the emittance to calculate rms alignment tolerances that will limit the emittance with a 95% confidence.

In the next section, we will describe the beam emittance and then list the effect of the vertical dispersion and the betatron coupling. Since we are considering a weakly coupled machine, our expressions will differ slightly from the more common expressions. Then, in Section 3, we will evaluate the expected value of the emittance due to random errors. Finally, in Section 4, we calculate the distribution density of the value of the emittance and the location of the 95% confidence point.

2. Emittance

A particle beam consists of particles distributed in 6-dimensional phase space. When the beam is uncoupled, the rms vertical emittance is simply given by:

$$\epsilon_y = \sqrt{\langle y^2\rangle\langle y'^2\rangle - \langle yy'\rangle^2} \quad . \tag{1}$$

But, when the beam is coupled, through either vertical dispersion or transverse betatron coupling, the normal modes of oscillation rotate from the horizontal, vertical, and longitudinal planes. In weakly coupled e^+/e^- rings, this coupling has two effects: it increases the projected vertical emittance, the larger horizontal and longitudinal emittances are projected into the vertical phase space, and it couples the "vertical" normal mode emittance to the synchrotron radiation noise, leading to an increase in the normal mode emittance.

The projected emittance depends upon the coupling and can fluctuate from point to point around the ring while the equilibrium normal mode emittance is invariant. In a damping ring, the normal mode emittance is the more relevant quantity since, in theory, the beam can be fully uncoupled after it is extracted from the ring; in this case, the vertical emittance equals "vertical" normal mode emittance. Thus, we will only consider this normal mode emittance, hereafter.

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The vertical dispersion describes coupling between the vertical phase space and the energy deviation. Thus, it directly couples the vertical plane to the energy fluctuations due to the synchrotron radiation. In the limit of weak transverse coupling, the "vertical" normal mode emittance is nearly aligned to the vertical plane and we can neglect the rotation. In this case, the equilibrium "vertical" emittance due to the vertical dispersion is[1]

$$\epsilon_y = \frac{C_q \gamma^2}{\mathcal{J}_y} \frac{\oint |G^3(s)| \mathcal{H}_y(s) ds}{\oint G^2(s) ds} \tag{2}$$

where $\mathcal{H}_y$ is the dispersion invariant:

$$\mathcal{H}_{x,y}(s) = \frac{1}{\beta_{x,y}} (\eta_{x,y}^2 + (\beta_{x,y}\eta'_{x,y} + \alpha_{x,y}\eta_{x,y})^2) \quad , \tag{3}$$

and $C_q = 55\hbar/(32\sqrt{3}mc) = 3.84 \times 10^{-13}$ meter, G is the inverse bending radii of the bend magnets $G = eB_y/p_0$, and $\mathcal{J}_y$ is the vertical damping partition. For a ring in the horizontal plane $\mathcal{J}_y = 1$; in the limit of weak coupling, the change in $\mathcal{J}_y$ due to errors is negligible.

In addition, the betatron coupling couples the "vertical" emittance to the synchrotron radiation noise via the horizontal dispersion. In the limit of weak coupling, *i.e.*, when far from the coupling resonances, this leads to an increase in the "vertical" normal mode emittance that can be expressed:[2]

$$\epsilon_y = \frac{C_q \gamma^2}{16\mathcal{J}_y \oint G^2 ds} \int_0^C ds \mathcal{H}_x |G^3| \left[\sum_{\pm} \frac{|Q_\pm(s)|^2}{\sin^2 \pi\Delta\nu_\pm} + 2\Re \frac{Q_+(s)Q_-(s)}{\sin \pi\Delta\nu_+ \sin \pi\Delta\nu_-} \right] \tag{4}$$

where

$$Q_\pm(s) = \int_s^{s+C} dz g \sqrt{\beta_x \beta_y} e^{i\left[\left(\psi_x(s) \pm \psi_y(s)\right) - \left(\psi_x(z) \pm \psi_y(z)\right) + \pi(\nu_x \pm \nu_y)\right]} \quad . \tag{5}$$

Here, $g = (K_2 y - \widetilde{K_1})$, the sum over $\pm$ denotes a sum over both the $+$ term (sum resonance) and the $-$ term (difference resonance) while $\Delta\nu_+ = \nu_x + \nu_y$ and $\Delta\nu_- = \nu_x - \nu_y$, and the operator $\Re$ yields the real portion of the expression.

At this point, we should note that Eq. (4) differs from other expressions for the emittance due to betatron coupling.[3,4] There are two reasons for this: first, we have considered the case far from the coupling resonances. It is common to calculate the coupling coefficients by considering the fourier components at the sum and difference resonances. This is not valid in our case since we have assumed that one is far from resonance. In this case, the coupling coefficients $Q_\pm(s)$ depend upon s much as the closed orbit does. In fact, the real part of $Q_\pm / \sin \pi\Delta\nu_\pm$ has exactly the same form as the vertical dispersion or the closed orbit but with a phase advance of $\psi_x \pm \psi_y$ instead of ψ_y. Similarly, the imaginary portion of $Q_\pm(s)/\sin \pi\Delta\nu_\pm$ is analogous to $(\beta_y y'_c + \alpha_y y)$. Thus, $|Q_\pm|^2/\sin^2 \pi\Delta\nu_\pm$ is completely analogous to $\mathcal{H}_y$.

Second, we have written the expression Eq. (4) in a form similar to Eq. (2), the emittance due to the vertical dispersion. This explicitly shows that the emittance depends upon the average of the coupling in all of the bending magnets and has implications for correction of the emittance. In theory, one can fully uncouple the beam at a specified location with four independent skew quadrupoles. But, to fully correct the emittance contribution from the betatron coupling in an e^+/e^- machine, one needs to correct the coupling at *every* bending magnet; 2 this is much harder to do!

Finally, we need to examine the independence of the two processes described by Eqs. (2) and (4). The change in the vertical betatron motion due to the emission of a photon can be written:

$$y_\beta = (\eta_y + c\eta_x + c'\eta'_x)\frac{u}{E_0} \qquad y'_\beta = (\eta'_y + d\eta_x + d'\eta'_x)\frac{u}{E_0} \quad , \tag{6}$$

where u/E_0 is the relative energy loss due to the photon and c, c', d, and d' are coupling coefficients. The two processes are independent if the coefficients $\langle c\eta_y\eta_x \rangle$, $\langle c'\eta_y\eta'_x \rangle$, *etc.*, are all equal to zero. Obviously, this will not be true at any one location, but, because η_x is roughly constant while η_y and the coupling terms (due to random errors) will oscillate with periods of roughly ν_y and $\nu_x \pm \nu_y$, the average of the coefficients around the ring will be zero. Thus, it is completely valid to treat the effects independently and the emittance contributions of Eqs. (2) and (4) just add.

3. Expected Values for Random Errors

We can quickly evaluate Eqs. (2) and (4) to calculate the expected value of the emittance due to random (gaussian) coupling errors; the effect of closed orbit errors is more complicated and is discussed in Ref. 2. Assuming random quadrupole rotations and random vertical sextupole misalignments, we find

$$\langle\epsilon_y\rangle = \frac{C_q\gamma^2}{\mathcal{J}_y}\frac{\oint |G^3(s)|ds}{\oint G^2(s)ds}\frac{1}{4\sin^2\pi\nu_y}\left[\sum_{quad}(K_1L)^2 4\langle\Theta^2\rangle\beta_y\eta_x^2 + \sum_{sext}(K_2L)^2\langle y_m^2\rangle\beta_y\eta_x^2\right] \qquad (7)$$

and

$$\langle\epsilon_y\rangle = \frac{\epsilon_x}{4}\frac{\mathcal{J}_x}{\mathcal{J}_y}\frac{(1-\cos 2\pi\nu_x\cos 2\pi\nu_y)}{(\cos 2\pi\nu_x - \cos 2\pi\nu_y)^2}\left[\sum_{quad}(K_1L)^2 4\langle\Theta^2\rangle\beta_x\beta_y + \sum_{sext}(K_2L)^2\langle y_m^2\rangle\beta_x\beta_y\right] \quad . \qquad (8)$$

Notice that in Eq. (8), the sum of $1/\sin^2\pi(\nu_x \pm \nu_y)$ has been written in terms of $\cos 2\pi\nu_x$ and $\cos 2\pi\nu_y$ and we have simplified the expression with the *equilibrium* horizontal emittance. In addition, notice that the cross terms have been not included in Eq. (8); these terms add contributions that are at least $1/2\pi\nu_{x,y}$ smaller than the contributions from the individual resonances and thus they will be neglected in all future calculations.

These expressions for the expected values have been derived repeatedly over the years. Estimates of the dispersion due to random errors dates back to some of the original work in this field and Eq. (8) is the same as that used in Ref. 6.

4. Distributions and Tolerances

In Section 3, we have listed the *expected* values of the vertical emittance. Naively, one could simply invert these equations to solve for alignment tolerances but the emittance due to any specific set of errors could deviate substantially from this expected value. Thus, when specifying tolerances, one should include a "confidence level" (CL); this is the probability that, given the specified tolerances, any specific machine will be less than the design limit. Typically, one wants to specify a large CL so that there is a small probability of exceeding the design limit. In this section, we will calculate the location of the 95% CL as a function of the expected value.

Calculating the CL requires a detailed knowledge of the distribution of the values of the emittance in an ensemble of machines. It is well known that the mean square amplitude of the normalized orbit due to random errors with gaussian distributions should have an exponential distribution function.[7] Since, as noted in Section 2, the equations for the closed orbit are similar to those of the dispersion function and the betatron coupling the same result applies to the amplitudes of $\mathcal{H}_y(s)$ and $|Q_\pm(s)|^2$. But, the vertical emittance is equal to the average of these functions in the bending magnets, and thus, we will consider the effect of averaging $\mathcal{H}_y(s)$ and $|Q_\pm(s)|^2$ over s in the next sections.

4.1 Emittance due to Vertical Dispersion

The actual distribution function for the values of the vertical emittance due to random errors is a very complicated function. Thus, we will derive an approximate form that can be integrated to solve for the location of the 95% CL. We will do this by solving for the moments of the distribution of emittances. The vertical emittance due to dispersion is given by Eq. (2). Assuming identical bending magnets and expressing this in complex notation, we find

$$\epsilon_y = \frac{\mathcal{J}_\epsilon\sigma_\epsilon^2}{4\sin^2\pi\nu_y}\overline{\left|\int_s^{s+C}\sqrt{\beta_y(z)}e^{i\psi_y(z)}F(z)dz\right|^2} \quad , \qquad (9)$$

where σ_ϵ is the rms relative energy spread,[1] F is the driving term for the dispersion function: $F = G_y + K_1 y + 2K_1\Theta\eta_x + K_2 y\eta_x + \cdots$, and the bar denotes the average around the ring.

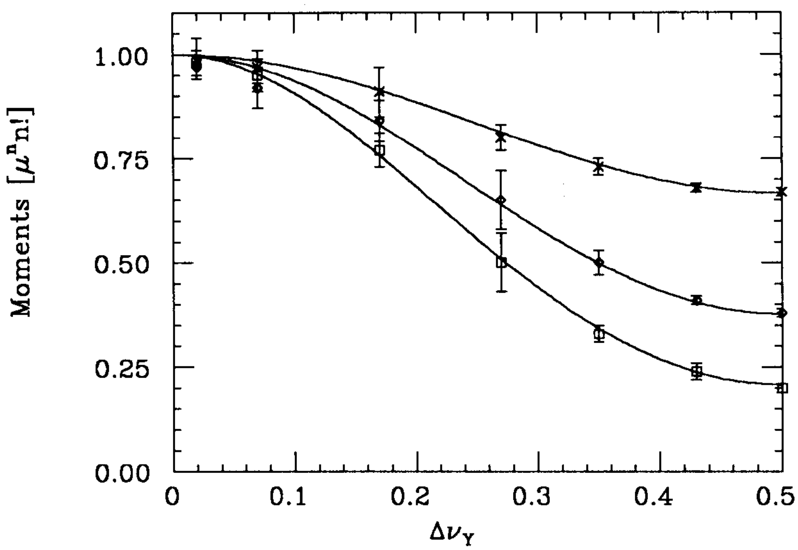

Fig. 1. Second, third, and fourth normalized moments of the distribution for ϵ_y from dispersion due to random errors versus the fractional tune; the second moment is the largest and the fourth moment is the smallest. The points are the results of simulations.

Now, we solve for the moments of Eq. (9) assuming that the random errors F have gaussian distributions. This yields

$$\begin{aligned}
\langle\epsilon_y\rangle &= \mu \\
\langle\epsilon_y^2\rangle &= 2\mu^2\left(1-\frac{1}{3}\sin^2\pi\nu_y\right) \\
\langle\epsilon_y^3\rangle &= 6\mu^3\left(1-\frac{2}{3}\sin^2\pi\nu_y+\frac{2}{45}\sin^4\pi\nu_y\right) \\
\langle\epsilon_y^4\rangle &\approx 24\mu^4\left(1-\sin^2\pi\nu_y+\frac{1}{3}\sin^4\pi\nu_y-\frac{2}{15}\sin^6\pi\nu_y\right) \quad ,
\end{aligned} \tag{10}$$

where μ is the expected value of the emittance due to the dispersion. The first three moments were calculated from Eq. (9), while the fourth moment was fit to data from simulations. These are shown in Fig. 1 where the second, third, and fourth moments, normalized by $n!\mu^n$, are plotted.

Notice that the moments only depend upon the first moment μ and the fractional vertical tune. When the vertical tune is close to an integer, the moments have the form $\mu_n = n!\mu^n$. These are the moments of an exponential distribution as noted in Ref. 7. As the fractional tune increases, the moments decrease, implying that the probability of large emittance values is decreased.

We could attempt to construct a distribution directly from these moments, but, instead, we simply notice that these moments are close to those of a modified χ-squared distribution where the number of degrees of freedom is a function of $\sin^2\pi\nu_y$. In particular, the distribution density can be approximated by

$$g(\epsilon_y) \approx \frac{n}{2\mu}\frac{e^{-\epsilon_y n/2\mu}}{\Gamma(\frac{n}{2})}\left(\frac{\epsilon_y n}{2\mu}\right)^{\frac{n}{2}-1} \quad , \tag{11}$$

where μ is the expected value of the emittance and n is the number of degrees of freedom which depends upon $\sin^2\pi\nu_y$:

$$\frac{n}{2} = \frac{1}{1-\frac{2}{3}\sin^2\pi\nu_y} \quad . \tag{12}$$

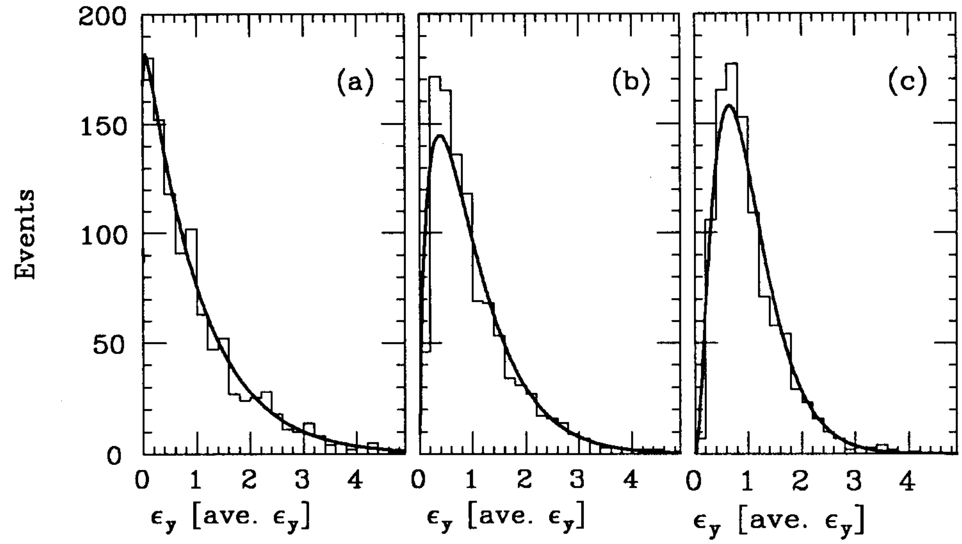

Fig. 2. Events versus ϵ_y due to the vertical dispersion in the NDR lattice. Histograms are calculated from 1000 simulations of random vertical sextupole misalignments with ring tunes of: (a) $\nu_y = 3.07$, (b) $\nu_y = 3.275$, and (c) $\nu_y = 3.43$; the curves are calculated from Eq. (11).

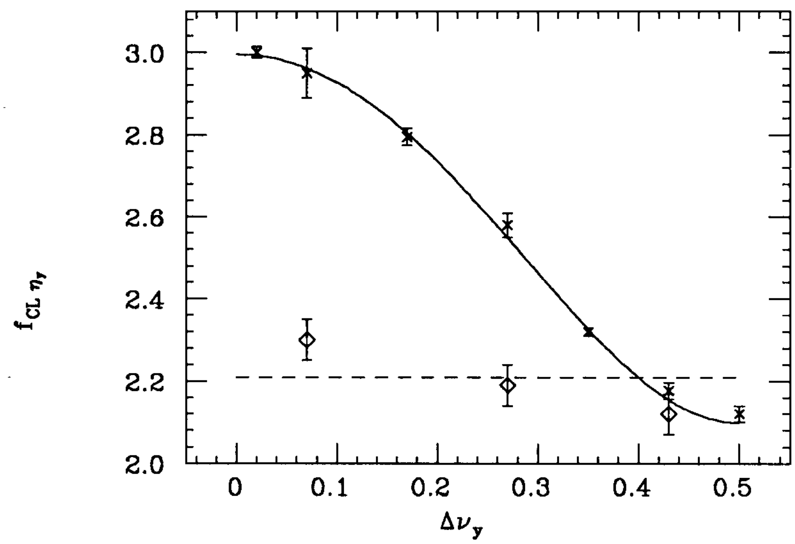

Fig. 3. 95% confidence level for ϵ_y due to dispersion versus the fractional tune; the dashed line is the result after correction - see Ref. 2.

With these definitions, this distribution has the same first and second moments as the value of the vertical emittance, Eq. (10). Furthermore, when the tune is integral, Eq. (11) is correctly equal to the density of an exponential distribution, and, when the fractional tune increases to 0.5, the relative error of the third and fourth moments of Eq. (11) is less than 2% and 8%, respectively.

These distributions are illustrated in Fig. 2 where the distribution density of the vertical emittance, arising from random errors, has been plotted for three different tunes. All of the histograms are generated from 1000 simulations of 150 μm vertical sextupole misalignments in the Stanford Linear Collider North Damping Ring (NDR) while the curves are calculated from Eq. (11). In Fig. 2(a), the tune is $\nu_y = 3.07$, while in Figs. 2(b) and 2(c) the tunes are $\nu_y = 3.275$ and $\nu_y = 3.43$. One can see that there is fairly good agreement between the simulations and the approximation.

At this point, we can calculate the location of the 95% CL for the distributions. This found by integrating the distribution density

$$\int_0^{f_{\rm CL}(\epsilon_y)} g(\epsilon_y)d\epsilon_y = 0.95 \quad , \tag{13}$$

where $f_{\rm CL}$ is the location of the 95% CL in units of the expected vertical emittance. The results are plotted in Fig. 3 as a function of the fractional vertical tune $\Delta\nu_y$. The solid curve is calculated from Eq. (11), while the simulation results are plotted as crosses. One can see that there is very close agreement between the simulation and the approximation results.

Finally, it is important to note the following: first, the curves for $f_{\rm CL}$ are universal. The only dependence comes from the fractional vertical tune. The value of $f_{\rm CL}$ is independent of the type of errors, the lattice type, and the integral portion of the tune. The data in Fig. 3 has been compared with simulations run on the ALS:[9] a Triple Bend Achromat lattice with an integral tune of 8, and a future damping ring design:[8] a FODO lattice with an integral tune of 11. In both cases, excellent agreement was found with the curve in Fig. 3.

Second, our calculations have assumed that the errors are random with gaussian distributions. A more realistic error distribution is a gaussian distribution where the tails are cutoff at $\pm 2\sigma$; it is doubtful that large alignment errors, values that are many σ, would go undetected. This will reduce $f_{\rm CL}$ even further, making Fig. 3 a conservative estimate of $f_{\rm CL}$.

And lastly, notice that there are two advantages of increasing the fractional tune towards a half-integer: the expected value of the emittance decreases, *and* the probability of large deviations above this expected value also decreases.

4.2 Emittance due to Betatron Coupling

Now, we can use the results of the previous section to calculate the distribution of the value of the vertical emittance arising from betatron coupling. Ignoring the cross term in Eq. (4), the emittance is the sum the two quantities $\overline{|Q_\pm|^2}$. As noted earlier, these two values have the same form as $\overline{\mathcal{H}_y}$ and thus they should each have approximate distributions given by Eq. (11). Furthermore, if $\overline{|Q_+|^2}$ and $\overline{|Q_-|^2}$ are mutually independent, then the distribution of their sum is just the convolution of the two individual distributions.

Since we have assumed that the errors have gaussian distributions, Q_+ and Q_- will be independent if[10]

$$\int_s^{s+C} dz\langle q^2(z)\rangle\beta_x\beta_y e^{i2\psi_x} = 0 \qquad\qquad \int_s^{s+C} dz\langle q^2(z)\rangle\beta_x\beta_y e^{i2\psi_y} = 0 \quad . \tag{14}$$

Both of these conditions will be (approximately) satisfied if there are many errors in a betatron period, $N \gg \nu_{x,y}$, and if the tunes are large, $\nu_{x,y} \gg 1$; this is typical of high tune (low emittance) rings.

Convolving the two individual distributions for $\overline{|Q_+|^2}$ and $\overline{|Q_-|^2}$, we find an approximate distribution for the value of the vertical emittance:

$$g(\epsilon_y) \approx \left(\frac{n_+}{2\mu_+}\right)^{\frac{n_+}{2}} \left(\frac{n_-}{2\mu_-}\right)^{\frac{n_-}{2}} \frac{e^{-\epsilon_y n_-/2\mu_-}}{\Gamma(\frac{n_+}{2})\Gamma(\frac{n_-}{2})} \int_0^{\epsilon_y} dx e^{-x\left(\frac{n_+}{2\mu_+}-\frac{n_-}{2\mu_-}\right)} x^{\frac{n_+}{2}-1}(\epsilon_y - x)^{\frac{n_-}{2}-1} \quad , \tag{15}$$

where n_+ and n_- are

$$\frac{n_\pm}{2} = \frac{1}{1-\frac{2}{3}\sin^2\pi\Delta\nu_\pm} \quad , \tag{16}$$

and $\mu_\pm$ are the expected values of the contributions from the sum and difference resonances. Although the integral in Eq. (15) can be expressed in terms of the degenerate hypergeometric

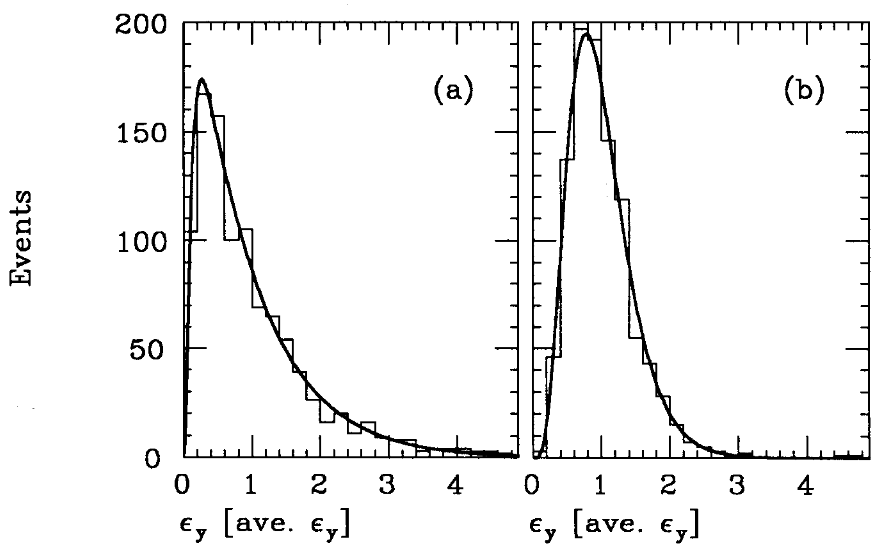

Fig. 4. Events versus ϵ_y due to the linear coupling in the NDR lattice. Histograms are calculated from 1000 simulations of random vertical sextupole misalignments for tunes of: (a) $\Delta\nu_+ = 0.35$ and $\Delta\nu_- = 0.10$, and (b) $\Delta\nu_+ = 0.35$ and $\Delta\nu_- = 0.50$; curves are calculated from Eq. (15).

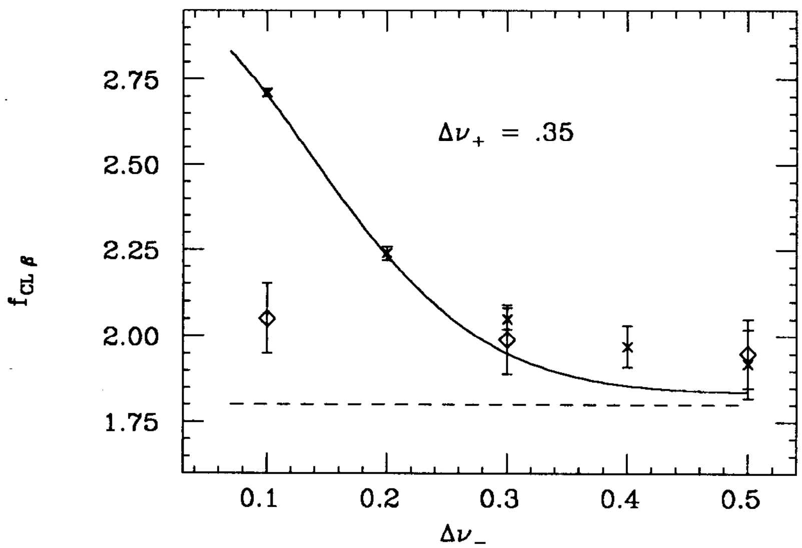

Fig. 5. 95% confidence level for ϵ_y due to betatron coupling versus the distance from the difference coupling resonance for $\Delta\nu_+ = .35$; the dashed line is the result after correction - see Ref. 2.

function, sometimes called Kummer's function, there is no simple evaluation and is thus left as is.

The distribution of the emittances is illustrated in Fig. 4 where the distribution density is plotted for two sets of tunes. In Fig. 4(a) the tunes are $\nu_x = 8.375$ and $\nu_y = 3.275$ so that $\Delta\nu_+ = 0.35$ and $\Delta\nu_- = 0.10$, while in Fig. 4(b) the tunes are $\nu_x = 8.425$ and $\nu_y = 2.925$ so that $\Delta\nu_+ = 0.35$ and $\Delta\nu_- = 0.50$. As before the histograms are found from 1000 simulations of random sextupole errors and the curves are calculated from Eq. (15). Again, there is very good agreement between the simulations and the approximation.

Now, we can calculate the location of the 95% CL which, in the case of the betatron coupling, is a function of both $\Delta\nu_+$ and $\Delta\nu_-$. This is illustrated in Fig. 5 where f_{CL} is plotted as a function of $\Delta\nu_-$, for $\Delta\nu_+ = 0.35$. The crosses are the results of simulations and the solid line is calculated from Eq. (15). One can see that there is very good agreement between the simulated results and the approximation when $\Delta\nu_-$ is small, but there is a significant discrepancy as $\Delta\nu_-$ increases. In particular, as $\Delta\nu_-$ increases toward the half-integer, the value of f_{CL} appears to depend upon the horizontal and vertical tunes in addition to $\Delta\nu_+$ and $\Delta\nu_-$. For example, when the tunes are $\nu_x = 8.575$ and $\nu_y = 3.075$ ($\Delta\nu_+ = 0.35$ and $\Delta\nu_- = 0.50$), f_{CL} equals 2.05. In contrast, when the tunes are $\nu_x = 8.425$ and $\nu_y = 2.925$ ($\Delta\nu_+ = 0.35$ and $\Delta\nu_- = 0.50$), f_{CL} equals 1.86. Thus, there is a substantial difference in f_{CL} even through $\Delta\nu_\pm$ are the same in the two cases. This difference could be explained by the cross term in Eq. (4) which depends upon $\sin 2\pi\nu_x$ along with $\sin \pi\Delta\nu_\pm$.

4.3 Tolerances

Finally, one can use the results of this section to calculate tolerances. We have found that the 95% CL occurs at a value between roughly two and three times the expected emittance. To calculate alignment tolerances with a 95% CL, we simply solve for tolerances that yield expected values that are a factor f_{CL} smaller than the design values.

Actually, the factors f_{CL} were calculated for the dispersive contribution and coupling contribution individually. Strictly, to calculate the f_{CL} for the sum of the two contributions requires convolving both distributions. Fortunately, one usually finds that either the dispersive or the coupling contribution dominates and thus the separate values f_{CL} can be used accurately. However, if both contributions are of equal magnitude, this method will result in tolerances that are slightly too severe.

5.0 Summary

In this paper, we have discussed the dominant low current contributions to the vertical emittance in e^+/e^- storage rings, namely, the vertical dispersion and the betatron coupling. The vertical dispersion and the betatron coupling are generated by both magnet alignment errors and a non-zero beam trajectory; we have only considered the effect of random alignment errors.

We have calculated alignment tolerances to limit the vertical emittance from the vertical dispersion and the betatron coupling. In particular, we have calculated approximate distribution functions for the values of the emittance in an ensemble of machines. From these distributions, we found tolerances that limit the vertical emittance with a 95% confidence level. In general, these are a factor of $\sqrt{2}$ to $\sqrt{3}$ more severe than tolerances simply calculated from the expected values of the emittance and beam size.

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