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# Hadrons of Arbitrary Spin in the Heavy Quark Effective Theory*

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## ABSTRACT

The symmetries of the heavy quark effective theory are used to identify the reduced set of form factors which describe the weak decays of heavy hadrons of arbitrary spin.

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# 1. Introduction

It has recently been emphasized [1, 2] that hadronic systems containing a single heavy quark ($m \gg \Lambda_{\rm QCD}$) admit additional symmetries not present in the full QCD lagrangian. In particular, such a system is most conveniently viewed as a freely propagating point-like color source (the heavy quark), dressed by strongly interacting "brown muck" bearing appropriate color, flavor, baryon number, energy, angular momentum and parity to make up the observed physical state. All interesting properties of the system are properties of the brown muck and its confining interaction with the source of color. Since an infinitely massive heavy quark does not recoil from the emission and absorption of soft ($E \approx \Lambda_{\rm QCD}$) gluons, and since magnetic interactions of such a quark fall off as $1/m$ and are hence negligible, neither its mass (*i.e.*, flavor) nor its spin affect the state of the light degrees of freedom. This results in a remarkable simplification of the description of transitions in which a hadron containing a heavy quark, with velocity $v^\mu$, decays into another hadron containing a heavy quark of a different flavor. To the heavy quark, this looks like a free decay (up to pertubative QCD corrections), in which the light dressing plays no role. The brown muck, on the other hand, knows only that its point-like source of color is now recoiling at a new velocity $v'^\mu$, and it must reassemble itself about it in some configuration. This reassembly will involve nonperturbative strong interactions in a horrible and incalculable way (in particular it may produce a multi-body final state), but the only property of the heavy quarks on which it will depend is the recoil velocity of the decay product.

The decay of the heavy quark occurs through the action of some external current and is calculable in perturbation theory. The transition of the brown muck to a boosted state, involving as it does low energy strong interactions, is incalculable and must be parameterized, but the result is independent of the mass and spin of the initial and final heavy quarks. Essentially, the matrix elements of a heavy quark current between hadronic states may be factorized schematically into heavy and light matrix elements as follows:

$$\begin{aligned}\langle \Psi'(v') \big| J(q) \big| \Psi(v) \rangle &= \\ &\langle\, Q'(v'), \pm\tfrac{1}{2}\, | J(q) |\, Q(v), \pm\tfrac{1}{2}\,\rangle \times \langle\, \text{light}, v', j', m_j' \,|\, \text{light}, v, j, m_j \,\rangle.\end{aligned} \tag{1.1}$$

The velocity of the outgoing light subsystem is the same as the velocity of the final heavy quark; hence the light matrix element depends on the change in heavy quark velocity $v'^\mu - v^\mu$ induced by the current $J = \overline{Q'}\Gamma Q$ but not on its Lorentz structure. We will neglect possible corrections to this form from the emission of hard gluons.

As written, eq. (1.1) is not directly applicable to physical processes, because in general physical heavy hadrons will be linear combinations of the two heavy quark spin states

$$|\,Q(v), -\tfrac{1}{2}\,\rangle, \quad |\,Q(v), \tfrac{1}{2}\,\rangle,$$

and the $2j+1$ light spin states

$$|\,\text{light}, v, j, -j\,\rangle, \ldots, |\,\text{light}, v, j, j\,\rangle,$$

so as to form eigenstates of total angular momentum. Hence eq. (1.1) can be used to extract relations between hadronic decays at the price of keeping track of some Clebsch-Gordan coefficients. It is straightforward, if tedious, to organize these coefficients for specific values of the brown muck angular momentum $j$. Such direct analysis has yielded relations between form factors arising in the decays of $j = \frac{1}{2}$ heavy mesons [1,3], and their excited $j = \frac{3}{2}$ states [4]. The heavy baryons ($j = 0, 1$) have been treated as well [3,5]; however, the complexity of such calculations grows significantly with increasing $j$. The same results have also been obtained through a consideration of the transformation properties of heavy hadrons under the heavy quark spin operators, which in the heavy quark limit are well-defined and independent of the heavy quark mass $[6-10]$. The method is to construct heavy current matrix elements between physical states in such a way that the invariance of $\langle\,\text{light}, v', j', m'_j\,|\,\text{light}, v, j, m_j\,\rangle$ under heavy quark spatial rotations is manifest. The form factor relations then emerge directly.

In this paper we will rederive this latter formalism, in such a way that its extension to brown muck with $j > 1$ is obvious. We will then use this result to explore some properties of transitions for general $j$, including to verify rules, first derived by Politzer [2], for counting the number of independent form factors in a given transition.

## 2. Brown Muck With Integral $j$

There are basic features of the formalism which will differ between the two cases of light dressing with integral and half-integral total spin $j$. Since these cases correspond in turn to even and odd fermion number, there are no transitions between the subclasses and they may be treated separately. It will be most convenient to consider first the case of integral light angular momentum, which may be thought of for concreteness as the case of the baryons and their excited states. For the cases $j = 0$ and $j = 1$ these results have previously been obtained by Georgi [8] and by Mannel, Roberts and Ryzak [9].

## 2.1 Representations of States

The baryons are built out of a heavy and a light component, each of which has a well-defined transformation under the Lorentz group. The heavy component is a single spinor $u_h$, satisfying the subsidiary condition

$$\not{v} u_h = u_h, \tag{2.1}$$

where $v^\mu$ is the velocity of the baryon (and of its heavy point-like constituent). This condition, in the $m \to \infty$ limit, ensures that the quark has positive energy and no lower components in the rest frame. (Heavy antiquarks would satisfy $\not{v} v_h = -v_h$.) The light constituent is an object of integral spin $j$, and is represented by a totally symmetric tensor of the form $A^{\mu_1 \cdots \mu_j}$, subject to the constraints of transversality and tracelessness:

$$v_{\mu_1} A^{\mu_1 \cdots \mu_j} = 0, \qquad A^\mu{}_\mu{}^{\mu_3 \cdots \mu_j} = 0. \tag{2.2}$$

These may then be combined into a single object representing the composite state,

$$\psi^{\mu_1 \cdots \mu_j} = A^{\mu_1 \cdots \mu_j} u_h. \tag{2.3}$$

This object has well-defined behavior under Lorentz transformations $\Lambda$,

$$\psi^{\mu_1 \cdots \mu_j} \to \Lambda^{\mu_1}{}_{\nu_1} \cdots \Lambda^{\mu_j}{}_{\nu_j} D(\Lambda) \psi^{\nu_1 \cdots \nu_j}, \tag{2.4}$$

as well as separately under heavy quark spin rotations,

$$\psi^{\mu_1 \cdots \mu_j} \to D(\widetilde{\Lambda}) \psi^{\mu_1 \cdots \mu_j}, \tag{2.5}$$

where $D(\Lambda) = e^{-\frac{i}{4}\sigma_{\mu\nu}S^{\mu\nu}}$ is the usual spinor representation, and in eq. (2.5) $\widetilde{\Lambda}$ is restricted to spatial rotations and refers only to the heavy quark spinor. However, for $j > 0$, $\psi^{\mu_1 \cdots \mu_j}$ does not transform *irreducibly* under the Lorentz group; instead it is a linear combination of an object with spin $j + \frac{1}{2}$ and one with spin $j - \frac{1}{2}$:

$$\psi^{\mu_1 \cdots \mu_j} = \psi^{\mu_1 \cdots \mu_j}_{j+1/2} + \psi^{\mu_1 \cdots \mu_j}_{j-1/2}. \tag{2.6}$$

These will correspond to a pair of physical states which differ only in the orientation of the heavy quark spin to the spin of the brown muck. They are degenerate in the heavy quark limit, as the chromomagnetic interaction vanishes as $m \to \infty$. We will now identify these two components for a few low values of $j$ before writing down the general expressions.

The $j = 0$ case is trivial, since in this case the heavy quark carries all of the angular momentum of the baryon. So we just have

$$\psi = u_h. \tag{2.7}$$

There is no decomposition to be done.

The case $j = 1$ is more interesting. Here

$$\psi^\mu = A^\mu u_h \tag{2.8}$$

must be decomposed into spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ states. We accomplish this by recalling the properties of a spin-$\frac{3}{2}$ Rarita-Schwinger vector-spinor $R^\mu$:

$$\not{v} R^\mu = R^\mu, \qquad v_\mu R^\mu = 0, \qquad \gamma_\mu R^\mu = 0. \tag{2.9}$$

If we define

$$\begin{aligned} \psi^\mu_{3/2} &= \left[\delta^\mu_\nu - \tfrac{1}{3}(\gamma^\mu + v^\mu)\gamma_\nu\right] \psi^\nu, \\ \psi^\mu_{1/2} &= \tfrac{1}{3}(\gamma^\mu + v^\mu)\gamma_\nu \psi^\nu, \end{aligned} \tag{2.10}$$

then it is straightforward to verify that $\psi^\mu_{3/2}$ satisfies the conditions in eq. (2.9) and is indeed a spin-$\frac{3}{2}$ object.* That $\psi^\mu_{1/2}$ has spin $\frac{1}{2}$ is most easily seen by rewriting it in the form

$$\psi^\mu_{1/2} = \tfrac{1}{\sqrt{3}}(\gamma^\mu + v^\mu)\gamma^5 \widetilde{\psi}, \tag{2.11}$$

where

$$\widetilde{\psi} = \tfrac{1}{\sqrt{3}}\gamma^5 \gamma_\mu \psi^\mu_{1/2} \tag{2.12}$$

is a positive energy spinor moving at velocity $v^\mu$ ($\not{v}\widetilde{\psi} = \widetilde{\psi}$) and provides the most convenient description of the physical state. Spin sums for these objects can be derived directly from the known relations for $A^\mu$ and $u_h$,

$$\sum_{i=1,2} u_h^{(i)} \overline{u}_h^{(i)} = \frac{1 + \not{v}}{2} \equiv \Lambda_+,$$

$$\sum_{i=1,2,3} A_\mu^{*(i)} A_\nu^{(i)} = -g_{\mu\nu} + v_\mu v_\nu.$$

* An equivalent representation of the spin-$\frac{3}{2}$ field is provided (up to normalization) by the antisymmetric tensor $P_{\mu\nu} = i\epsilon_{\mu\nu\alpha\beta} v^\alpha \psi^\beta_{3/2}$. See ref. [9].

We obtain

$$\sum_{i=1,\ldots,4} \psi^\mu_{3/2}\overline{\psi}^\nu_{3/2} = \Lambda_+ \left[-g^{\mu\nu} + v^\mu v^\nu + \tfrac{1}{3}(\gamma^\mu - v^\nu)(\gamma^\nu + v^\nu)\right],$$

$$\sum_{i=1,2} \psi^\mu_{1/2}\overline{\psi}^\nu_{1/2} = -\tfrac{1}{3}\Lambda_+(\gamma^\mu - v^\mu)(\gamma^\nu + v^\nu). \tag{2.13}$$

The normalizations also check:

$$-g_{\mu\nu}\,\mathrm{Tr}\sum_{i=1,\ldots,4} \psi^\mu_{3/2}\overline{\psi}^\nu_{3/2} = 4,$$

$$-g_{\mu\nu}\,\mathrm{Tr}\sum_{i=1,2} \psi^\mu_{1/2}\overline{\psi}^\nu_{1/2} = 2, \tag{2.14}$$

accounting for a total of $4+2 = 3\times 2$ independent states. Of course these properties of spin-$\frac{3}{2}$ Rarita-Schwinger objects are all well known [11]; we have rederived and checked them here to give the reader confidence in the generalization to brown muck of higher spin $j$.

We turn now to the case $j = 2$. Here the baryon is of the form

$$\psi^{\mu\nu} = A^{\mu\nu} u_h,$$

which must be decomposed into degenerate spin-$\frac{5}{2}$ and spin-$\frac{3}{2}$ states. Here and later we will need the conditions for a spin-$(n+\frac{1}{2})$ Rarita-Schwinger tensor-spinor $R^{\mu_1\cdots\mu_n}$ (totally symmetric and traceless) [11]:

$$\not{v} R^{\mu_1\cdots\mu_n} = R^{\mu_1\cdots\mu_n}, \qquad v_{\mu_1}R^{\mu_1\cdots\mu_n} = 0, \qquad \gamma_{\mu_1}R^{\mu_1\cdots\mu_n} = 0. \tag{2.15}$$

Defining

$$\psi^{\mu\nu}_{5/2} = \left[\delta^\mu_\alpha\delta^\nu_\beta - \tfrac{1}{5}(\gamma^\mu + v^\mu)\gamma_\alpha\delta^\nu_\beta - \tfrac{1}{5}\delta^\mu_\alpha(\gamma^\nu + v^\nu)\gamma_\beta\right]\psi^{\alpha\beta},$$

$$\psi^{\mu\nu}_{3/2} = \left[\tfrac{1}{5}(\gamma^\mu + v^\mu)\gamma_\alpha\delta^\nu_\beta + \tfrac{1}{5}\delta^\mu_\alpha(\gamma^\nu + v^\nu)\gamma_\beta\right]\psi^{\alpha\beta}, \tag{2.16}$$

we check directly that $\psi^{\mu\nu}_{5/2}$ satisfies the conditions (2.15) and is indeed spin-$\frac{5}{2}$. In turn, $\psi^{\mu\nu}_{3/2}$ may be seen to be spin-$\frac{3}{2}$ by writing it in the form

$$\psi^{\mu\nu}_{3/2} = \tfrac{1}{\sqrt{10}}(\gamma^\mu + v^\mu)\gamma^5\widetilde{\psi}^\nu + \tfrac{1}{\sqrt{10}}(\gamma^\nu + v^\nu)\gamma^5\widetilde{\psi}^\mu, \tag{2.17}$$

where

$$\widetilde{\psi}^\mu_{3/2} = \sqrt{\tfrac{2}{5}}\gamma^5\gamma_\nu\psi^{\mu\nu}_{3/2} \tag{2.18}$$

satisfies the conditions (2.9). Again, this is the most useful form.

It is now straightforward to extend these results to general $j$. The composite of a spin-$\frac{1}{2}$ heavy quark and spin-$j$ brown muck decomposes into degenerate spin-$(j \pm \frac{1}{2})$ states as follows:

$$\begin{aligned} \psi^{\mu_1\cdots\mu_j}_{j+1/2} &= \psi^{\mu_1\cdots\mu_j} - \psi^{\mu_1\cdots\mu_j}_{j-1/2}, \\ \psi^{\mu_1\cdots\mu_j}_{j-1/2} &= \tfrac{1}{2j+1}\big[(\gamma^{\mu_1}+v^{\mu_1})\gamma_{\nu_1}\delta^{\mu_2}_{\nu_2}\cdots\delta^{\mu_j}_{\nu_j} \\ &\qquad + \cdots + \delta^{\mu_1}_{\nu_1}\cdots\delta^{\mu_{j-1}}_{\nu_{j-1}}(\gamma^{\mu_j}+v^{\mu_j})\gamma_{\nu_j}\big]\psi^{\nu_1\cdots\nu_j}. \end{aligned} \tag{2.19}$$

Furthermore, we can reduce $\psi^{\mu_1\cdots\mu_j}_{j-1/2}$ to an object with one fewer index by writing

$$\begin{aligned} \psi^{\mu_1\cdots\mu_j}_{j-1/2} = \sqrt{\tfrac{1}{j(2j+1)}}\big[&(\gamma^{\mu_1}+v^{\mu_1})\gamma^5\widetilde{\psi}^{\mu_2\cdots\mu_j} + (\gamma^{\mu_2}+v^{\mu_2})\gamma^5\widetilde{\psi}^{\mu_1\mu_3\cdots\mu_j} \\ &+ \cdots + (\gamma^{\mu_j}+v^{\mu_j})\gamma^5\widetilde{\psi}^{\mu_1\cdots\mu_{j-1}}\big], \end{aligned} \tag{2.20}$$

where

$$\widetilde{\psi}^{\mu_1\cdots\mu_{j-1}} = \sqrt{\tfrac{j}{2j+1}}\gamma^5\gamma_{\mu_j}\psi^{\mu_1\cdots\mu_j}_{j-1/2}. \tag{2.21}$$

## 2.2 Matrix Elements

We may now use these representations to calculate matrix elements of heavy quark currents between baryon states, making use of the factorization property (1.1). Recall that we may calculate the heavy quark transition in perturbation theory, while the best we can do for the light transition is to absorb our ignorance of strong dynamics into Lorentz-invariant parameters in a covariant form factor decomposition. Yet we will see that analyzing the light transition after making use of eq. (1.1), rather than decomposing the baryon matrix elements directly, results in a tremendous reduction in the number of incalculable form factors which we must introduce.

We will first consider the light transitions in isolation, although we know that in fact the "states" we are using are actually linear combinations of physical states. Replacing the light states by their tensor representations, we have

$$_\beta\langle\,\mathrm{light}, v', j', m'_j\,|\,\mathrm{light}, v, j, m_j\,\rangle_\alpha = A'^{\nu_1\cdots\nu_{j'}}A^{\mu_1\cdots\mu_j}\zeta^{\alpha\beta}_{\nu_1\cdots\nu_{j'};\mu_1\cdots\mu_j}. \tag{2.22}$$

Here $\alpha$ and $\beta$ represent all other quantum numbers associated with the light states (such as mass and flavor), and the quantity $\zeta^{\alpha\beta}_{\nu_1\cdots}$ is a function of the Lorentz invariant $v\cdot v'$. From here on we shall suppress the indices $\alpha$ and $\beta$, but *one must not forget that they are there.* There are many distinct light states with the same total angular momentum; the replacement in a matrix element of one such state with another introduces an entirely new set of form factors.

The quantity $\zeta_{\nu_1\cdots;\mu_1\cdots}$ represents the amplitude for a light state with given spin quantum numbers and moving at velocity $v^\mu$ to make a transition to a state with a possibly different spin and moving at velocity $v'^\mu$. It absorbs all of our ignorance about the details of the strong interactions. Given the properties (2.2) of $A^{\mu_1\cdots\mu_j}$ and $A'^{\nu_1\cdots\nu_{j'}}$, the most general form for $\zeta_{\nu_1\cdots;\mu_1\cdots}$ is, taking $j'-j\geq 0$ (without loss of generality) and $w^\mu = v'^\mu - v^\mu$,

$$\begin{aligned}\zeta_{\nu_1\cdots\nu_{j'};\mu_1\cdots\mu_j} = (-1)^j w_{\nu_{j+1}}\cdots w_{\nu_{j'}}\big[&C_0^{(j',j)}(v\cdot v')g_{\nu_1\mu_1}\cdots g_{\nu_j\mu_j}\\ &+C_1^{(j',j)}(v\cdot v')w_{\nu_1}w_{\mu_1}g_{\nu_2\mu_2}\cdots g_{\nu_j\mu_j}\\ &+\cdots+C_j^{(j',j)}(v\cdot v')w_{\nu_1}w_{\mu_1}\cdots w_{\nu_j}w_{\mu_j}\big].\end{aligned} \tag{2.23}$$

Note that because of the transversality of $A'^{\nu_1\cdots}$ and $A^{\mu_1\cdots}$, the factors $w_{\nu_i}w_{\mu_i}$ reduce to the less apparently symmetric form $v'_{\nu_i}v_{\mu_i}$. The factor $(-1)^j$ is inserted because the indices on the space-like tensors $A'^{\nu_1\cdots}$ and $A^{\mu_1\cdots}$ should be contracted with $-g_{\nu_i\mu_i}$. This formula identifying the independent form factors becomes more transparent if we take a few specific cases:

(i) $j=0,\ j'=2$:

$$\zeta_{\nu_1\nu_2} = C_0^{(2,0)}(v\cdot v')w_{\nu_1}w_{\nu_2}; \tag{2.24}$$

(ii) $j=1,\ j'=1$:

$$\zeta_{\nu;\mu} = -C_0^{(1,1)}(v\cdot v')g_{\nu\mu} - C_1^{(1,1)}(v\cdot v')w_\nu w_\mu; \tag{2.25}$$

(iii) $j=2,\ j'=2$:

$$\begin{aligned}\zeta_{\nu_1\nu_2;\mu_1\mu_2} = {}&C_0^{(2,2)}(v\cdot v')g_{\nu_1\mu_1}g_{\nu_2\mu_2} + C_1^{(2,2)}(v\cdot v')g_{\nu_1\mu_1}w_{\nu_2}w_{\mu_2}\\ &+C_2^{(2,2)}(v\cdot v')w_{\nu_1}w_{\mu_1}w_{\nu_2}w_{\mu_2}.\end{aligned} \tag{2.26}$$

We can now calculate the matrix elements of a heavy quark current $J(q) = \overline{Q'}\Gamma Q$ between dressed "states" as follows:

$$\langle\Psi'(v')|\,J(q)\,|\Psi(v)\rangle = -\sqrt{4mm'}A'^{\nu_1\cdots\nu_{j'}}\overline{u}'_h\Gamma u_h A^{\mu_1\cdots\mu_j}\zeta_{\nu_1\cdots\nu_{j'};\mu_1\cdots\mu_j}. \tag{2.27}$$

The factor $-\sqrt{4mm'}$ arises due to the non-relativistic normalization of the heavy states which we are using [1]. However, since we would like to have physical baryons as the states $|\Psi(v)\rangle$, we must perform the decomposition (2.6); then eq. (2.27) becomes

$$-\sqrt{4mm'}\,\overline{\Psi}'^{\nu_1\cdots\nu_{j'}}_{j'\pm1/2}\ \Gamma\ \psi^{\mu_1\cdots\mu_j}_{j\pm1/2}\ \zeta_{\nu_1\cdots\nu_{j'};\mu_1\cdots\mu_j}. \tag{2.28}$$

Note that *four* classes of decays are described by the same set of form factors. For example, if we identify the degenerate heavy baryons $\Sigma_h$ and $\Sigma_h^*$ with the composite of a heavy quark and the lowest energy $j = 1$ light state, all of the form factors arising in the decays $\Sigma_h \to \Sigma_{h'}$, $\Sigma_h \to \Sigma^*_{h'}$, $\Sigma^*_h \to \Sigma_{h'}$ and $\Sigma^*_h \to \Sigma^*_{h'}$ are related to the two quantities $C_0^{(1,1)}(v \cdot v')$ and $C_1^{(1,1)}(v \cdot v')$ defined in eq. (2.25).

The non-recoil limit, $v' = v$, is particularly simple from the point of view of the brown muck. If the heavy transition is from one immovable color source to another at the same velocity, nothing whatsoever happens to the light dressing. The amplitude vanishes for non-trivial transitions of the light degrees of freedom. Indeed, we see that as $w \to 0$ only the $C_0^{(j',j)}$ form factors contribute, and then only for $j' = j$. In this case eq. (2.22) for $m'_j = m_j$ reduces to the normalization of the light state, yielding the condition

$$C_0^{(j,j)}(v \cdot v' = 1) = \begin{cases} 1 & \text{for } \alpha = \beta; \\ 0 & \text{for } \alpha \neq \beta. \end{cases} \tag{2.29}$$

Here we have assumed implicitly that $C_p^{(j',j)}(v \cdot v')$ has no pole as $w^2 \to 0$, or at least no pole as strong as $w^{-2p}$. Of course we see no obvious mechanism for producing such a pole, for example, no Goldstone boson to which the transition could couple as in the derivation of the Goldberger-Treiman relation. However we can also turn the argument of the previous paragraph around. Since the simultaneous $m \to \infty$, $v = v'$ limit is one in which absolutely nothing happens to the light degrees of freedom, the orthogonality of the basis of light states *requires* that only a form factor proportional to $g_{\nu_1\mu_1} \cdots g_{\nu_j\mu_j}$ can be nonvanishing. Hence such poles in $C_p^{(j',j)}$ are excluded, except insofar as they can be reabsorbed into a finite contribution to $C_0^{(j,j)}$, in which case they have already been accounted for.

## 3. Brown Muck With Half-integral $j$

We turn now to the case of light constituents with half-integral angular momentum. The physical arguments are the same as with $j$ an integer, while the formalism is nominally more complicated to develop, so we will focus on the formalism. In particular, the two cases of orbital angular momentum $\ell = j \pm \frac{1}{2}$ will have to be treated distinctly.

### 3.1 REPRESENTATIONS OF STATES

Let us consider the construction for $j = \frac{1}{2}$. Once again, the heavy component of the hadron is a positive energy quark $u_h$ satisfying $\not{v} u_h = u_h$, but now the representation of the light component will depend on whether $\ell = 0$ or $\ell = 1$. Note that the energy splitting between states of different $\ell$ does *not* vanish in the heavy quark limit. We will first take the case $\ell = 0$. Here the brown muck transforms under the Lorentz group simply as an antiquark $\overline{v}_\ell$ satisfying

$$\overline{v}_\ell \not{v} = -\overline{v}_\ell. \tag{3.1}$$

(The subscript $\ell$ on $\overline{v}_\ell$ here just denotes "light".) The composite, which we may write

$$u_h \overline{v}_\ell,$$

is a linear combination of objects with total angular momentum $j \pm \frac{1}{2} = (0, 1)$. It is easiest to identify these in the rest frame, where the spin operator takes the simple form

$$S^i = \frac{1}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}.$$

It is also convenient to work with the rest frame spinor basis

$$u_h^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u_h^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_\ell^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_\ell^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3.2}$$

Then, using $\boldsymbol{S}(u_h \overline{v}_\ell) = (\boldsymbol{S} u_h)\overline{v}_\ell - u_h(\overline{v}_\ell \boldsymbol{S})$, we find

$$\begin{aligned} \boldsymbol{S}\left[u_h^{(1)}\overline{v}_\ell^{(1)} + u_h^{(2)}\overline{v}_\ell^{(2)}\right] &= 0, \\ S^3\left[u_h^{(1)}\overline{v}_\ell^{(1)} - u_h^{(2)}\overline{v}_\ell^{(2)}\right] &= 0, \\ S^3\left[u_h^{(1)}\overline{v}_\ell^{(2)}\right] &= u_h^{(1)}\overline{v}_\ell^{(2)}, \\ S^3\left[u_h^{(2)}\overline{v}_\ell^{(1)}\right] &= -u_h^{(2)}\overline{v}_\ell^{(1)}. \end{aligned} \tag{3.3}$$

From the basis (3.2) we calculate (note the lowered spatial indices)

$$
\begin{aligned}
\frac{1}{\sqrt{2}}\left(u_h^{(1)}\overline{v}_\ell^{(1)} + u_h^{(2)}\overline{v}_\ell^{(2)}\right) &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{I} \\ 0 & 0 \end{pmatrix}, \\
\frac{1}{\sqrt{2}}\left(u_h^{(1)}\overline{v}_\ell^{(1)} - u_h^{(2)}\overline{v}_\ell^{(2)}\right) &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sigma_3 \\ 0 & 0 \end{pmatrix}, \\
u_h^{(1)}\overline{v}_\ell^{(2)} &= \frac{1}{2}\begin{pmatrix} 0 & \sigma_1 + \mathrm{i}\sigma_2 \\ 0 & 0 \end{pmatrix}, \\
u_h^{(2)}\overline{v}_\ell^{(1)} &= \frac{1}{2}\begin{pmatrix} 0 & \sigma_1 - \mathrm{i}\sigma_2 \\ 0 & 0 \end{pmatrix},
\end{aligned}
\tag{3.4}
$$

leading us to the Lorentz covariant identification of pseudoscalar ($P$) and vector meson ($V$) states:

$$
P = -\tfrac{1}{\sqrt{2}}\Lambda_+\gamma^5, \qquad V(\eta^\mu) = \tfrac{1}{\sqrt{2}}\Lambda_+\eta^\mu\gamma_\mu, \tag{3.5}
$$

where $\eta^\mu$ is the polarization of the vector meson ($\eta^\mu v_\mu = 0$). The transformation of these states under the Lorentz group is given by

$$
\begin{aligned}
P &\to D(\Lambda)PD^\dagger(\Lambda), \\
V(\eta^\mu) &\to D(\Lambda)V(\Lambda^\mu{}_\nu\eta^\nu)D^\dagger(\Lambda),
\end{aligned}
\tag{3.6}
$$

while under spatial rotations $\widetilde{\Lambda}$ of the heavy quark

$$
\begin{aligned}
P &\to D(\widetilde{\Lambda})P, \\
V(\eta^\mu) &\to D(\widetilde{\Lambda})V(\eta^\mu).
\end{aligned}
\tag{3.7}
$$

For $j = \frac{1}{2}$ but $\ell = 1$, the light degrees of freedom are the spin-$\frac{1}{2}$ combination of a vector and a spinor. Using the negative energy analogue of eqs. (2.10)–(2.12), we write

$$
\begin{aligned}
R_\ell^\mu &= \tfrac{1}{3}(\gamma^\mu - v^\mu)\gamma_\nu A^\nu v_\ell \\
&\equiv \tfrac{1}{\sqrt{3}}(\gamma^\mu - v^\mu)\gamma^5\widetilde{v}_\ell.
\end{aligned}
\tag{3.8}
$$

Decomposing $u_h\overline{\widetilde{v}}_\ell$ as in eq. (3.4), we obtain (up to irrelevant signs) the pair of states

$$\tfrac{1}{\sqrt{2}}\Lambda_+\tfrac{1}{\sqrt{3}}(\gamma^\mu - v^\mu) \qquad \text{and} \qquad \tfrac{1}{\sqrt{2}}\Lambda_+\gamma^5\eta^\nu\gamma_\nu\tfrac{1}{\sqrt{3}}(\gamma^\mu - v^\mu). \tag{3.9}$$

It is convenient to eliminate the spurious index by contracting on the right with $\frac{1}{\sqrt{3}}\gamma_\mu$, after which we have the scalar ($S$) and pseudovector ($B$) states:

$$S = \tfrac{1}{\sqrt{2}}\Lambda_+, \qquad\qquad B(\eta^\mu) = \tfrac{1}{\sqrt{2}}\Lambda_+\gamma^5\eta^\mu\gamma_\mu. \tag{3.10}$$

These should be compared to eq. (3.5).

For half-integral $j > 1$, the light antispinor becomes a generalized Rarita-Schwinger tensor-antispinor $\overline{R}_\ell^{\mu_1\cdots\mu_k}(\gamma^5)$, where $k = j - \frac{1}{2}$ and the $\gamma^5$ is present if $j = \ell - \frac{1}{2}$ (after removing the spurious index). In either case $\overline{R}_\ell^{\mu_1\cdots\mu_k}$ satisfies

$$\overline{R}_\ell^{\mu_1\cdots\mu_k}\not{v} = -\overline{R}_\ell^{\mu_1\cdots\mu_k} \tag{3.11}$$

as well as the other conditions listed in and above eq. (2.15) (note that Dirac matrices now all act on the *right*). Then the object which must be decomposed into its spin-($j \pm \frac{1}{2}$) pieces is $u_h\overline{R}_\ell^{\mu_1\cdots\mu_k}(\gamma^5)$, which transforms in the obvious way under $\Lambda$ and $\widetilde{\Lambda}$. Let us first work out the case $j = \frac{3}{2}$ and $\ell = 1$. Recalling eq. (3.8), we write

$$R_\ell^\mu = \left[\delta_\nu^\mu - \tfrac{1}{3}(\gamma^\mu - v^\mu)\gamma_\nu\right] A^\nu v_\ell \tag{3.12}$$

and use eq. (3.5) to decompose the $u_h\overline{v}_\ell$ part of $u_h\overline{R}_\ell^\mu$. A straightforward rearrangement of the Dirac matrices yields the spin-2 object

$$V^\mu(\eta_+^{\mu\nu}) = \tfrac{1}{\sqrt{2}}\Lambda_+\eta_+^{\mu\nu}\gamma_\nu, \tag{3.13}$$

where $\eta_+^{\mu\nu} = \eta_+^{\nu\mu}$, $\eta_{+\mu}^{\mu} = 0$, and $v_\mu\eta_+^{\mu\nu} = 0$, and the spin-1 object

$$P^\mu(\eta_-^\nu) = \sqrt{\tfrac{3}{4}}\Lambda_+\gamma^5\eta_-^\nu\left[\delta_\nu^\mu - \tfrac{1}{3}\gamma_\nu(\gamma^\mu - v^\mu)\right], \tag{3.14}$$

where $\eta_-^\mu v_\mu = 0$. (We derive $\eta_+^{\mu\nu}$ entirely from the $V$ part of eq. (3.5), while $\eta_-^\mu$ is a linear combination of both terms. This is as we would expect, since after the identification (3.5) we have either a spin-0 object ($P$) or a spin-1 object ($V$) to combine with the spin-1 vector $A^\nu$; $\eta_+^{\mu\nu}$ must be the symmetric combination of the vectors $V$ and $A^\nu$.) The $j = \frac{3}{2}$ states with $\ell = 2$ are identical to $V^\mu$ and $P^\mu$, but with opposite parity:

$$\begin{aligned} B^\mu(\eta_+^{\mu\nu}) &= \tfrac{1}{\sqrt{2}}\Lambda_+\gamma^5\eta_+^{\mu\nu}\gamma_\nu, \\ S^\mu(\eta_-^\nu) &= \sqrt{\tfrac{3}{4}}\Lambda_+\eta_-^\nu\left[\delta_\nu^\mu - \tfrac{1}{3}\gamma_\nu(\gamma^\mu + v^\mu)\right]. \end{aligned} \tag{3.15}$$

The generalization to arbitrary half-integral $j$ is now straightforward. We construct objects of spin $j \pm \frac{1}{2}$ as follows, for $j = \ell + \frac{1}{2}$:

$$\begin{aligned} V^{\mu_1\cdots\mu_k} &= \tfrac{1}{\sqrt{2}}\Lambda_+\eta_+^{\mu_1\cdots\mu_{k+1}}\gamma_{\mu_{k+1}}, \\ P^{\mu_1\cdots\mu_k} &= \sqrt{\tfrac{2k+1}{2k+2}}\Lambda_+\gamma^5\eta_-^{\nu_1\cdots\nu_k}\left[\delta_{\nu_1}^{\mu_1}\cdots\delta_{\nu_k}^{\mu_k} - \tfrac{1}{2j+1}\gamma_{\nu_1}(\gamma^{\mu_1} - v^{\mu_1})\delta_{\nu_2}^{\mu_2}\cdots\delta_{\nu_k}^{\mu_k}\right. \\ &\qquad \left. - \cdots - \tfrac{1}{2j+1}\delta_{\nu_1}^{\mu_1}\cdots\delta_{\nu_{k-1}}^{\mu_{k-1}}\gamma_{\nu_k}(\gamma^{\mu_k} - v^{\mu_k})\right] \end{aligned} \tag{3.16}$$

Here $\eta_\pm^{\mu_1\cdots}$ are symmetric, traceless and transverse to $v^\mu$. We obtain the coefficients in eqs. (3.14) and (3.16) by requiring that $V^{\mu_1\cdots\mu_k}$ and $P^{\mu_1\cdots\mu_k}$ be normalized correctly in terms of $\eta_\pm^{\mu_1\cdots}$. The $j = \ell - \frac{1}{2}$ states $B^{\mu_1\cdots\mu_k}$ and $S^{\mu_1\cdots\mu_k}$ are their counterparts of opposite parity, as in eq. (3.15). Recall that in the heavy quark limit the states $P$ and $V$ are degenerate, as are the states $B$ and $S$, but the pair $(P, V)$ is split from $(S, B)$.

## 3.2 Matrix Elements

We can now use these representations to construct matrix elements of heavy quark currents between meson states, in exact analogy to the case of baryons. Because the brown muck now carries a spinor index, the light matrix element takes the form

$$_\beta\langle\, \text{light}, v', j', m_j' \,|\, \text{light}, v, j, m_j \,\rangle_\alpha = \text{Tr}\left[R'^{\nu_1\cdots\nu_{k'}}\overline{R}^{\mu_1\cdots\mu_k}\xi^{\alpha\beta}_{\nu_1\cdots\nu_{k'};\mu_1\cdots\mu_k}\right]. \tag{3.17}$$

From here on, as with the baryons, we will suppress (but not forget) the indices $\alpha$ and $\beta$. In principle $\xi_{\nu_1\cdots;\mu_1\cdots}$ has Dirac structure, and it could include terms proportional to $\not{v}$. However in the trace $\not{v}$ reduces to zero (for $(P, V) \to (P, V)$ and $(S, B) \to (S, B)$ matrix elements) or to $\pm 2$ (for $(P, V) \to (S, B)$ matrix elements), so such terms are redundant. Since there is no other vector in the problem, and terms with $\gamma^5$ are excluded by parity conservation of the strong interactions, $\xi_{\nu_1\cdots;\mu_1\cdots}$ has no nontrivial Dirac structure after all. Hence the formula (2.23) may be extended to half-integral $(j', j)$. Again, we write out a few examples to make

things clear:
(i) $j = \frac{1}{2}$, $j' = \frac{1}{2}$:

$$\xi = C_0^{(\frac{1}{2},\frac{1}{2})}(v \cdot v'). \tag{3.18}$$

This is the famous function originally identified by Isgur and Wise [1].
(ii) $j = \frac{1}{2}$, $j' = \frac{3}{2}$:

$$\xi_\nu = C_0^{(\frac{3}{2},\frac{1}{2})}(v \cdot v') w_\nu; \tag{3.19}$$

(iii) $j = \frac{3}{2}$, $j' = \frac{3}{2}$:

$$\xi_{\nu;\mu} = -C_0^{(\frac{3}{2},\frac{3}{2})}(v \cdot v') g_{\nu\mu} - C_1^{(\frac{3}{2},\frac{3}{2})}(v \cdot v') w_\nu w_\mu. \tag{3.20}$$

The suppression of the upper indices on $\xi^{\alpha\beta}_{\nu_1\cdots;\mu_1\cdots}$ should not obscure the fact that there are an entirely separate sets of form factors $C_i^{(j'j)}$ for $(P,V) \to (P,V)$, $(B,S) \to (B,S)$ and $(P,V) \to (B,S)$ transitions.

The matrix element of a heavy quark current $J(q) = \overline{Q'}\Gamma Q$ between physical meson states is then given by

$$\langle \Psi'(v') | \, J(q) \, | \Psi(v) \rangle = -\sqrt{4mm'} \, \mathrm{Tr} \left[ \overline{M'}^{\nu_1 \cdots \nu_{k'}} \; \Gamma \; M^{\mu_1 \cdots \mu_k} \right] \xi_{\nu_1 \cdots \nu_{k'};\mu_1 \cdots \mu_k}, \tag{3.21}$$

where $M^{\alpha\cdots} = P^{\alpha\cdots}$, $V^{\alpha\cdots}$, $S^{\alpha\cdots}$ or $B^{\alpha\cdots}$. For example, this reduces in the $j = j' = \frac{1}{2}$, $\ell = 0$ case to the familiar result [6] [7] for heavy pseudoscalar and vector meson decays,

$$\langle \Psi'(v') | \, \overline{Q'}\Gamma Q \, | \Psi(v) \rangle = -\sqrt{4mm'} \xi(v \cdot v') \, \mathrm{Tr} \, M' \Gamma M. \tag{3.22}$$

We can also reproduce compactly the results of ref. [4] for the matrix elements of excited meson states. Finally, we recover the extension to half-integral $j$ of the normalization condition (2.29), $C_0^{(j,j)}(1) = 1$ for $\alpha = \beta$.

# 4. Other Applications

## 4.1 Counting Form Factors

We may now consider the question of how many independent form factors appear for given values of $j$ and $j'$. This question was first addressed by Politzer, in ref. [2]. His method is to go to the "brick wall" frame ($\boldsymbol{v}' = -\boldsymbol{v}$) and quantize the angular momentum of the brown muck about the spatial axis defined by $\boldsymbol{v}$. In this frame the action of parity is particularly simple, since here $w^\mu$ has no time-component and simply changes sign. Thus the form factor $\zeta_{\nu_1\cdots\nu_{j'};\mu_1\cdots\mu_j}$ is multiplied by $(-1)^{(j'-j)}$, as is $\xi_{\nu_1\cdots\nu_{j'};\mu_1\cdots\mu_j}$. Assuming the transition $A^{\mu_1\cdots} \to A'^{\nu_1\cdots}$ involves no change in intrinsic parity, the light matrix element (2.22) picks up the same factor (otherwise it picks up an additional minus sign). The rotational symmetry about $\boldsymbol{v}$ of the light system then yields three rules:

1. Angular momentum about this axis is conserved, so $m_j = m_j'$.
2. Amplitudes for $m_j$ are equal (up to a phase) to those for $-m_j$.
3. Amplitudes for $m_j = 0$ vanish if $j' - j$ is not an even integer.

In the absence of the third, the first two rules would imply that there are $j+1$ independent form factors for $j \le j'$ integral, and $j + \frac{1}{2}$ for $j \le j'$ half-integral. The same result would follow immediately from the form of eq. (2.23). However, we must now account for the additional restriction implied by Rule 3, which is *not* implied by our results so far. Imposing it by explicitly inserting the $m_j = 0$ states into eqs. (2.22) and (2.23), we then derive the following condition on the form factors $C_i^{(j',j)}$ for $j$ integral and $j' - j \ge 0$ odd:

$$f(j)C_0^{(j',j)} + f(j-1)w^2 C_1^{(j',j)} + \cdots + f(0)w^{2j} C_j^{(j',j)} = 0, \tag{4.1}$$

where $f(0) = 1$ and $f(k) = (2k-1)!!/k!$ for $k \ge 1$. In this case there are only $j$ independent form factors. For example, we find that $j = 1$, $j' = 2$ transitions are governed by a single form factor,

$$\xi_{\nu_1,\nu_2;\mu_1} = C_1^{(2,1)} w_{\nu_2} \left[ -w^2 g_{\nu_1\mu_1} + w_{\nu_1} w_{\mu_1} \right], \tag{4.2}$$

and that, for $j'$ odd and $j = 0$,

$$\xi_{\nu_1\cdots\nu_{j'}}(v \cdot v') = 0. \tag{4.3}$$

### 4.2 Sum Rules for General Spin

If the picture of heavy hadrons which we have been using is correct, in particular the factorization property (1.1), then it should be possible to prove sum rules for these "spectator" decays. This has been done for $(j,j') = (0,0)$ and $(\frac{1}{2},\frac{1}{2})$ by Bjorken, Dunietz and Taron [12]. They show that the squared matrix element for the decay of one heavy quark to another, summed over final spin states,

$$\sum_{s=\pm 1/2} \left\langle Q(v), \pm\tfrac{1}{2}\right| J^{\dagger}(q) \left|Q'(v'), s\right\rangle \left\langle Q'(v'), s\right| J(q) \left|Q(v), \pm\tfrac{1}{2}\right\rangle , \tag{4.4}$$

is equal to the same quantity with heavy hadrons replacing heavy quarks, summed over all possible final hadron states of velocity $v'^{\mu}$. This justifies the idea that it is consistent to think of the decay of the heavy quark and the rearrangement of the light degrees of freedom about the decay product as independent processes.

The machinery is now clearly in place to extend their proof to arbitrary $j$ and $j'$. However the task is probably more tedious than enlightening, especially as we consider the result to be intuitively compelling. We will restrict ourselves here to commenting on one aspect of these sum rules, namely their form as $v' \to v$, in an expansion in $(v \cdot v' - 1)$. (See refs. [4,12,13] for more detailed discussion of the cases $j = 0, \frac{1}{2}$.) At zeroth order, of course, only the form factor $C_0^{(j,j)}$ (for $\alpha = \beta$) contributes to the squared matrix element. At linear order, there are also positive definite contributions proportional to $(v \cdot v' - 1)|C_0^{(j\pm 1,j)}|^2$. (Note that nonresonant final states with more than one particle may be included here.) Canceling the quantity (4.4) from both sides of the sum rule, one obtains in this limit an expression of the form

$$1 = h_j(v \cdot v')|C_0^{(j,j)}(v \cdot v')|^2 + \cdots , \tag{4.5}$$

where the elided terms are all nonnegative or vanish at least as $(v \cdot v' - 1)^2$. The factor $h_j(v \cdot v')$ comes from taking the product of a polarization state of spin $j$ with its Lorentz boosted counterpart. We find $h_0(v \cdot v') = 1$, while for $j = \frac{1}{2}$ we have

$$h_{1/2}(v \cdot v') = \tfrac{1}{2}(1 + v \cdot v'). \tag{4.6}$$

The positive derivative of $h_{1/2}(v \cdot v')$ at $v \cdot v' = 1$, together with the relation (4.5), yields an essentially *kinematical* restriction on the slope of the Isgur-Wise function at the endpoint. Defining

$$\xi(v \cdot v') = C_0^{(\frac{1}{2},\frac{1}{2})}(v \cdot v') = 1 - \rho^2(v \cdot v' - 1) + \cdots , \tag{4.7}$$

one finds $\rho \geq \frac{1}{2}$. That no such restriction arises for $j = 0$ had led to the speculation [4] that this suppression was associated with the zitterbewegung of the brown muck. If this were true, we would expect to find this effect only for half-integral $j$. However, for $j$ an integer we can now use the tensor representations to calculate explicitly the nontrivial condition $\rho^2 \geq j^2/(2j-1)$ for $j \geq 0$. Hence the suppression cannot have this particular origin.

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