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Linear Instability of Non-vacuum Spacetimes*

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ABSTRACT

We study the problem of linear instability in non-vacuum spacetimes. For vacuum spacetimes linear instability occurs when the spacetime has Killing vectors. In the non-vacuum case, one must prescribe how the sources are to vary. For one natural choice, we show that the signal for instability is the existence of Integral Constraint Vector fields. These vector fields lead, as in the vacuum case, to nonlinear constraints on the first order perturbations to the metric and momentum. For other choices for variations of the sources, we show how to modify the definition of Integral Constraint Vectors appropriately. Since Robertson-Walker spacetimes have Integral Constraint Vectors our results may have cosmological applications.

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1. INTRODUCTION

This paper will deal with the linear stability of solutions to the Einstein equation

$$G_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{1}$$

A solution to the vacuum equations ($T_{\mu\nu} = 0$) is said to be linearization stable [1] if every solution to the linearized equations about that solution is tangent to a curve of exact solutions. For vacuum spacetimes with compact spatial surfaces it was shown by Fisher and Marsden [1] and Moncrief [2] that a necessary and sufficient condition for the linear stability of an exact solution is the absence of Killing vectors. For spacetimes with Killing vectors Moncrief [3] showed that additional constraints on the linearized solutions arise at second order in perturbation theory, as a requirement for the solubility of the second order equations. These constraints require that a certain charge, which is given by the integral of a quantity quadratic in the first order fields, must vanish. There is one such constraint for each Killing vector. It may happen that a solution to the linear equations fails one or more of these nonlinear tests. Brill and Deser [4] gave a striking example of linear instability in a vacuum spacetime having the flat 3-torus for spatial hypersurfaces. They found that if the overall size of the 3-torus is held fixed, then no perturbations of the metric at all are possible.

For non-vacuum spacetimes there are a number of possible definitions of linear stability. If one allows the stress-energy to be freely adjusted, then every spacetime is trivially linearization stable, since any metric solves the Einstein equations with the stress-energy tensor given by the right hand side of (1). From this point of view the only constraints would come from energy conditions imposed on the stress tensor. If instead we consider solutions of (1) with prescribed perturbations to the sources at some initial time then an interesting structure emerges which extends that found by Fisher and Marsden and Moncrief in the vacuum case. Note that in vacuum, perturbations to the stress-energy vanish by definition. So the definition of linear stability, which we use in this paper, is consistent with that used in the vacuum case. We will show that a necessary condition for a solution of (1) to be linearization stable in this sense is the absence of Integral Constraint Vectors (ICV's), which were defined in [5]. The presence of ICV's in the background spacetime leads to nonlinear constraints on first order perturbations at second order in perturbation theory similar to those found by Moncrief [3] in the vacuum case. In vacuum the ICV's are precisely the Killing vectors, so our results reduce to those of refs. [1,2,3].

In practise the physical situation in which one is interested will determine an appropriate notion of linearization stability. For example, if one wants to study perturbations which are pure gravitational radiation in which the sources are initially unperturbed, then the above definition would be appropriate. This would be particularly natural in deSitter spacetimes.

The additional nonlinear conditions on the solutions of the linear equations are expessed in terms of conditions on a charge Q, which is an integral of terms quadratic in $\delta g_{ij}, \delta\pi_{ij}$, and their spatial derivatives. In the vacuum case, when the ICV's are actually Killing vectors, Q is "Taub's conserved charge" [6]; it is the time component of a conserved current constructed from a Killing vector and the second variation of the Einstein tensor. For example, when the Killing vector is timelike, Q can be interpeted as the "energy in the gravitational perturbation". One interesting thing in the non-vacuum case is that there are constraints on the perturbations in the absence of a spacetime symmetry–the ICV's are not killing vectors. Q is not conserved, but evolves in a well-defined way. Though Robertson-Walker spacetimes do not have a timelike killing vector, the charge associated with one of the constraint vectors is like an energy in the gravitational field, as we shall describe.

In Section 2 we review the definition of integral constraint vectors given in reference [5]. In Section 3 we describe the consequences of the existence of ICV's at first and second orders in perturbation theory. At second order we see the relevance of ICV's to the problem of linearization stability defined in an apropriate way. Section 4 contains examples showing the nontriviality of the nonlinear constraints on the first order variations of the initial data. In Section 5 we discuss a definition of a generalized ICV, which is appropriate for studying the linear stabilty of particular matter theories coupled to gravity. In Section 6 we comment on the relationship between the nonlinear charge Q and the quadrupole formula. In Section 7 we make some concluding remarks. Appendix A contains formulas relevant to the computations of the varied constraint operators. Appendix B gives the explicit forms of the ICV's in Robertson-Walker spacetimes.

2. INTEGRAL CONSTRAINT VECTORS

As in the vacuum case, we proceed by studying solutions of the Einstein constraint equations, which determine the space of initial data for the evolution equations. Initial data consists of a 3-metric g_{ij} and a momentum π^{ij} specified on a spatial hypersurface Σ. These must satisfy the Hamiltonian and momentum constraint equations

$$H[g,\pi] \equiv \frac{1}{g}\left(\pi^{ij}\pi_{ij} - \tfrac{1}{2}\pi^2\right) - R = -16\pi\rho, \tag{2}$$

$$H_k[g,\pi] \equiv -\frac{2}{\sqrt{g}} D_i \pi^i{}_k = -16\pi J_k, \tag{3}$$

where $g = det(g_{ij})$ and D_i and R are the 3-dimensional covariant derivative operator and scalar curvature. The constraints and sources will be denoted collectively by $\Psi = (H, H_k)$ and $S = -16\pi(\rho, J_k)$.

Suppose there exists a one parameter family of exact solutions to the constraint equations $g(\lambda), \pi(\lambda), S(\lambda)$, with expansions around $\lambda = 0$

$$\begin{aligned} g_{ij}(\lambda) &= \bar{g}_{ij} + \lambda h_{ij} + \tfrac{1}{2}\lambda^2 \tilde{h}_{ij} + \dots \\ \pi^{ij}(\lambda) &= \bar{\pi}^{ij} + \lambda p^{ij} + \tfrac{1}{2}\lambda^2 \tilde{p}^{ij} + \dots \\ S(\lambda) &= \bar{S} + \lambda \delta S + \tfrac{1}{2}\lambda^2 \delta^2 S + \dots \end{aligned} \tag{4}$$

Expand the constraint equations $\Psi[g(\lambda), \pi(\lambda)] = S(\lambda)$ around $\lambda = 0$. At first order we have

$$D\Psi(h,p) \equiv \frac{\delta\Psi}{\delta g_{ij}}\cdot h_{ij} + \frac{\delta\Psi}{\delta\pi^{ij}}\cdot p^{ij} = \delta S, \tag{5}$$

where the functional derivatives are evaluated at the background metric and momentum $\bar{g}_{ij}$, $\bar{\pi}^{ij}$. Similarly at second order

$$D^2\Psi(h,p)^2 + D\Psi(\tilde{h},\tilde{p}) = \delta^2 S. \tag{6}$$

Now ask when the first order variation of the constraints can be assembled into a total 3-divergence. Does there exist a function F and a 3-vector field β^k on the surface such that for all variations of the metric and momentum h_{ij}, p^{ij}

$$(F,\beta^k)\cdot D\Psi(h,p) \equiv FDH(h,p) + \beta^k DH^k(h,p) = D_l B^l(h,p). \tag{7}$$

This is the defining property of an ICV [5]. In terms of the four dimensional spacetime this requires that

$$2V^\mu n_\alpha DG^\alpha{}_\mu = D_l B^l,$$

where $V^a = Fn^a + \beta^a$, n^a is the unit normal to the three surface Σ, and $\beta^a n_a = 0$. Integrating (7) over the hypersurface Σ

$$\int_\Sigma (F,\beta^k)\cdot D\Psi(h,p) = \int_\Sigma (h,p)\cdot D\Psi^*(F,\beta^k) + \int_{\partial\Sigma} da_l B^l(h,p), \tag{8}$$

we see that a vector field V^a is an ICV if and only if

$$D\Psi^*(F,\beta^k) = 0, \tag{9}$$

that is, (F,β^k) is in the kernel of the adjoint of the constraint operator. Explicit calculation gives the following set of differential equations which the constraint vector field must satisfy [5]

$$\begin{aligned}\left(-D_iD_j + \bar{g}_{ij}D_lD^l + \bar{R}_{ij}\right)F &= \bar{g}_{ij}D^k(\beta^l\bar{K}_{lk}) - D_l(\beta^l\bar{K}^{ij}) \\ &\quad + \bar{K}_i{}^lD_{[l}\beta_{j]} + \bar{K}_j{}^lD_{[l}\beta_{i]}, \\ D_i\beta_j + D_j\beta_i &= 2F\bar{K}_{ij}\end{aligned} \tag{10}$$

For (F, β^k) satisfying (10) the boundary term in (8) is then given by

$$\begin{aligned}16\pi B^i(h,p) = &\left(D^ih - D_lh^{il}\right)F - hD^iF + h^{il}D_lF \\ &+ \frac{1}{\sqrt{g}}\left(\bar{\pi}^{jk}h_{jk}\bar{g}^i{}_l - 2\bar{\pi}^{ik}h_{kl} - 2p^i{}_l\right)\beta^l,\end{aligned} \tag{11}$$

where on the right hand side $h = \bar{g}^{ij}h_{ij}$ and indices are raised and lowered using the background metric $\bar{g}_{ij}$.

In vacuum the ICV's are the Killing vectors and equation (7) holds independent of the choice of hypersurface [5]. In general the existence of an ICV depends on the choice of slicing. Robertson-Walker spacetimes with the preferred slicing all have ten ICV's [5]. Six of these are the spatial Killing vectors. The remaining four have nonzero time components. In deSitter spacetimes the time dependence of these vector fields can be chosen so that they are Killing fields. In general though Killing vectors are not necessarily ICV's.

3. 1st and 2nd Order Constraints

The existence of an ICV for an exact solution $\bar{g}_{ij}$, $\bar{\pi}_{ij}$ of the constraints has a number of consequences for perturbations around this solution. If h_{ij}, p^{ij} is a solution to the linear equations with perturbed sources given by δS, then at first order in perturbation theory equations (5) and (8) give

$$\int_V (F, \beta^k)\cdot\delta S = \int_{\partial V} da_l B^l(h,p) \tag{12}$$

If the boundary integral on the right hand side vanishes, then (12) is a restriction on possible perturbations to the sources – a necessary condition which the sources must satisfy in order that there exist solutions to the linearized equations. The boundary term will be zero when the spatial hypersurface is compact and the integration volume is taken to be the entire hypersurface. The boundary term will also vanish on open hypersurfaces for perturbations which are created by local, causal processes, if the integration region is taken to be sufficiently large. This situation is quite similiar to the relation between charge density and electric field imposed by Gauss' Law. The result for closed surfaces is analogous to the requirement in E&M that the net charge vanish on a closed hypersurface.

In general for isolated systems (12) gives a relation between moments of the source and the behavior of the field far from the source. A familiar example is the case of a nonvacuum spacetime which is almost Minkowski. One of the constraint vectors is the time-translation Killing vector, and the boundary term (11) is just the ADM mass [7], which perturbatively is equal to a volume integral of the mass density. Implications of these first order integral constraints for cosmology were discussed in [8,9].

At second order in perturbation theory we find a result which bears on the issue of linear stability of the background spacetime. Integrating (6) over a spatial volume and using equation (8) we have

$$\int_{\partial V} da_l B^l(\tilde{h},\tilde{p}) = -\int_V (F,\beta^k)\cdot D^2\Psi(h,p)^2 + \int (F,\beta^k)\cdot \delta^2 S \qquad (13)$$

We will refer to the first integral on the right hand side of (13) as $Q(h,p)^2$,

$$Q(h,p)^2 \equiv \int_V (F,\beta^k)\cdot D^2\Psi(h,p)^2.$$

Q is quadratic in h_{ij} and p^{ij} and their spatial derivatives. Explicit expressions for the second order variations of the constraint operators are given in appendix A. If the boundary term in (13) vanishes, then we are left with

$$Q(h,p)^2 = \int\limits_V (F, \beta^k) \cdot \delta^2 S \tag{14}$$

which relates the first order variations in the metric and momentum to the second order variation of the sources. In the vacuum case (14) gives the result of Moncrief [3] that on compact spatial surfaces the charge Q must vanish. So, we can see in a straightforward way how linearization instability occurs: Suppose the linearized vacuum equations have been solved for h_{ij}, p^{ij}. This solution must now satisfy the additional nonlinear constraint (14), which may fail to be true. That this is nontrivial will be discussed below, but already we have the example of Brill and Deser [4] for the vacuum case.

The non-vacuum case is more complicated because there are more possibilities for the perturbed sources – we must define what rules the perturbed sources are to obey. This will depend on what physical problem one wants to solve. The simplest case is the initial value problem with prescribed sources (like specifying the charge and current densities in E&M) taking

$$\rho = \bar{\rho} + \lambda\delta\rho(x), \qquad j_k = \bar{j}_k + \lambda\delta j_k(x). \tag{15}$$

Of course, one can not usually solve this problem exactly, so one uses a perturbative approach for small λ. In perturbation theory, the variation of the sources are then by definition first order in λ. Higher order variations of the sources, such as $\delta^2 S$ on the right hand side of equation (6) are absent. For sources specified in this way equation (13) becomes

$$Q(h,p)^2 = -\int\limits_{\partial V} da_l B^l(\tilde{h}, \tilde{p}) \tag{16}$$

On compact spatial surfaces, or for peturbations which have been created by local, causal processes (with sufficiently large region of integration), we then have the requirement

$$Q(h,p)^2 = 0. \tag{17}$$

For example, in a closed Friedman universe any linearized solution must satisfy $Q = 0$. The perturbed fields which describe a loop of cosmic string, or some other lump of matter, "appearing" in an open Freidman universe, must satisfy $Q = 0$, if the boundary is outside the forward light cone of the initial perturbation.

A special case is when there are no sources perturbations, only metric perturbations - for example, transverse-traceless gravitational waves in Robertson-Walker spacetimes. In deSitter spacetimes this is particuarly of interest, since it is natural to consider metric perturbations with the cosmological constant fixed.

The expression for Q simplifies considerably for transverse traceless modes in Robertson-Walker. One finds

$$\begin{aligned} Q = \int_V dv F \left(\frac{1}{g} p_{ij} p^{ij} - 2 \left(\frac{\dot{a}^2}{a^2} - 2\frac{k}{a^2} \right) h_{ij} h^{ij} - \frac{2}{\sqrt{g}} \frac{\dot{a}}{a} h_{ij} p^{ij} + K_{tt} \right) \\ - 2 \int_V dv \frac{1}{\sqrt{g}} \beta^k \delta \Gamma_{kjl} p^{jl} \end{aligned} \tag{18}$$

K_{TT} is the spatial-gradient kinetic energy in the transverse traceless modes, given in general in appendix A. For flat spatial sections

$$\int_V K_{tt} = (2\pi)^3 \int d^3k \frac{3}{4} \mathbf{k}^2 |\epsilon_{ij}(k)|^2 \tag{19}$$

where ϵ_{ij} is the polarization tensor.

If the background is Minkowski spacetime ($\dot{a} = 0$, $k = 0$ above), Q is precisely the energy of transverse traceless fluctuations, as defined by the energy-momentum pseudotensor (see e.g. [10]). In an expanding universe, with a nonzero background energy density, Q has positive definite contributions from the momentum squared and the wave vector squared, but there are also negative definite (and sign indefinite) contributions to Q from the expansion. In the negatively curved Robertson-Walker spacetimes, which approach Minkowski spacetime at late times, the ICV's approach the four translational killing vectors, and Q does approach the usual expression, which is positive definite and conserved in the background spacetime. In this flat space limit Q can be interpeted as an energy. In general though Q is not positive and is not conserved. Q evolves in a well defined (but not particuarly transparent) way. Motivated by the connection for assymptotically flat spacetimes between Q and the standard quadrupole formulae discussed below, it would be interesting to see if the evolution of the boundary term to which Q is equal, provides a useful notion of energy flux in spacetimes which are asymptotically Robertson-Walker.

So far we have been discussing the implications of equation (13) when the sources ρ and j_k are considered as fixed, and when the boundary term vanishes. When the boundary term is nonzero, we do not learn anything about the linear stability. In this case equation (13) is a relation between the volume integral defining Q and the second order far field⋆.

4. An example of the loss of superposition

We will now give an example of linearization instability in a Robertson-Walker spacetime with flat spatial sections and periodic boundary conditions. Flat spatial sections make the integrals for Q transparent and the toroidal topology imposes the requirement that $Q = 0$. For the torus the integral constraint vectors are not so interesting - the only constraint vectors consistent with periodic boundary conditions are the spatial translations. However, this is sufficient for an example of the loss of superposition of linearized gravitational radiation.

⋆ Indeed, summing equations (12) and (13)together with all higher orders gives an equality between a volume integral of the perturbed matter fields and the nonlinear terms in H and H_k and a boundary term.

Consider the two transverse traceless metric perturbations

$$h^I_{ij} = f^I_k(t)e_{ij}sin\mathbf{k}\cdot\mathbf{x}, \qquad h^{II}_{ij} = f^{II}_k(t)e_{ij}cos\mathbf{k}\cdot\mathbf{x} \tag{20}$$

where $k_i e^i{}_j = 0$, $e^i{}_i = 0$, and f^I, f^{II} are two linearly independent solutions to $\nabla_a\nabla^a f(t) = 0$ in flat Robertson-Walker, where ∇_a is the four dimensional covariant derivative. Then with $k_i = n_i\pi/L_i$, these are solutions to the linearized Einstein equations with the correct boundary conditions and with the stress-energy unperturbed.

For the constraint vector which is translation in the x-direction and transverse traceless perturbations,

$$Q = \frac{1}{2}\int_V dv(\dot{h}_{jl} + 2\frac{\dot{a}}{a}h_{jl})\partial_x h^{jl}. \tag{21}$$

Therefore, for each of the above waves, Q vanishes, $Q(h^I) = 0$ and $Q(h^{II}) = 0$, but for the sum

$$Q(A_I h^I + A_{II}h^{II}) = \frac{1}{4}A_I A_{II} V k_x(\dot{f}^{II} f^I - \dot{f}^I f^{II}) \tag{22}$$

which in general is not equal to zero, since the waves may be out of phase.

The torus is a simple example, but the 3-sphere is much the same. Perturbations in a universe with closed spatial sections must satisfy $Q = 0$, and it is straightforward to verify that this is nontrivial: take for a solution to the linear constraints the transverse traceless perturbations generated by the tensor spherical harmonic with eignevalue n (as described e.g. in Lifschitz and Khalatnikov [11]). The weighting funciton F in the integrand of Q is the second order scalar spherical harmonic $Q^{(2)}$ (see appendix B and [11]), or equivalently, $F \sim \mathcal{D}^{1/2}$, where the $\mathcal{D}^L_{NI}$ are the hyperspherical harmonics, which form $2L+1$ dimensional irreducible representations of $SU(2)$ [12,13]. The transverse traceless perturbations generated by a pure $\mathcal{D}^L$ mode is a solution to the linear constraints. Such a solution also satisfies $Q = 0$, since the tensor product $L \otimes L$ contains no $L = 1/2$. However, the sum of two such solutions with L values differing by $1/2$ have a nonzero Q, since $L \otimes (L + 1/2) \simeq (1/2) \oplus \cdots$. In terms of the tensor spherical harmonics in [11], this corresponds to the principal eignevalue n differing by one.

5. General Conditions for Linear Stability

If the stress energy is described by a particular field theory, one may want to specify a particular field configuration at an intial time, rather than the mass and current densities. For example, for matter described by a scalar field ϕ one could take the field $\phi = \bar{\phi} + \lambda\delta\phi$ and its time derivative as initial data. This in turn implies what the initial perturbation to the background stress energy is. Here the field perturbation is order λ, so there are order λ^2 contributions to the stress energy, which can in general come both from the matter field and the perturbed metric. For a free scalar field in a background Robertson-Walker spacetime, the perturbed stress-energy is independent of the perturbed metric through terms of second order in λ:

$$\rho = \bar{\rho} + \lambda\left(\dot{\bar{\phi}}\delta\dot{\phi}\right) + \frac{1}{2}\lambda^2\left(\delta\dot{\phi}^2 + \bar{g}^{ij}(\partial_i\delta\phi)(\partial_j\delta\phi)\right) \tag{23}$$

$$J_k = -\lambda\dot{\bar{\phi}}(\partial_k\delta\phi) - \lambda^2\delta\dot{\phi}(\partial_k\delta\phi), \tag{24}$$

where we are using gaussian normal coordinates. Hence in this example there is a second order contribution to the perturbed sources, but it is fixed by the initial data on the matter fields only. So again, there is an additional nonlinear constraint on the linear metric and momentum perturbations– again, when the boundary term vanishes, then h_{ij}, p^{ij} must satisfy $Q(h,p)^2 = \int(F, \beta^k) \cdot \delta^2 S$, which is some fixed number.

In the general case though, the second order sources may depend on the first and second order metric perturbations. In this case the analysis of the previous section tells us nothing about linear stability. The right hand side of equation (14) will depend on second order fields. We simply have a relation which presents no apparent obstruction to finding a solution.

These observations point to a more general criterion for linearization stability. The general strategy is to "leave on the right hand side" of the constraint equations (5) and (6) terms in the perturbed stress-energy which are independent of the metric perturbations, and "move to the left hand side" everything else. The "left hand side" is now an operator $D\mathcal{O}$ which acts on a set of fields. If the kernel of the adjoint $D\mathcal{O}^*$ is nonempty, then there will be constraints on perturbations analogous to those discussed above. Explicitly, if the matter stress-energy has direct dependence on the metric, $S = S(g_{ij}, A)$, where A denotes the matter fields (not necessarily scalars), then

$$D\mathcal{O}(h_{ij}, p^{ij}, A) = D\Psi - \frac{\delta S}{\delta g_{ij}} h_{ij}. \tag{25}$$

There are restrictions on the perturbations of the matter if there is a vector $V^a = fn^a + b^k$ in the kernel,

$$D\mathcal{O}^* \cdot (f, b^k) = D\Psi^* \cdot (f, b^k) - \left(\frac{\delta S}{\delta g_{ij}}\right)^* (f, b^k) \cdot (1, 0) = 0 \tag{26}$$

If a constraint vector (f, b_k) exists, then again at linear order the matter perturbation must satisfy

$$\int_V (f, b_k) \cdot \frac{\delta S}{\delta A} \delta A = \mathit{boundary\ term} \tag{27}$$

in order that solutions to the linear equations exist. And at second order, there are additional nonlinear constraints on the solutions to the linear equations,

$$-Q' + \int (f, b^k) \frac{\delta^2 S}{\delta A^2} \cdot (\delta A)^2 = \mathit{boundary\ term}, \tag{28}$$

where we have set the second order variation of the matter fields $\delta^2 A$ equal to zero, and

$$Q' = \int_V (f, b^k) \cdot \left(D^2\Psi(h,p)^2 - \frac{\delta^2 S}{\delta g_{ij}^2} \cdot (h_{ij})^2 - \frac{\delta^2 S}{\delta g_{ij} \delta A} \cdot (h_{ij}, A) \right) \tag{29}$$

is the new second order charge.

As an example of when $D\mathcal{O}^*$ is relevant for linearization stability, rather than $D\Psi^*$, suppose the stress energy is generated by electromagnetic fields. The energy density is given by $16\pi\rho = g^{ij}V_{ij}$, where $V_{ij} = 2(E_iE_j + B_iB_j)$ (the notation is meant to emphasize that the stress energy is generated by a vector field, rather than a scalar, which at least in Robertson-Walker, leads to a difference between $D\Psi$ and $D\mathcal{O}$). The current density is $4\pi J_k = \epsilon_{klm}E^lB^m$. In this case, unlike that of the scalar field, the first order metric perturbation does appear in the first order variations of the energy and current densities. These are given by

$$\delta\rho = \frac{1}{16\pi}\left(-h^{ij}\bar{V}_{ij} + g^{ij}\delta V_{ij}\right) \tag{30}$$

$$\delta j_k = \tfrac{1}{2}h\bar{J}_k + \frac{1}{4\pi}\bar{\epsilon}_{klm}\left(\bar{E}^l\delta B^m + \delta E^l\bar{B}^m - h^l{}_n\bar{E}^n\bar{B}^m - h^m{}_n\bar{E}^l\bar{B}^n\right) \tag{31}$$

In this case, the generalized ICV (f, b^k) must satisfy equation (10) with the left hand side of the first equation replaced by

$$L_{ij}f - \bar{V}_{ij}f + 16\pi\left(\tfrac{1}{2}\bar{g}_{ij}\bar{J}_k + \frac{1}{4\pi}\bar{\epsilon}_{km(i}\bar{E}_{j)}\bar{B}^m - \frac{1}{4\pi}\bar{\epsilon}_{km(i}\bar{E}^m\bar{B}_{j)}\right)b^k, \tag{32}$$

where L_{ij} is the operator on the left hand side of the first equation in (10).

We can now check to see whether or not Robertson-Walker spacetimes filled with radiation can have constraint vectors in this generalized sense. For Robertson-Walker the stress-energy can't have the above form, since this picks out preferred directions. However, we can take a statistical average of radiation propagating in various directions, so that

$$\bar{V}_{ij} = 2 < \bar{E}_i \bar{E}_j + \bar{B}_i \bar{B}_j >= \frac{16}{3} \pi \bar{\rho} \bar{g}_{ij}, \tag{33}$$

and the zeroth order current vanishes. b^j satisfies the same equation (10) as before, but f must satisfy

$$D_i D_j f = g_{ij} \left(\frac{\dot{a}}{a} \right)^2 f, \tag{34}$$

for both the open and closed universes. This is the equation for a conformal killing vector, and it can be checked that the conformal killing vectors for the three-surfaces of constant curvature do not satisfy this equation. Hence there are no solutions for these vector fields, and therefore no linearization instability in the Einstein-ensemble averaged Maxwell system. This example is contrived in the sense that one has to "bend over backwards" to respect the Robertson-Walker symmetries, and generate the stress-energy in a way that the metric peturbation enters at first order. In this averaged system, the perturbation to the sources from the Maxwell fields is zero until second order, since $< \Delta E_i \bar{E}_j >= 0$.

We note that Arms [14,15] has also studied the linearization stability of solutions to the Einstein-Maxwell equations. Her definition of linearization stability and hence also her results differ from ours. In Arms' approach the variations of the sources are not prescribed, either by specifying $\delta\rho$ and δj_k or by specifying δE_i and δB_i. The analogue of the operator $D\mathcal{O}$ then depends on the variation of the matter fields as well as on the variation of the metric and momentum. The problem defined in this way is similiar to the vacuum case, with a larger set of fields. Arms finds [14,15] that, under a number of conditions, linearization instability of the Einstein-Maxwell system can occur if the background solution has continuous spatial symmetries.

We have looked at a different question, where one fixes something about the sources. If one prescribes the variations of the electric and magnetic fields δE_i and

δB_i away from their background values, then for any solution h_{ij}, p^{ij} to the linearized constraint equations with these sources, do there always exist higher order variations in the metric and momentum such that the constraints are satisfied? In this case one looks for solutions f and b^k in $KerDO^*$ equation (32)(instead of the operator $D\Phi^*$ considered in [14]). One finds that solutions with nonzero f are possible (unlike the result in [14]). That is, there may be an instability, even if there are no simultaneous symmetries of all the fields on the initial value surface.

6. Vacuum case, Taub's conserved charge, and Quadrupole formula

In the vacuum case the charge is conserved because it is the spatial integral of the time component of a conserved current. Let $\bar{g}_{ab}$ be a solution to the vacuum Einstein equation, and assume that this spacetime has a killing vector ξ^a. Taub [6] noted that the current $J^a = \xi^b D^2 G^a{}_b(h)^2$ is conserved in the background spacetime $\bar{g}_{a\bar{b}}$, if h_{cd} is a solution to the linearized, four dimensional Einstein equation. This follows because the Bianchi identity implies that if the background is vacuum, and if h_{cd} is a solution to the linear vacuum equations, then the second variation of the Einstein tensor is covariantly conserved, $\nabla_a D^2 G^a{}_b(h)^2 = 0$. In the non-vacuum case, even with a killing vector, the second variation is not conserved. The time component of J^a is exactly the constraint integrand, so Taub's conservation law states that

$$\frac{d}{dt}Q = \frac{d}{dt}\int J^\alpha n_\alpha dv = -\int_{\partial V} \xi^\alpha D^2 G^k{}_\alpha da_k \tag{35}$$

If the killing vector is timelike, then one can call the charge the energy in the gravitational perturbations, in the sense that Q is conserved.

On the other hand, we know that not only is the time rate of change of Q a boundary term, but since Killing vectors in vacuum are constraint vectors, Q itself is also a boundary term,

$$Q = \int_{\partial V} da_l B^l(\tilde{h}, \tilde{p}) \tag{36}$$

Hence as Moncrief [3] noted, for vacuum spacetimes with compact spatial surfaces, the charge actually vanishes, $Q = 0$. Further, we see that the time rate of change of the boundary term of the second order perturbation is equal to the flux defined by the pseudotensor,

$$\frac{d}{dt}\int da_l B^l(\tilde{h},\tilde{p}) = -\int da_l \xi^\alpha D^2 G^l{}_\alpha \tag{37}$$

The right hand side of (35) or (37) is the usual starting point to find the flux of gravitational radiation crossing some big sphere, and yields the usual quadrupole formula for perturbations off Minkowski spacetime. This suggests that there is a relation between the quadrupole formula and the time rate of change of the boundary term defined by the constraint vectors. The "quadrupole formula" is supposed to be the energy radiated due to sources, so the above formulas, true for vacuum, do not immediately apply. With time dependent sources, but considered as small, one still looks for solutions for the metric which are perturbations off Minkowski–at zeroth order, the stress energy and hence Einstein tensor vanishes, and at first and second order there are perturbative corrections to the stress energy. Since the background Einstein tensor vanishes, the Bianchi identities imply that $\nabla_a DG^a{}_b = 0$ as an identity. Therefore at second order,

$$\nabla_a(D^2G^a{}_b \cdot h^2 - \delta^2 T^a_b) = -\nabla DG^a_b \cdot \tilde{h} = 0. \tag{38}$$

One can then check that the current defined by this quantity and the time translation Killing vector ξ^a of Minkowski is conserved. Therefore, if the sources are compact, and vanish on ∂V, then

$$\frac{d}{dt}\int_V \xi^\alpha n_\beta(D^2G^\beta{}_\alpha - \delta^2 T^\beta{}_\alpha)dv = -\int_{\partial V} \xi^\alpha D^2 G^k{}_\alpha da_k \tag{39}$$

Equation (39) relates a flux in the gravitational perturbations to the time rate of change of Q minus the time rate of change of the second order sources. Suppose the sources are independent of time after some time. Then during the "quiet time", the gravitational flux is just determined by $\dot{Q}$, that is, equation (39) reduces to (35). So now we can relate the boundary term defined by the constraints in equation (36), to the quadrupole formula,

$$\frac{d}{dt}\int_{\partial V} B^l(\tilde{h},\tilde{p}) = -\frac{1}{45}\left(\frac{d^3}{dt^3}q\right)^2 \tag{40}$$

where q is the traceless quadrupole moment tensor of the source, evaluated at retarded time.

7. Conclusion

We have seen that if a spacetime has an integral constraint vector, then it is linearization unstable. Some of the solutions to the linearized constraint equations must be discarded, although at linear order they appear to be good solutions. This occurs, for example, in Robertson-Walker cosmologies. The definition of linear stability must include a specification of how the sources are allowed to vary; the simplest case is when the mass and momentum densities are prescribed as part of the initial data.

It will be of interest to answer some obvious questions about the boundary term (11). Can it provide a useful notion of differential mass, in the spirit of the ADM mass? Is the time rate of change of the boundary term related to the flux of gravitational energy?

One of the original motivations for this study was to understand how linearization instability must be taken into account in a functional integral approach to gravity. In the functional integral, though the fields are not necessarily solutions to the equations of motion, they are required to satisfy the constraints. Since in practice one uses an expansion to approximate the integral, spurious solutions to the linearized constraints should presumably be excluded. Recall, for example, that for deSitter spacetime with 3-spheres for spatial sections, solutions to the linear constraints must also satisfy $Q = 0$. This question has been studied for a number of vacuum spacetimes in [16,17].

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Appendix A

Here we collect some formulae on second order variations of the constraints. Barred symbols, such as $\bar{g}_{ij}$ and $\bar{\pi}^{ij}$, denote tensors in the background geometry. Indices in the varied equations are raised and lowered with the background metric, $g = det(g_{ij})$, and $h = h_{ij}\bar{g}^{ij}$, $p = p^{ij}\bar{g}_{ij}$. With the exception of the exact constraint equations, the three dimensional covariant derivative operators D_l are with respect to the background metric.

$$H = \frac{1}{g}\left(\pi^{ij}\pi_{ij} - \tfrac{1}{2}\pi^2\right) - R \equiv \frac{1}{g}P - R \tag{41}$$

$$H_k[g,\pi] = -\frac{2}{\sqrt{g}}D_i\pi^i_k = -16\pi J_k, \tag{42}$$

$$Dg = \bar{g}h \tag{43}$$

$$D^2 g = \bar{g}\left(h^2 - h^{ij}h_{ij}\right) \tag{44}$$

$$DP = 2\bar{\pi}_{ij}p^{ij} - \bar{\pi}p + \left(2\bar{\pi}^{ik}\bar{\pi}^j{}_k - \bar{\pi}\bar{\pi}^{ij}\right)h_{ij} \tag{45}$$

$$DR = -D^l D_l h + D^i D^j h_{ij} - \bar{R}^{ij}h_{ij} \tag{46}$$

$$DH = -\frac{1}{\bar{g}}\bar{P}h + \frac{1}{\bar{g}}DP - DR \tag{47}$$

$$DH_k = \frac{-2}{\sqrt{\bar{g}}} \left(D_i p^i{}_k + h_{ki} D_j \pi^{ij} + \pi^{li} \delta\Gamma_{kil}[h] \right) + \frac{h}{\sqrt{\bar{g}}} D_i \pi_k{}^i \tag{48}$$

$$\delta\Gamma_{kil}[h] = \tfrac{1}{2} \left(D_i h_{kl} + D_l h_{ki} - D_k h_{il} \right) \tag{49}$$

$$\begin{aligned} D^2 P = {} & p^{ij} p_{ij} + \bar{\pi}^{ij} \bar{\pi}^{kl} h_{ik} h_{jl} \\ & + 4 \bar{\pi}^j{}_k \bar{g}^i{}_l p^{kl} h_{jl} - \tfrac{1}{2} p^2 - \tfrac{1}{2} \left(\bar{\pi}^{ij} h_{ij} \right)^2 \\ & - \bar{\pi} p^{ij} h_{ij} - \bar{\pi}^{ij} p h_{ij} \end{aligned} \tag{50}$$

$$D^2 H = \left(h^2 + h_{ij} h^{ij} \right) \frac{1}{\bar{g}} \bar{P} - 2 \frac{h}{\bar{g}} DP + \frac{1}{\bar{g}} D^2 P - D^2 R \tag{51}$$

$$D^2 R = \bar{g}^{ij} D^2 R_{ij} - 2 h^{ij} D R_{ij} + h^i{}_n h^{nj} R_{ij} \tag{52}$$

$$D R_{ij} = D_k D_{(i} h^k{}_{j)} - \tfrac{1}{2} D_i D_j h - \tfrac{1}{2} D_l D^l h_{ij} \tag{53}$$

$$\begin{aligned} D^2 R_{ij} = {} & \frac{1}{4} D_i h_{lm} D_j h^{lm} + \tfrac{1}{2} D^l h_{in} \left(D_l h_j^n - D^n h_{jl} \right) \\ & + \tfrac{1}{2} h^{mn} \left(D_i D_j h_{mn} - D_n D_i h_{mj} - D_n D_j h_{mi} + D_m D_n h_{ij} \right) \\ & + \tfrac{1}{2} \left(\tfrac{1}{2} D^n h - D_m h^{mn} \right) \left(D_i h_{nj} + D_j h_{ni} - D_n h_{ij} \right) \end{aligned} \tag{54}$$

For Robertson-Walker spacetimes

$$DP = 2 \frac{\dot{a}}{a} \sqrt{\bar{g}} p - 4 \left(\frac{\dot{a}}{a} \right)^2 \bar{g} h \tag{55}$$

$$DR = -D^2 h + D^i D^j h_{ij} - \frac{2k}{a^2} h \tag{56}$$

$$DH = 2 \left(\frac{k}{a^2} + \frac{\dot{a}^2}{a^2} \right) h + \frac{2}{\sqrt{\bar{g}}} \frac{\dot{a}}{a} p + D^2 h - D^i D^j h_{ij} \tag{57}$$

$$\begin{aligned} D^2 P = {} & p^{ij} p_{ij} + 4 \bar{g} \left(\frac{\dot{a}}{a} \right)^2 h^{ij} h_{ij} - 2 \sqrt{\bar{g}} \frac{\dot{a}}{a} p^{ij} h_{ij} \\ & - \tfrac{1}{2} p^2 - 2 \bar{g} \left(\frac{\dot{a}}{a} \right)^2 h^2 + 2 \sqrt{\bar{g}} \frac{\dot{a}}{a} h p \end{aligned} \tag{58}$$

$$\sqrt{\bar g}D^2H = \frac{1}{\sqrt{\bar g}}\left(p^{ij}p_{ij} - \tfrac{1}{2}p^2\right) - 2\frac{\dot a}{a}\left(h_{ij}p^{ij} + hp\right) - \frac{2k}{a^2}h^2 + h_{ij}h^{ij}\left(\frac{4k}{a^2} - 2\frac{\dot a^2}{a^2}\right) + K_s + K_{tt}. \tag{59}$$

$$K_s = -3\sqrt{\bar g}h^{ij}\left(\tfrac{1}{2}D_iD_jh - D_iD^lh_{lj}\right) + \sqrt{\bar g}\left(\tfrac{1}{2}D_jh - D_ih^i{}_j\right)\left(\tfrac{1}{2}D^jh - D_lh^{lj}\right) \tag{60}$$

$$K_{tt} = \sqrt{\bar g}\left(-\frac{3}{2}h^{ij}D_lD^lh_{ij} - \frac{3}{4}D_ih_{nm}D^ih^{nm} + \tfrac{1}{2}D_lh_{ij}D^jh^{il}\right) \tag{61}$$

$$\sqrt{\bar g}D^2H_k = -2p^{jl}\delta\Gamma_{kjl} - 2h_{jk}D_ip^{ij} + 2hD_ip^i{}_k - 4\frac{\dot a}{a}\sqrt{\bar g}h\left(D_lh^l{}_k - \tfrac{1}{2}D_kh\right) \tag{62}$$

Appendix B

Explicit formulae for the four Robertson-Walker ICV's which are not purely spatial Killing vectors (from ref. [5]).

For closed and open spatial sections ($k = +1, -1$ respectively) define the following harmonic functions

$$k = +1: \quad Q^{(0)} = \cos\chi, \quad Q^{(1)} = \sin\chi\cos\Theta, \quad Q^{(2)} = \sin\chi\sin\Theta\cos\phi, \quad Q^{(3)} = \sin\chi\sin\Theta\sin\phi. \tag{63}$$

$$k = -1: \quad Q^{(0)} = \cosh\chi, \quad Q^{(1)} = \sinh\chi\cos\Theta, \quad Q^{(2)} = \sinh\chi\sin\Theta\cos\phi, \quad Q^{(3)} = \sinh\chi\sin\Theta\sin\phi. \tag{64}$$

where χ, Θ, ϕ are the standard hypersherical (hyperbolic) angular coordinates. The four ICV's are then given by

$$\vec{V}_{(a)} = Q^{(a)}\frac{\partial}{\partial t} + k\dot{a}aD^jQ^{(a)}\frac{\partial}{\partial x^j}, \qquad a = 0,1,2,3. \tag{65}$$

For flat spatial sections,

$$\begin{aligned}\vec{V}_{(0)} &= \frac{\partial}{\partial t} - \frac{\dot{a}}{a}x^i\frac{\partial}{\partial x^i} \\ \vec{V}_{(k)} &= x^k\frac{\partial}{\partial t} + \frac{\dot{a}}{a}\left(\tfrac{1}{2}\delta^{ki}r^2 - x^ix^k\right)\frac{\partial}{\partial x^i}, \qquad k = 1,2,3,\end{aligned} \tag{66}$$

where x^1, x^2, x^3 are standard Euclidean coordinates.

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