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Self-Gravitating Strings In 2 + 1 Dimensions*

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ABSTRACT

We present a family of classical spacetimes in $2+1$ dimensions. Such a spacetime is produced by a Nambu-Goto self-gravitating string. Due to the special properties of three-dimensional gravity, the metric is completely described as a Minkowski space with *two* identified worldsheets. In the flat limit, the standard string is recovered. The formalism is developed for an open string with massive endpoints, but applies to other boundary conditions as well. We consider another limit, where the string tension vanishes in geometrical units but the end-masses produce finite deficit angles. In this limit, our open string reduces to the free-masses solution of Gott, which possesses closed timelike curves when the relative motion of the two masses is sufficiently rapid. We discuss the possible causal structures of our spacetimes in other regimes. It is shown that the induced worldsheet Liouville mode obeys (*classically*) a differential equation, similar to the Liouville equation and reducing to it in the flat limit. A quadratic action formulation of this system is presented. The possibility and significance of quantizing the self-gravitating string, is discussed.

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1. Introduction

Much attention has been devoted in recent years to the study of low dimensional gravity (two or three spacetime dimensions). In two dimensions, gravitation is classically trivial, but the quantum conformal anomaly renders the Liouville mode dynamical, and the structure of the theory (with various forms of matter) is both rich and tractable [1]. Two-dimensional gravity is useful both as a toy model to guide our thoughts about the (as yet nonexistent) four-dimensional theory [2][3], and as a new tool for studying noncritical string theory and string field theory.

In $2+1$-dimensional Einstein gravity, there is still no dynamical graviton, but the classical theory is already nontrivial: a nonvanishing energy-momentum tensor may exist †, and can be remotely detected by parallel-transporting tensor quantities around the matter distribution. Another globally-nontrivial aspect of spacetime, when moving sources are present, is an obstruction to defining global Minkowskian time, which results in 'time shifts' experienced by observers (external or part of the system) travelling along some closed paths [4]. Some matter distributions, although satisfying the energy-positivity conditions, give rise to sufficient time shifts to allow 'naked' closed timelike or lightlike curves[5]. In addition, a topological mass-term is allowed in three dimensions, which *does* render the graviton dynamical.

Classical $2+1$ dimensional gravity is also relevant as an approximation scheme (dimensional reduction) in the study of cosmic strings; an idealized infinite, straight cosmic string is a three-dimensional pointlike particle.

Quantum Einstein gravity has also been studied in three dimensions, both in the continuum[6] and discretized[7] versions.

In this paper, we initiate the study of classical three-dimensional spacetimes produced by a particular type of matter — a single, Nambu-Goto string. In other words, the action is the Einstein-Hilbert term plus the Nambu-Goto action, the latter being covariantized in the three target-space dimensions. We shall refer to

† Even with no cosmological constant.

this as the self-gravitating, or Nambu-Goto-Einstein, string. There are several reasons for considering this system. It has a richer dynamics than the classical matter distributions thus far considered in three-dimensional gravity[4][8] . In addition, if an *open* string with two massive endpoints is considered, one recovers in the zero-tension limit, the spacetime recently considered by Gott[5]. This spacetime simply consists of two point-masses (conical singularities) moving past each other. When the relative motion is sufficiently rapid, the spacetime possesses closed time-like curves. It is therefore of interest to study the causal structure of spacetime in the case where the masses are tethered together elastically — since then their relative motion is nonuniform. The mathematically simplest way to implement this generalization, seems to be via the Nambu-Goto-Einstein string.

As a final motivation, we wish to raise the question of whether the self-gravitating string can be consistently quantized[‡] — and if so, whether it is relevant to ordinary string theory.

The main results reported here are as follows. The geometry of our classical spacetime is expressed as a flat Minkowski space, with the region between two worldsheets excised and the two worldsheets identified. The equations of motion of the worldsheet are found, and are reduced to a Liouville-like equation for the induced worldsheet Liouville mode. The flat limit and the zero-tension limit are worked out for the case of open string with massive endpoints. It is shown how to expand solutions in the geometrized string tension. The two-worldsheet formalism is recast, using auxiliary fields, as an action principle on the worldsheet. The new action is quadratic, with quadratic constraints on the fields, and is perhaps amenable to consistent quantization. It is suggested that the *first* quantized self-gravitating string could be a step towards understanding nonperturbative string field theory in a continuum setting, if such a quantization is possible.

Further work concerning the possible global structures (causal and topological)

‡ Since this is a 3d string theory, we might well have to add a CFT on the worldsheet as a necessary condition for the theory to be consistent.

of our classical spacetimes, and the feasibility of the proposed quantization scheme, is in progress.

Throughout, we concentrate mostly on the case of open string with massive endpoints, because our original motivation was in extending the Gott solution. In the context of quantization, of course, one should work either with standard open strings (that is, having lightlike endpoints), or with closed strings; The treatment differs only in the boundary conditions, and we indicate the requisite changes at the appropriate places.

The remainder of this paper is organized as follows. In section 2, the geometry of our spacetime is described, and the attendant formalism set up. In section 3 the string equations of motion are derived, and the geometric meaning of the masses at the ends of the open string is established. In section 4 we state how the equations of motion can be reduced to a (*classical*) Liouville-like equation; the details of this demonstration are relegated to appendix B. Section 5 contains a discussion of the asymptotic form of the three-metric in 2+1 dimensional gravity, and of the global structure of spacetime. In section 6 we treat perturbation theory in the string tension, and derive the flat limit and the zero-tension limit for the case of open string with massive endpoints. In section 7, the possible causal structures of our spacetimes are discussed, as well as the general significance of a 'nonphysical' causal structure. In section 8, the quadratic-action formulation is presented; some off-shell results from appendix B are used. In section 9 we discuss quantization, and section 10 summarizes the conclusions. Appendix A lists our Minkowski-space, curvature and units conventions. In appendix C we present a proof that the geometrically-defined endpoint masses are conserved, whereas appendix D treats the flat open string with endmasses, and presents a concrete configuration of such a string.

2. Geometry of Self-Gravitating String.

As explained in the introduction, we shall mostly treat the case of open string with massive endpoints, but will indicate how the more familiar cases (open string with lightlike ends, and closed string) are to be handled. The only differences are in the boundary conditions imposed, and in the global properties of the resulting spacetime[⋆].

The spacetime we are interested in has the topology of R^3; its metric has Minkowskian signature, and is everywhere flat except on a worldsheet of an open string. Fig. 1 depicts the string, which is an equal-time section of the worldsheet. We denote its two endpoints P and Q. We also use this notation for the *worldlines* of the respective endpoints, which constitute the boundary of the worldsheet (the string is assumed to exist in the infinitely remote past or future)[†]. We denote the worldsheet by S; it is assumed to have the topology $I \times R$, where I is the closed unit interval[‡]. For simplicity, we remove from spacetimes infinitesimal neighborhoods of the singularities P and Q.

The energy-momentum tensor is nonzero only on the worldsheet, so due to Einstein's field equations and the dimensionality of spacetime, space is locally flat everywhere else. We therefore choose the following convenient coordinatization

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⋆ Since the open string with massive endpoints is not a familiar case, even in the flat limit, the reader is referred to appendix D for a brief treatment of the flat case for such a string.

† Since we are interested in possible closed timelike curves, caution should be exercised in using terms such as 'section', 'past' or 'future'. Since spacetime is assumed homeomorphic to R^3, a global coordinate system can be chosen. A global time coordinate does not always exist, but we assume that it *does* exist in some open neighborhood of the worldsheet itself. Thus it makes sense, in our spacetimes, to talk about equal-time sections of the worldsheet, but not necessarily of the entire manifold; it is in this sense that Fig.1 is to be understood. Of course, there is in general no global killing vector, temporal or other, so the equal-time sections of the worldsheet depend on the time coordinate chosen.

‡ We choose to disallow self-intersecting worldsheets. The reason is that if the string cuts itself, a new curvature singularity is created at the intersection, and in addition the global structure of spacetime changes. This is in marked contrast with standard string theory, where self-intersections of the embedding of S are irrelevant. Therefore, *some* of the configurations resulting from our construction may not be physical solutions; conversely, the self-intersecting solutions will not be globally reproduced by our construction. We will further discuss these global considerations in section 5.

of the manifold. Let an orientation be defined on the manifold; This induces an orientation on any equal-time section of a neigborhood V of the worldsheet (see the previous footnote). The worldsheet is two-sided, and we define its *bottom* side as that through which any directed curve, lying within the section and winding around S in a positive sense, passes on the segment from P to Q. The opposite side will be referred to as the *top* of the worldsheet. Now, we erect a flat Minkowskian coordinate system on the bottom side of S, and extend it outside of V and around P and Q. However, the coordinates $\{z^\mu\}$ assigned in this way to a point on the top side, will depend on whether one continues around P or Q. Therefore, transition functions are needed on the top side of S[§]. In fact, the transition region can be chosen to intersect V anywhere on the top side, but must run from S to infinity. Unlike S, it does not correspond to any physical singularity, so both $g_{\mu\nu}$ and $\Gamma^\lambda_{\mu\nu}$ are continuous across it. Thus, the transition functions must constitute an isometry of the Minkowski metric, that is, a Lorentz transformation combined with a shift[4]. We call this coordinate atlas M. It describes the manifold with S itself removed; the *entire* manifold is M with a homeomorphism that identifies the top and bottom sides of S.

In M, the coordinates of the worldsheet differ on the top and bottom sides, whether one continues around P or Q; we adopt the continuation around P. Let the bottom coordinates be $x^\mu = x^\mu(u)$, where u^a is a parametrization of the worldsheet. We denote by $\hat{x}^\mu(u)$ the coordinates of the point, on the top of S, that is to be identified with $x^\mu(u)$. Clearly, at P the two sets of coordinates must agree; thus one boundary condition is

$$x^\mu(P) = \hat{x}^\mu(P) \tag{1}$$

The Minkowski space, with the appropriate identifications, is depicted in Fig. 2; the transition region, T, has been chosen to run from Q to infinity, along an arbitrary continuation of the worldsheet. If the surface T is considered as part of

§ And also in some region outside V.

S, then for a point u^a on the surface T or its boundary Q, x and $\hat{x}$ are related by the linear transition function:

$$x(u) = \bar{L}\hat{x}(u) + \bar{b} \quad on\ T\ , \tag{2}$$

where $\bar{L}$ is a fixed Lorentz transformation, and $\bar{b}$ is a fixed three-vector. At Q, eq.(2) becomes the second boundary condition:

$$x(Q) = \bar{L}\hat{x}(Q) + \bar{b} \tag{3}$$

We will occasionally refer to x,$\hat{x}$ as the two worldsheets — bottom and top, respectively; although they describe the same surface. Clearly, we could just as well have chosen the transition region from P to infinity, and continued around Q— in which case the boundary condition at Q would have been the simpler one.

Following standard string practice, we choose the parametrization u^a to be orthonormal on the bottom worldsheet; then, by the continuity of the three-dimensional metric across S, u^a are orthonormal on the top worldsheet as well*. We denote:

$$u^0 = \tau,\ u^1 = \sigma,\ u^{\pm} = u^0 \pm u^1 \tag{4.a}$$

$$\partial_{\pm} \equiv \partial/\partial u^{\pm} \tag{4.b}$$

The orthonormality condition then reads (see appendix A for conventions),

$$(\partial_{\pm}x)^2 = (\partial_{\pm}\hat{x})^2 = 0\ . \tag{5}$$

Eq.(5) incorporates the continuity of the $++$ and $--$ components of the 3d metric

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$\star$ $g_{\alpha\beta}(z)$ are not continuous functions in the $\{z\}$ coordinates. To obtain continuous components, we need to continue $\{z\}$ *across* S, rather than around P. We shall define *two* such coordinate systems, below eq.(8d).

at S; the continuity of the $+-$ component implies,

$$E \equiv \partial_+ x \cdot \partial_- x = \partial_+ \hat{x} \cdot \partial_- \hat{x} \tag{6}$$

where $2E$ is the induced conformal scale factor on S: the induced metric is $ds^2 = 2E du^+ du^-$.

There is still a residual parametrization freedom, which we partially use up in choosing P,Q to be $\sigma = 0,\pi$, respectively†.

Recall that we have assumed the worldsheet has global equal-time sections. In terms of the u^a coordinates, we require that the lines $\tau =$ *const* are spacelike, whereas $\sigma =$ *const* are timelike. Since u^a are orthonormal, this is equivalent to

$$E(u) < 0 \ . \tag{7}$$

It is useful to define the normal to S. We define $n(u)$ as the unit (spacelike) three-vector normal to S on its bottom side, and pointing into the *top* side, at point u. Since $\partial_\pm x$ span the local tangent plane to S, we find (components in $\{z\}$ coordinates)

$$n_\mu = \frac{1}{E} \epsilon_{\mu\alpha\beta} \partial_+ x^\alpha \partial_- x^\beta \ , \tag{8a}$$

$$n^2 = 1, \quad n \cdot \partial_\pm x = 0 \ . \tag{8b}$$

Similarly, the unit normal vector to the top side, also pointing into the top side, is in the same coordinates (by virtue of eq. (6))

$$\hat{n}_\mu = \frac{1}{E} \epsilon_{\mu\alpha\beta} \partial_+ \hat{x}^\alpha \partial_- \hat{x}^\beta \ , \tag{8c}$$

$$\hat{n}^2 = 1, \quad \hat{n} \cdot \partial_\pm \hat{x} = 0 \ . \tag{8d}$$

In order to study the three-geometry in the neighborhood V of the worldsheet, we extend the bottom-side coordinates, $\{z\}$, across S into the top side, and call the

† There still remains the freedom to reparametrize via $u^+ \to f(u^+)$,$u^- \to f(u^-)$, where $f(v) - v$ is periodic with period 2π.

new coordinate system $\{x\}$. Another useful coordinate system, $\{\hat{x}\}$, is obtained by extending $\{z\}$ from the *top* side across S. This notation extends in a consistent way our use of $\hat{x}, x$ to denote the top and bottom coordinates of the surface S itself, into V. Thus, we have $\hat{x}(x(u)) = \hat{x}(u)$.

Below S, $g_{\mu\nu}(x) = \eta_{\mu\nu}$, whereas above S, $\hat{g}_{\mu\nu}(\hat{x}) = \eta_{\mu\nu}$. But the metric g, in the x coordinate system, must be continuous at S. The continuity of the tangential components was already encoded in eqs.(5),(6) above; that of the normal-normal and mixed components, reads

$$(\frac{\partial \hat{x}}{\partial n})^2 = 1, \tag{9a}$$

$$\frac{\partial \hat{x}}{\partial n} \cdot \partial_{\pm} \hat{x} = 0 \tag{9b}$$

where we denote $\partial/\partial n \equiv n^{\alpha} \partial/\partial x^{\alpha}$. From eqs.(8d),(9a) and (9b) we easily find:

$$\frac{\partial \hat{x}}{\partial n} = \hat{n}. \tag{9c}$$

The connection, in x coordinates, is

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda} (g_{\alpha\lambda,\beta} + g_{\beta\lambda,\alpha} - g_{\alpha\beta,\lambda}) \ . \tag{10}$$

The curvature on S causes a discontinuity in Γ. Since the worldsheet is that of a Nambu-Goto string, and since $g_{\mu\nu}$ is Minkowskian at S, the energy-momentum tensor in the bulk of the worldsheet (that is, away from P,Q) is given in $\{x\}$ coordinates by

$$T^{\mu\nu}(y) = \kappa \int du^+ du^- \big(\partial_+ x^{\mu} \partial_- x^{\nu} + \partial_- x^{\mu} \partial_+ x^{\nu}\big) \delta(y - x(u)), \tag{11}$$

where $\kappa > 0$ is the string tension‡. We work in geometrized units, in which the

‡ This $T_{\mu\nu}$ satisfies the weak, strong and dominant energy positivity conditions[9].

three-dimensional Newton's constant is

$$8\pi G = 1,$$

masses are dimensionless, and κ has dimensions of inverse length. The field equations are thus (see appendix A)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -T_{\mu\nu} \tag{12}$$

Now, $g_{\alpha\beta,\lambda}$ vanishes on the bottom side of S; let its dicontinuity across S (top value minus bottom value) be[§]

$$\Delta g_{\alpha\beta,\lambda} \equiv n_\lambda P_{\alpha\beta}(u) \tag{13a}$$

Then by eq. (10),

$$\Delta\Gamma^\mu_{\alpha\beta} = \frac{1}{2}(n_\beta P_\alpha^{\ \mu} + n_\alpha P_\beta^{\ \mu} - n^\mu P_{\alpha\beta}) \tag{13b}$$

Let $P^{(t)}$ denote the projection of P on the plane locally tangent to S at u. Eqs. (11)-(13) imply:

$$P^{(t)}_{\mu\nu} = 4\kappa(n_\mu n_\nu - \eta_{\mu\nu}) + \frac{2\kappa}{E}(\partial_+ x_\mu \partial_- x_\nu + \partial_- x_\mu \partial_+ x_\nu) \tag{14}$$

§ The discontinuity of the gradient of a function that is a constant on one side of S, is normal to the surface S.

3. The String Equations of Motion.

The equations of motion of the string are easily found by writing the Nambu-Goto action in the background metric $g_{\mu\nu}$, and varying this action w.r.t. x. We obtain, in $\{x\}$ coordinates,

$$\partial^2 x^\mu + \bar{\Gamma}^\mu_{\alpha\beta}\partial_+ x^\alpha \partial_- x^\beta = 0, \tag{15a}$$

where $\bar{\Gamma}$ is half the discontinuity,

$$\bar{\Gamma}^\mu_{\alpha\beta} \equiv \frac{1}{2}\Delta\Gamma^\mu_{\alpha\beta}. \tag{15b}$$

$\bar{\Gamma}$ is also the average of the values of Γ on the top and bottom sides. It is easy to verify that eqs.(15) ensure the local conservation of $T_{\mu\nu}$ in the bulk of the worldsheet.

We contract indices μ,ν in eq.(14) with $\partial_+ x$,$\partial_- x$ respectively, and use eq.(13b) to obtain

$$\Delta\Gamma^\mu_{\alpha\beta}\partial_+ x^\alpha \partial_- x^\beta = \kappa E n^\mu \ . \tag{16}$$

Thus, by eq.(8a), we find the equation of motion for S, (see appendix A for notation)

$$\partial^2 x = -\frac{\kappa}{2}\partial_+ x \times \partial_- x \tag{17}$$

Repeating the above derivation in $\hat{x}$ coordinates, we find the corresponding equation for $\hat{x}(u)$:

$$\partial^2 \hat{x} = \frac{\kappa}{2}\partial_+ \hat{x} \times \partial_- \hat{x} \tag{18}$$

We shall refer to eqs.(17),(18) as the *vector* equations of motion; there is another, scalar equation, which we now derive. Eqs.(17),(18) describe the motion of the string under its own gravitational field. To render this picture consistent, we must determine $P^{(t)}$ geometrically, and impose its equality to the expression (14) found from the gravitational field equations. On the top side of S, $\hat{\Gamma} = 0$, and Γ is therefore known from the transformation law of the connection. This readily gives:

$$P_{\mu\nu} = \frac{\partial}{\partial n}(\frac{\partial \hat{x}^\lambda}{\partial x^\mu})\frac{\partial \hat{x}_\lambda}{\partial x^\nu} + \frac{\partial}{\partial n}(\frac{\partial \hat{x}^\lambda}{\partial x^\nu})\frac{\partial \hat{x}_\lambda}{\partial x^\mu} \tag{19}$$

From the definition of $\partial/\partial n$ (below eq.(9b)), we have

$$\partial_\pm(\frac{\partial \hat{x}}{\partial n}) = \partial_\pm n^\mu \frac{\partial \hat{x}}{\partial x^\mu} + \partial_\pm x^\mu \frac{\partial}{\partial n}(\frac{\partial \hat{x}}{\partial x^\mu}) \tag{20}$$

From eqs.(19),(20) and the expressions[⋆] (B.5) for $\partial_\pm n$, we can find $P^{(t)}$; equating it to eq.(14), we get the conditions

$$\partial_\pm \hat{x}^\mu \partial_\pm(\frac{\partial \hat{x}_\mu}{\partial n}) = -n \cdot \partial_\pm^2 x \ , \tag{21a}$$

$$\sum_\pm \partial_\mp \hat{x}^\mu \partial_\pm(\frac{\partial \hat{x}_\mu}{\partial n}) = -\kappa E \ . \tag{21b}$$

The left-hand sides of eqs.(21) can be found by applying ∂_+, ∂_- to eq.(9b), and using eq.(18). The result is that eq.(21b) is an identity, whereas eq.(21a) becomes the additional, *scalar* equation of motion (actually two scalar equations):

$$n \cdot \partial_\pm x = \hat{n} \cdot \partial_\pm \hat{x} \tag{22}$$

This equation, like the continuity condition (6), relates $x(u)$ to $\hat{x}(u)$.

The dynamics of the worldsheet, and therefore of spacetime, is determined by the orthonormality conditions (5), the continuity condition (6), the equations of motion (17),(18),(22) and the boundary conditions (1) and (2). For a *closed* string, the treatment is unchanged, except for the boundary conditions. As usual in string theory, we would then scale u^a so $x(u)$,$\hat{x}(u)$ are periodic in σ, with period 2π. We

⋆ With k set to its on shell value, $-\frac{\kappa}{2}$ (eq.(B.14)).

still need to choose an arbitrary point P on the string, so that the coordinate-transition region runs from P to spatial infinity (see discussion in section 2). We may choose $\sigma = 0$ at P. Then eqs.(1)-(3) still apply, if one understands Q to mean the line $\sigma = 2\pi$. S has the topology of a cylinder, and we assume that the region of spacetime interior to it has the topology of the interior of a cylinder[†‡].

For a single, free point mass, equal-time sections of spacetime in a rest frame of the mass are cones, with the deficit angle (in our units of $8\pi G = 1$) equal to the mass. In the case of our self-gravitating (open) string, the mass at each end-point, if nonzero, has the same geometrical interpretation[§]. Alternatively, the endmasses may be *physically* defined, either as the strength of the boundary δ-function terms that must be added to eq.(11) to ensure $T_{\mu\nu}$ conservation, or as Lagrangian parameters in an action formulation (see eq.(43) below). At first glance, it is not clear that the geometrical definition yields τ- independent masses at P and Q, but this is indeed the case, as proven in appendix C. The endmasses must satisfy some other constraints, discussed in section 5.

4. Integration using the Liouville Mode.

We now show how the dynamics of the self-gravitating string can be largely reduced to solving a single nonlinear wave equation on the worldsheet.

Let us define the induced Liouville mode, ϕ, as follows: ($E < 0$ by eq.(7))

$$E(u) = -e^{\phi(u)} \tag{23}$$

where E is the induced orthonormal scale factor (eq.(6)). As shown in appendix B (part II), the equations of motion imply the following differential equation for ϕ

† Also, we again assume that S does not self-intersect.

‡ Ref. 8 also considers the case where a section of the interior metric has the topology of a punctured disk.

§ We impose the requirement that the end-masses be non-negative; the origin of this constraint is not geometrical, but rather physical — to ensure energy positivity at the ends of the string.

(eq. (B.22)):

$$\partial^2\phi + se^{-\phi} = \frac{\kappa^2}{4}e^{\phi} \tag{24}$$

where s is a sign which, at least locally on the worldsheet, is undetermined. It is important to note that eq.(24) does not necessarily hold in the worldsheet coordinates defined below eq.(6). That is, (24) always holds locally for *some* orthonormal parametrization u^a, but in general, this will not be the same u^a for which $\sigma = \ const$ at P and Q¶.

In the flat case $\kappa = 0$, (24) reduces to the Liouville equation when $s = -1$. In the general case, eq.(24) is not soluble in closed form. It appears surprising, at first glance, that ϕ satisfies the Liouville equation in the flat case. But actually, this is a straightforward consequence of the equation of motion, which becomes $\partial_+\partial_- x = 0$, and of the three-dimensionality of target space; Let us demonstrate this fact. Since $\partial_\pm x$ are null vectors and $\partial_\pm x$,n span three dimensional space, it is easy to see that

$$\partial_\pm x \times \partial^2_\pm x = \beta_\pm \partial_\pm x$$

and $\beta_\pm$, defined in (B.5b), depends only on $u^\pm$, since so does $\partial_\pm x$*. Clearly

$$\partial^2 E = \partial^2_+ x \cdot \partial^2_- x \ .$$

Now, $\partial^2_\pm x$ must be a linear combination of the two vectors $\partial_\pm x$ and n, so we easily find

$$E\partial^2 \ln(-E) = (\partial^2_+ x \cdot n)(\partial^2_- x \cdot n) = \beta_+\beta_-,$$

and by locally setting $\beta_\pm$ to the signs $s_\pm$ via appropriate reparametrizations (that preserve orthonormality), we recover eq.(24) in this (flat) case.

¶ For the closed string, however, the functions $\beta_+(u_+)$, $\beta_-(u_-)$ which govern the requisite reparametrization (see eqs.(B.18)-(B.19)) are themselves periodic with period 2π, so for closed string the new orthonormal coordinates respect the boundary conditions.

* A less trivial fact, proven in appendix B, is that $\partial_\mp\beta_\pm = 0$ even for $\kappa \neq 0$.

Once the Liouville mode ϕ is known, we can (in principle) solve for the entire configuration $\{x^\mu(u)\}$,$\{\hat{x}^\mu(u)\}$, as follows**. Since $\beta_\pm = s_\pm$, we have by virtue of eqs.(8) and (B.10),

$$\partial_\pm x \times \partial_\pm^2 x = \pm s_\pm \partial_\pm x \tag{25a}$$

In appendix B (part III), it is shown how to express $x^\mu(u)$ in terms of two functions on S, namely $\gamma_\pm(u)$. (See eq.(B.25)). We can 'almost' determine these two functions: they satisfy the equations

$$\partial_\pm \gamma_\mp = -\frac{\kappa}{2} E s_\mp \partial_\mp \gamma_\mp, \tag{25b}$$

which tell us the *directions* of the worldsheet vectors $\partial_a\gamma_+$,$\partial_a\gamma_-$ at each point u^a on S. This can be used to write a nonlinear first-order differential equation for the function $u^- = u^-(u^+)$ describing any $\gamma_+ =$ *const* line on the worldsheet, and similarly another such equation for $\gamma_- =$ *const* lines. (These differential equations are not in general soluble in closed form.). Once the $\gamma_\pm = c_\pm$ lines are known ($c_\pm$ constants), say in the form $u^- = f_\pm(u^+; c_\pm)$, we can solve for $c_\pm$ as functions of u^a, and this gives a *particular* solution of eqs.(25b); let us denote it by $\Gamma_\pm(u)$. The general solution of (25b) is then,

$$\gamma_\pm = h_\pm(\Gamma_\pm(u)) \tag{26}$$

where $h_\pm$ are two arbitrary functions of a single argument; the eq.(B.28) then furnishes a functional equation for these two unknown functions***. Thus, we see that knowledge of $E(u)$ 'almost' determines the configuration $x^\mu(u)$, in the sense that the remaining equations (two first-order differential equations and one functional equation) have, as unknowns, functions of a *single* variable. The equations which yield $\hat{x}^\mu(u)$ from E are the same, except that κ is replaced by $-\kappa$.

** We describe the procedure for the bottom worldsheet; the determination of $\hat{x}(u)$ proceeds in the same way. In addition, in what follows we ignore boundary conditions and global aspects of the solution.

*** Since eq.(B.28) holds for any u^+,u^-, it should in general be powerful enough to determine two functions of a *single* variable.

5. Global Structure and Asymptotic Form of Three-Metric.

In four-dimensional gravity, and for a physical system sufficiently localized in space to have an asymptotically Minkowskian spacetime at spatial infinity, the ADM procedure[10] can be used to define and compute the total momentum and angular momentum of the system, in terms of the approach to asymptotia. A similar definition can be employed in 2+1 dimensional gravity, but with two differences:

I. Firstly, an 'asymptotic' form of the metric holds *exactly* in the region exterior to any world-tube that contains the support of $T_{\mu\nu}$[****]. By 'asymptotic form' we mean a standardized metric, depending on a finite number of degrees of freedom, and in a *single* coordinate patch[†] which encompasses the entire exterior of the said worldtube.

II. Such an asymptotic form does *not* approach a Minkowski metric at spatial infinity.

The standardized asymptotic form depends on two scalar constants. This can be understood as follows[*]: In the region exterior to the worldtube, consider a Minkowskian coordinate system, such as our $\{z\}$. Since the exterior is not simply connected, transition functions are needed from M to itself when travelling around a noncontractable loop; these functions are a Poincare transformation, which is just eq.(2). The Lorentz transformation $\bar{L}$ can either be boosted to become a pure rotation, in which case we will call it 'rotationlike', or it cannot — in the latter case it is 'boostlike'. The boostlike case is the Gott regime, where arbitrarily large closed timelike curves occur in the exterior region; we will return to this case later. Here, let us restrict attention to the nonpathological, rotationlike case. By

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**** Such a worldtube clearly exists for our system.

† *Neither* of the three coordinate systems $\{z\}$,$\{x\}$ and $\{\hat{x}\}$ is an example of such a coordinate system.

∗ For a more thorough treatment of the issue of asymptotic spacetime in three-dimensional gravity, including the implications for a canonical formulation (which is relevant to our sections 8 and 9), see Brown and Henneaux[11] . These authors allow a cosmological constant.

combining a shift with a boost, we can then choose new Minkowski coordinates in which

$$\bar{L} = R(2\pi(1-a)),\ \bar{b} = (2\pi\beta, 0, 0)\ , \tag{27}$$

where $R(\alpha)$ denotes a spatial rotation by angle α. This is the simplest form to which eq.(2) can be brought, and the following single-patch exterior metric encodes† it:

$$ds^2 = -(dt + \beta d\varphi)^2 + dr^2 + a^2 r^2 d\varphi^2\ . \tag{28}$$

This can be taken as the standardized asymptotic metric, for the case of rotationlike $\bar{L}$.

Returning to a general Minkowski frame, the three-dimensional Lorentz transformation $\bar{L}$ is described by a three-vector $\bar{w}^\mu$, as follows:

$$(\bar{L})_{\mu\nu} = (1 + \frac{1}{2}\bar{w}^2)\eta_{\mu\nu} - \sqrt{1 + \bar{w}^2/4}\ \bar{w}^\lambda \epsilon_{\mu\lambda\nu} - \frac{1}{2}\bar{w}_\mu \bar{w}_\nu \tag{29}$$

$\bar{L}$ is rotationlike (boostlike) when $\bar{w}^2$ is negative (positive). The three-vectors $\bar{w}$ and $\bar{b}$, which characterize the exterior metric, are the general-relativistic generalizations of the flat-space concepts of *momentum* and *angular momentum*, respectively; in the next section we will see that the two sets of parameters are in fact proportional to each other in the flat limit•.

Finally, we briefly discuss other global aspects of the Nambu-Goto-Einstein (NGE) spacetimes. As pointed out in section 2, the global construction of these spacetimes from $\{x(u),\ \hat{x}(u)\}$ (which was described in sections 2-3) breaks down

† Choosing the transition surface T to run along $\varphi =$ *const*, the coordinates (t, r, φ), valid throughout the exterior, are related to the Minkowski polar coordinates (t_m, r_m, φ_m) by

$$r_m = r,\ \varphi_m = \varphi,\ t_m = t + \beta\varphi\ .$$

The exterior region with metric eq.(28) is causal, provided this region is chosen to lie outside $r^2 = \beta^2/a^2$.

• This interpretation of the transformation eq.(2), was first elucidated in ref. 4.

if the string intersects itself. Thus, some constructions $\{x(u),\ \hat{x}(u)\}$ might not be physical; conversely, physical spacetimes with self-intersecting strings will not be globally reproduced by our construction.

Furthermore, even with no self-intersections, some constructions do not result in a spacetime with the assumed R^3 topology. As an illustration, consider two free, static masses with zero string tension[4]*. The top and bottom copies of the surface $S+T$ in $\{z\}$ coordinates, may be chosen as broken planes. Equal-time sections of two such spacetimes are depicted in Figs.3a and 3b, which are special cases of Fig.2. In both cases, the deficit angle at P satisfies

$$0 < m^{(P)} < \pi \ .$$

But at Q, Fig.3a shows the case $0 < m^{(Q)} < \pi$, whereas in Fig.3b, $\pi < m^{(Q)} < 2\pi$. It is easily seen that the excessive deficit angle at Q closes up the spatial section, so the spacetime of Fig.3b does not have the topology R^3. Returning to the case $m^{(P)} = m^{(Q)}$, the full constraint when $\kappa = 0$ is clearly $0 \leq m \leq \pi$. For nonzero κ, the upper bound may change, although in any case $0 \leq m < 2\pi$. The upper bound is configuration dependent, and is determined by the requirement that the top copy of $S+T$ will not turn back on itself to intersect the bottom copy (as occurs in Fig.3b).

This test extends to a general criterion, which determines whether some particular construction $\{x(u),\ \hat{x}(u)\}$ corresponds to a nonintersecting NGE space of topology R^3. The criterion is that *the two copies of $S+T$ are only allowed to cut themselves or each other in ways that do not close off any region of the section (Fig. 2) that is finite in $\{z\}$ coordinates.*

This criterion imposes constraints, both on the functions $x(u)$,$\hat{x}(u)$ and on $\bar{L}$. When $\bar{L}$ is rotationlike, the latter constraint is (in the frame for which eq.(27)

∗ This special case is dealt with in section 6.

holds)

$$a > 0 \ .$$

In the special case of Figs.3 (static masses and $\kappa = 0$), this simplifies to the condition $m < \pi$.

Note that the above criterion *allows* sections to have the structure of a plane with multiple Riemann sheets. Any intersection of a copy of S with itself, that conforms to the criterion, does *not* correspond to a self-intersection of the physical string.

6. Perturbation Theory and Special Limits.

The developments in this section depend on the fact that, for small κ, the equations of motion and boundary conditions may be solved in a perturbative series. In what follows, we shall restrict ourselves to the first order in κ. All components in this section are in $\{z\}$ coordinates continued around P (see section 2).

We solve the orthonormality condition and eq.(17) by expanding (b is a constant three-vector),

$$x(u) = b + A(u^+) + \tilde{A}(u^-) - \frac{\kappa}{2} A(u^+) \times \tilde{A}(u^-) + O(\kappa^2) \tag{30a}$$

where the vector functions A,$\tilde{A}$ are arbitrary, except that

$$(A')^2 = (\tilde{A}')^2 = 0. \tag{30b}$$

For $\kappa = 0$, this reduces to the standard flat-string separation into left- and right-moving modes.

$\hat{x}$ can be expanded in the same form, with A,$\tilde{A}$ replaced by different functions C,$\tilde{C}$; b replaced by $\hat{b}$; and $\kappa \rightarrow -\kappa$. However, due to the continuity condition eq.(6), the scalar product $A'(u^+) \cdot \tilde{A}'(u^-)$ must be the same as its top side counterpart, to $O(\kappa^0)$. Therefore we must have

$$L \cdot C(u^+) = A(u^+) + const. + O(\kappa) \ , \tag{31a}$$

$$L \cdot \tilde{C}(u^-) = \tilde{A}(u^-) + const. + O(\kappa) \tag{31b}$$

where L is a constant Lorentz transformation.

From eqs.(5),(6),(18) and (31), we find to order κ:

$$\begin{aligned} L \cdot \hat{x}(u) = \hat{b} + rA(u^+) + \frac{1}{r}\tilde{A}(u^-) - \kappa B(u^+) - \kappa\tilde{B}(u^-) \\ + \frac{\kappa}{2}A(u^+) \times \tilde{A}(u^-) + O(\kappa^2) \end{aligned} \tag{31c}$$

where $r = r(\kappa) = 1 + O(\kappa)$ is a constant to be determined, and

$$B' = -A \times A', \ \tilde{B}' = \tilde{A} \times \tilde{A}'. \tag{31d}$$

The functions B,$\tilde{B}$ are determined up to constant additive three-vectors, which can be absorbed into $\hat{b}$.

Unlike the Lorentz transformation $\bar{L}$ (eq. (2)), which is a physical observable, L is not: it is easy to see that x,$\hat{x}$ are unchanged under an order-κ change in L, if accompanied by suitable $O(1)$ shifts in A,$\tilde{A}$, b and $\hat{b}$.

In order to determine $r(\kappa)$, it is necessary to impose the scalar equation of motion, eq.(22). Using eqs. (8a),(8c) as well, we find that

$$r(\kappa) = 1 + O(\kappa^2) \tag{31e}$$

.

As shown in appendix B (part (II)), the content of the scalar equation of motion is only a single numerical condition, once the vector equations of motion are used; so it is not surprising we are able to adjust the constant $r(\kappa)$ so that eq.(22) is perturbatively satisfied.

Boundary Conditions. Specializing to the case of open string, we next impose on the perturbative solution (eqs.(30)-(31)) the boundary conditions—eqs.(1),(3). We will only treat (1) in detail, since the handling of the Q boundary condition proceeds in exactly the same way as that at P.

Eq.(1) reads, after differentiating it w.r.t. τ,

$$(L-1)\cdot(A'+\tilde{A}') - \frac{\kappa}{2}(L+1)(A\times\tilde{A})' = \kappa(A\times A' - \tilde{A}\times\tilde{A}') + O(\kappa^2)\ , \qquad (32a)$$

where the arguments of A,$\tilde{A}$ are τ, since $\sigma = 0$ at P.

The Flat Limit. This is the limit in which the end-masses m tend to zero, while the ratio m/κ is held fixed. (For simplicity, we will assume throughout that the two endmasses are equal.). This includes the special case of the standard open string, for which the ratio vanishes; but we will emphasize here the case of massive end-points.

Since an endmass corresponds to a deficit angle (see section 3), we expect the matrix $L-1$ to be of order m. Thus, the flat limit corresponds to taking $L\to 1$ while holding $(L-1)/\kappa$ fixed. Letting

$$L\cdot a \approx a - w\times a\ , \qquad (32b)$$

eq.(32a) becomes

$$(w-\kappa(\tilde{A}-A))\times(A'+\tilde{A}') = O(\kappa^2) \qquad (33a)$$

and therefore

$$\frac{w^\mu}{\kappa} + \psi(A'+\tilde{A}')^\mu \approx \tilde{A}^\mu - A^\mu \qquad (33b)$$

where ψ is a scalar function of τ. We decompose

$$\frac{w^\mu}{\kappa} \equiv \bar{\rho}\bar{c}^\mu, \quad \bar{c}^2 = -1 \tag{33c}$$

and set $\psi = -\bar{\rho}\bar{\varphi}$. In the flat limit, $\bar{\rho}$ and $\bar{c}$ are finite, and eq.(33b) becomes

$$\bar{\varphi}(A' + \tilde{A}') = \bar{c} - \frac{1}{\bar{\rho}}(\tilde{A} - A) \; . \tag{33d}$$

This implies that the square of the vector $\bar{\varphi}(A' + \tilde{A}')$ is τ-independent, and thus $\bar{\varphi} = \bar{\beta}\varphi$, where $\bar{\beta}$ is some constant and $\varphi(\tau)$ is given by eq.(D.9b). Thus eq.(33d) is exactly (D.9a), with the identifications

$$c^\mu = \bar{c}^\mu/\bar{\beta}, \;\; \rho = \bar{\rho}\bar{\beta}, \tag{33e}$$

and we have in the flat limit

$$\frac{w^\mu}{\kappa} \rightarrow \rho c^\mu \; . \tag{34a}$$

We have thus recovered the flat boundary condition at P. In the corresponding procedure at Q, the arguments of A,$\tilde{A}$ are $\tau+\pi$,$\tau-\pi$ respectively, and the matrix L is replaced by

$$L^{(Q)} \equiv L(\bar{L})^{-1} \; ; \tag{35a}$$

In the flat limit,

$$L^{(Q)} \cdot a \approx a + w^{(Q)} \times a \tag{35b}$$

for any vector a, and the counterpart of eq.(34a) is

$$\frac{w^{(Q)\mu}}{\kappa} \rightarrow \rho^{(Q)} d^\mu \; . \tag{34b}$$

For the flat string, $\rho = m^{(P)}/\kappa$ and $\rho^{(Q)} = m^{(Q)}/\kappa$, so these two numbers are equal when (as we assume) $m^{(P)} = m^{(Q)} = m$:

$$\rho = \rho^{(Q)} = m/\kappa \; . \tag{34c}$$

The constant vectors c^μ,d^μ are the three-velocities of P,Q respectively, in the limit of vanishing flat-string tension ($\rho \rightarrow \infty$). We conclude that the boundary conditions at both P and Q reduce to their correct flat limits. The equations of motion themselves simply become

$$x(u) = \hat{x}(u), \ \partial^2 x = 0,$$

which is the standard string equation of motion in orthonormal gauge.

Total Momentum and Angular Momentum. As discussed in section 5, and as first found in ref. 4, the Lorentz transformation $\bar{L}$ and the shift $\bar{b}$ are closely related in the flat limit to the system's total momentum and angular momentum, respectively. We use the perturbative results to make this statement more precise.

From eqs. (29),(32a),(34)-(35) we obtain to first order in κ,

$$\bar{w}^\mu \approx m(c^\mu + d^\mu) \tag{36}$$

On the other hand, the flat total momentum of the string is,(eq.(D.3))

$$p^\mu_{total} = \kappa \int_0^\pi d\sigma \frac{\partial x^\mu}{\partial \tau} + m\frac{dx^\mu}{ds}(P) + m\frac{dx^\mu}{ds}(Q) \ . \tag{37}$$

Finally, in the flat limit eq.(30a) becomes

$$x(u) = A(u^+) + \tilde{A}(u^-) \tag{38}$$

By using eqs.(D.5),(D.9) and the corresponding equations at Q, together with eqs.(36)-(38), we obtain to first order in κ,

$$\bar{w}^\mu \approx p^\mu_{total} \tag{39a}$$

in our geometrical units. Slightly more involved, yet straightforward, algebra yields for the shift vector:

$$\bar{b}^\mu \approx J^\mu_{total} \; , \tag{39b}$$

where the three-vector J_{total} is dual to the total flat angular-momentum tensor of the system:

$$(J_{total})_\mu = \epsilon_{\mu\alpha\beta}\{\kappa \int\limits_0^\pi d\sigma x^\alpha \frac{\partial x^\beta}{\partial \tau} + \sum_{P,Q} m(x^\alpha \frac{dx^\beta}{ds})\} \; . \tag{39c}$$

Here ds is an interval of proper time at either (massive) string-end. The above analysis can be repeated for the open string with lightlike endpoints (the case m=0), or for a closed string, and one obtains the same relations (39a)-(39b). In both cases, J_{total} and p_{total} will not contain any boundary terms.

Zero Tension Limit. This we define to be the limit where $\kappa \rightarrow 0$ with L,$L^{(Q)}$ fixed; actually, we shall consider a small but nonvanishing κ, so that the above perturbative expansion may be used. Thus, the string tension is small in geometrized units⋆, but the endmasses give rise to finite deficit angles in their respective rest frames. The boundary condition at P is again given by eq.(32a), where now $(L-1)$ is of order 1. As usual, a corresponding condition holds at Q.

These boundary conditions can be solved in power series in κ; the procedure is quite similar to the large-ρ expansion of the flat string, described in appendix D, and will not be presented here. When $\kappa = 0$, the solution of eq.(32a) is $A'(\tau) + \tilde{A}'(\tau) || w$; this is equivalent to the $\rho \rightarrow \infty$ limit of the flat string, and implies that the worldline P is rectilinear† — it is a free mass (as is Q). Since the rest-frame deficit angles at P and Q are m, it is easy to see that

$$w_\mu = 2 \sin \frac{m}{2} c_\mu, \;\; w^{(Q)}_\mu = 2 \sin \frac{m}{2} d_\mu \; , \tag{40}$$

where L is related to w^μ via the same relation (eq.(29)) as holds between $\bar{L}$ and $\bar{w}^\mu$,

⋆ Since κ has inverse-length dimensions in these units, what is meant is that $\kappa D << 1$, for some length or time scale D typical of the system.

† See comment below eq.(D.9b).

and $(L^{(Q)})^{-1}$ is again related in this way[‡] to the vector $w^{(Q)}$. The vectors c,d are the constant three-velocities of P and Q, respectively. m must satisfy $0 \leq m \leq \pi$ to ensure the R^3 topology of spacetime (see section 5).

Rearranging eq.(35a), we have

$$\bar{L} = (L^{(Q)})^{-1} L \tag{41}$$

From eqs.(40)-(41), we can determine the relative velocity that must be imparted to the two masses in order to enter the Gott regime: specializing the $\{z\}$ coordinate system to a center of mass frame, $c^0 = d^0$,$c^i = -d^i$ and we obtain:

$$\bar{w}^2 = 4(w^0)[\frac{(w^0)^2}{4} - 1] \tag{42}$$

The vector $\bar{w}$ becomes spacelike, and $\bar{L}$ ‘boostlike’, when $w^0 > 2$; In this regime, closed timelike curves exist sufficiently far away from the two masses[§], and there are CTC’s of *arbitrarily* large extent in space and time.

On the other hand, when $\bar{w}$ is timelike, *no* CTC’s or CLC’s exist in the zero-tension limit, and spacetime is completely causal[5].

When a small, nonzero string tension is turned on, the κ-perturbative corrections do not change these causal properties of the Gott spacetime. We thus conclude that slightly accelerating the two particles, at least via the mechanism of tying them to opposite ends of a Nambu-Goto string, *cannot* produce CTC’s in a previously-causal spacetime, since a spacetime with arbitrarily large CTC’s (in the above sense) cannot be initially causal.

‡ Eqs. (32), (35b) are the linearized versions of these relations.

§ Assuming that the two masses have a nonvanishing impact parameter, ‘sufficiently far’ means relative to the impact parameter. The CTC’s must also have a sufficiently large extent, in center-of-mass time, around the fiducial time — the instant in which the masses are at minimal distance from each other.

7. Causal Structure of Spacetime.

We have assumed that¶ some neighborhood of the worldsheet S has spacelike equal-time sections; such a neighborhood has a normal causal structure — that is, no closed timelike/lightlike curves exist within it.

The entire spacetime manifold, on the other hand, may have closed timelike curves (CTC's) or closed lightlike curves (CLC's), although the energy-momentum tensor (eq.(11), plus the boundary contributions) satisfies all standard energy-positivity criteria* and there are no horizons**. An example is when $\kappa << 1$ and $\bar{L}$ is boostlike (the Gott regime— see section 6). In that case arbitrarily large*** CTC's exist, but arbitrarily *small* CTC's do not exist. Another example of such a spacetime (also in $2 + 1$ dimensions) is Godel's solution[9] for a homogenous dust universe. However, one might take the attitude that such spacetimes should be legislated away. A more interesting structure is one in which the energy conditions are satisfied, there are 'naked' CTC's or CLC's****, *and* these curves do not extend before a finite, globally-defined time. In such a spacetime, the CTC's are 'produced' in a universe which was, until that epoch, causal*****. Familiar examples of such spacetime are the four-dimensional empty Taub-NUT space, and its two-dimensional analog — the Misner space[9]******.

If some physical process is found to generate CTC's which are accessible to (macroscopic or microscopic) observers, it will become important to ascertain the

¶ See discussion in section 2.
* These are the *weak, strong* and *dominant* energy conditions[9].
** The two-wormhole 'time machine' of Morris *et al*[12] is an example of a $3 + 1$-dimensional spacetime with CTC's, in which the weak energy condition is violated. The Kerr-Newman black hole with $a^2 + e^2 \leq m^2$ ($a \neq 0$) has CTC's, but they are hidden behind an event horizon[13] .
*** in both space and time; see last footnote of section 6.
**** That is, outside any horizon.
***** The significance of the ability to *produce* CTC's at finite time, was emphasized to me by Lenny Susskind.
****** These two manifolds, however, are geodesically incomplete near the CTC-production epoch (and at the time when CTC's end). Geodesic incompleteness (timelike or null) is a kind of singularity, in that it dooms some observers to finite affine lifetimes.

physical ramifications of these curves. Back-reaction effects might conspire to prevent CTC's from forming; alternatively, it might become necessary to study self-consistent propagation of fields in spaces with CTC's[14] .

Returning to the self-gravitating string, an interesting question is whether there exists a regime for which CTC production can occur, in a previously causal spacetime. As we learned in the previous section, such solutions can only exist, if at all, for large κ. Also, such a solution must have a rotationlike $\bar{L}$, since otherwise some CTC's are *guaranteed* to extend into past infinity, just as for the Gott solution. We do not yet know whether there are such CTC-producing solutions of the self-gravitating string†.

Finally, we note that unusual causal structures are of interest not only classically or semiclassically. Most of the work that has been done to date on quantum gravity uses, in the sum over spacetime histories, metrics of euclidean signature. An appeal is then made to some version of Wick rotation, in order to make statements about Minkowskian spacetime. It is thus important to attempt the direct study of path integrals over Minkowskian metrics, and such an integral may need to include a sum over causal structures, as well as over topologies.

† In the absence of general theorems to the contrary, it remains possible that small accelerations applied to two four-dimensional cosmic strings might enable the artificial generation of CTC's, as Gott suggests. But as a result of our perturbative analysis in section 6, we know that (for small accelerations) either the acceleration mechanism is different from the one we have considered, or the four-dimensionality of spacetime would enter in a crucial way in such a procedure — for instance, through the finite length of the cosmic string, or horizon formation. In a four-dimensional interpretation of our spacetime, the endmasses become cosmic strings, whereas *our* string should be re-interpreted as a membrane connecting them.

8. An Action Formulation.

In this section we shall start with the Nambu-Goto-Einstein action, eliminate the nonpropagating metric, and end up with a quadratic* action, the dynamical degrees of freedom all living on the worldsheet. These degrees of freedom are $x, \hat{x}, n$ and $\hat{n}$, and Lagrange multipliers are introduced to enforce geometric constraints, which are also quadratic in the worldsheet fields. For simplicity, we shall restrict ourselves in this section to spacetimes for which $\bar{L}$ is rotationlike (see section 7).

The Nambu-Goto-Einstein action, in an orthonormal worldsheet parametrization, is the sum of the three-dimensional Einstein-Hilbert action, the bulk worldsheet action, and terms for the endmasses†:

$$\begin{aligned} S_{NGE}(x,\hat{x},g) = &-\frac{1}{2}\int d^3x R\sqrt{-g} + \kappa\int du^+ du^- g_{\alpha\beta}\partial_+ x^\alpha \partial_- x^\beta \\ &+ \sum_{P,Q} m \int d\tau \sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}} \end{aligned} \tag{43}$$

The extremization of S_{NGE} does not quite yield the gravitational equation of motion: the discrepancy stems from the surface terms in the variation of the first (Einstein-Hilbert) term, which arise when integrating this variation by parts. Such surface terms depend only on the asymptotic form of the metric (see section 5). In four dimensions, and for asymptotically Minkowskian spacetime, the asymptotic form** is characterized by the total momentum of the system, and it is well known[10] how to modify the action and the Hamiltonian formalism to accomodate these degrees of freedom. Similarly, in three spacetime dimensions, the surface terms depend only on $\bar{w}$ and $\bar{b}$.

In what follows we shall circumvent this issue by constraining the variation $\delta g_{\alpha\beta}$ to be such as to preserve $\bar{w}$; as a consequence, the scalar equation (22) will

* The action is actually cubic if one views the Lagrange multipliers as fields.

† The Lagrangian mass parameter m in this action, is the same geometrical parameter given on shell by eq.(C.1).

** Or rather, its leading deviation from the Minkowski metric.

not result from extremizing our action, and must be put in by hand[•⋆]. Before leaving behind the issue of the surface terms, however, we make two observations. One is that although we fix a *vector* $\bar{w}^{\mu}$, only one scalar equation will be lost. The reason is that, by a suitable coordinate transformation, the asymptotic form can be chosen to be eq.(28), for which $\bar{w}$ has a single component $\bar{w}^0$ (see eq.(27)). In other words, two of the missing equations of motion are identities, thanks to three-dimensional general coordinate invariance. The other observation is that we need only fix $\bar{w}$ (not $\bar{b}$) in varying the metric, because the surface terms do not depend on $\bar{b}$. In fact, the surface terms are[9][†]

$$-\frac{1}{2}\int g^{\alpha\beta}\delta R_{\alpha\beta}\sqrt{-g}d^3x = \frac{1}{2}\int_{\Sigma}(g^{\alpha\beta}\delta\Gamma^{\gamma}_{\alpha\beta} - g^{\gamma\alpha}\delta\Gamma^{\lambda}_{\alpha\lambda})d\sigma_{\gamma} \ , \tag{44}$$

and using the asymptotic metric (28), we find that (44) equals

$$-2\pi\delta a\int dt$$

which does not depent on β.

We are now ready to eliminate the metric from the formalism. In the coordinate system $\{x\}$, $g_{\mu\nu}$ is Minkowskian on the worldsheet, so it drops out of the last two terms in eq.(43). The first term can be computed as follows. First, the procedure of section 3 (eqs.(19)-(22)) is repeated *off-shell*, i.e. assuming only orthonormality and metric continuity (that is, not using any equations of motion). This yields $P^{(t)}$, which together with eq.(13b) finally gives

$$-\frac{1}{2}\int d^3xR\sqrt{-g} = 2\int du^+du^-(n\cdot\partial^2x - \hat{n}\cdot\partial^2\hat{x}) \ . \tag{45a}$$

To summarize, $g_{\mu\nu}$ is eliminated, purely geometrically, in terms of the top and bottom worldsheets, and when the action S_{NGE} is expressed in terms of x and $\hat{x}$,

• Recall (appendix B) that the extra content in this equation is just one numerical condition.

⋆ For a general analysis of the canonical formulation of three-dimensional gravity which treats in detail the asymptotic degrees of freedom, see ref. 11.

† Here Σ is some cylindrical surface enclosing the worldsheet, and $d\sigma$ an outward-pointing normal to Σ with a magnitude equal to a surface-element area.

the result is:

$$S_1(x,\hat{x}) = 2\int du^+du^-(n\cdot\partial^2x - \hat{n}\cdot\partial^2\hat{x}) + \frac{\kappa}{2}\int du^+du^-(\partial_+x\cdot\partial_-x + \partial_+\hat{x}\cdot\partial_-\hat{x}) + \sum_{P,Q} m\int d\tau\sqrt{-(\frac{dx}{d\tau})^2} \tag{45b}$$

where the second (bulk string) term has been rendered top-bottom symmetric by use of eq.(6), and $n,\hat{n}$ are given by eqs.(8).

In (45b), the configuration $x(u),\hat{x}(u)$ is free, except for the orthonormality and metric-continuity constraints, which we rewrite,

$$(\partial_\pm x)^2 = (\partial_\pm\hat{x})^2 = 0 \tag{45c}$$

$$\partial_+x\cdot\partial_-x = \partial_+\hat{x}\cdot\partial_-\hat{x} \tag{45d}$$

and the boundary conditions, eqs.(1)-(3), which we rewrite thus:

$$x(P) = \hat{x}(P)\ , \tag{46a}$$

$$\frac{d}{d\tau}x(Q) = \bar{L}\cdot\frac{d}{d\tau}\hat{x}(Q)\ . \tag{46b}$$

We have replaced eq.(3) by its τ-derivative, since as explained above, we are only holding $\bar{w}$ fixed, not $\bar{b}$, so the weaker constraint (46b) is appropriate. The bulk constraints⋆, eqs.(45c)-(45d), and the end constraints eqs.(46), will be accounted for by adding to the action S_1 the appropriate Lagrange-multiplier terms. For the case of the closed string, the last term in S_1 is absent, and the boundary conditions eqs.(46) are modified as described in section 3.

⋆ We use the term *bulk* to refer to the interior of the worldsheet, i.e. away from the boundaries P and Q.

The action (45b) is nonpolynomial in x and $\hat{x}$, since $n, \hat{n}$ are; this makes it awkward to work with, and renders the task of quantization rather hopeless. Fortunately, this problem can be avoided if we elevate $n(u), \hat{n}(u)$ to the status of additional dynamical fields on the worldsheet; in this picture, however, we must impose the further bulk constraints,

$$n^2 = \hat{n}^2 = 1 \ , \tag{47a}$$

$$n \cdot \partial_\pm x = \hat{n} \cdot \partial_\pm \hat{x} = 0 \ . \tag{47b}$$

all the constraints (eqs. (45c)-(47b)) are quadratic, and so is the action S_1 in the new picture — except for the endmass terms†.

The quadratic action, obtained by adding to S_1 all the constraints weighted by their Lagrange multipliers, is:

$$\begin{aligned} S_2 = 2\int du^+ du^- (n \cdot \partial^2 x - \hat{n} \cdot \partial^2 \hat{x}) + \int du^+ du^- [(\frac{\kappa}{2} + \lambda_1)\partial_+ x \cdot \partial_- x \\ + (\frac{\kappa}{2} - \lambda_1)\partial_+ \hat{x} \cdot \partial_- \hat{x}] + \int du^+ du^- [\sum_\pm \lambda_2^\pm (\partial_\pm x)^2 + \sum_\pm \hat{\lambda}_2^\pm (\partial_\pm \hat{x})^2 \\ + \lambda_3 (n^2 - 1) + \hat{\lambda}_3 (\hat{n}^2 - 1) + \sum_\pm \lambda_4^\pm n \cdot \partial_\pm x + \sum_\pm \hat{\lambda}_4^\pm \hat{n} \cdot \partial_\pm \hat{x}] \\ + \int d\tau [\mu_1(\tau) \cdot (\hat{x}(P) - x(P)) + \mu_2(\tau) \cdot (\frac{dx(Q)}{d\tau} - \bar{L} \cdot \frac{d\hat{x}(Q)}{d\tau})] \\ + \sum_{P,Q} m \int d\tau \sqrt{-(dx/d\tau)^2} \end{aligned} \tag{48}$$

The multipliers $\lambda_1, \lambda_2^\pm, \hat{\lambda}_2^\pm, \lambda_3, \hat{\lambda}_3, \lambda_4^\pm, \hat{\lambda}_4^\pm$ are scalars, whereas μ_1, μ_2 are three-vectors.

† That is not a problem, however, since we would only be interested in quantizing the self-gravitating string for a closed string, or for an open string with massless ends; in either case there are no endmass terms.

In the remainder of this section, we shall ignore the boundary terms in S_2, and analyze only the bulk equations of motion resulting from varying the action (48). Before embarking upon this analysis, we briefly summarize the results. We shall find that these equations of motion are consistent with the correct vector equations (17)-(18). They must, in fact, be *equivalent* to them, since the extremization of S_2 is equivalent to that of the original S_{NGE}; from our direct analysis of (48), however, we were only able to show that this equivalence is plausible. We also find that the numerical parameter r_1 can be chosen arbitrarily, at least if one only varies the fields $x, n, \hat{x}, \hat{n}$ in the bulk of the worldsheet[‡]. Thus, the scalar equation $r_1 = 1$ is not reproduced by variation in the bulk. As discussed in the beginning of this section, we attribute this missing information to the fact that $\bar{w}$ must be held fixed in the variation; the condition $r_1 = 1$ must be imposed by hand.

We now proceed with the bulk extremization[§] of S_2.

Variation w.r.t. $n(u)$ gives,

$$2\partial^2 x + 2\lambda_3 n + \sum_{\pm} \lambda_4^{\pm} \partial_{\pm} x = 0$$

so from orthonormality,

$$\lambda_4^{\pm} = 0 \; . \tag{49a}$$

Thus $\partial^2 x = -\lambda_3 n$; whence, in the notation of appendix B,

$$\lambda_3 = -kE \tag{49b}$$

Similarly, $\delta / \delta \hat{n}$ yields

$$\hat{\lambda}_4^{\pm} = 0, \; \hat{\lambda}_3 = \hat{k} E \; . \tag{49c}$$

‡ We have not yet checked how the worldsheet surface terms affect this statement, but it seems they will not determine r_1, since eq.(22) was derived from *local* analysis of the equations of motion.

§ Integrations by part can be freely performed on the worldsheet, since we are ignoring boundary terms.

Extremizing S_2 w.r.t. x, we find

$$\partial^2 n - (\frac{\kappa}{2} + \lambda_1)\partial^2 x - \sum_{\pm} \partial_\pm(\lambda_2^\pm \partial_\pm x) - \frac{1}{2}\sum_{\pm}(\partial_\pm \lambda_1)\partial_\mp x = 0 \tag{50}$$

From this vector equation, three scalar equations are obtained, by dotting it with $\partial_\pm x$ and with n. Using the off-shell result (B.5), the $\partial_\pm x$ components of eq.(50) become:

$$2\partial_\mp(\beta_\pm - E\lambda_2^\mp) = E\partial_\pm \lambda_1 \ , \tag{51}$$

whereas the n component is

$$\partial_+ n \cdot \partial_- n = -(\frac{\kappa}{2} + \lambda_1)kE + \sum_{\pm} \lambda_2^\pm (\partial_\pm n)\cdot(\partial_\pm x) \ . \tag{52}$$

Using eq.(B.5a) again, this becomes

$$-2kE^2 f = \beta_- g_+ + \beta_+ g_- \ , \tag{53a}$$

where we define

$$f \equiv \lambda_1 + k + \frac{\kappa}{2} \ , \tag{53b}$$

$$g_\pm \equiv \beta_\pm - 2E\lambda_2^\mp \ . \tag{53c}$$

Next, use eqs.(B.8a) and (53b)-(53c) to rewrite (51) as follows:

$$\partial_\mp g_\pm = E\partial_\pm f \tag{54}$$

Extremizing S_2 w.r.t. $\hat{x}$, and following the same algebraic manipulations, we obtain

$$2\hat{k}E^2 \hat{f} = \hat{\beta}_- \hat{g}_+ + \hat{\beta}_+ \hat{g}_- \tag{55a}$$

$$\partial_\mp \hat{g}_\pm = E\partial_\pm \hat{f} \tag{55b}$$

where $\hat{\beta}_\pm$ are defined in eq.(B.6b), and

$$\hat{f} \equiv -\lambda_1 - \hat{k} + \frac{\kappa}{2} \ , \tag{55c}$$

$$\hat{g}_\pm \equiv \hat{\beta}_\pm - 2E\hat{\lambda}_2^\mp \ . \tag{55d}$$

These are all the bulk Euler-Lagrange equations resulting from S_2, apart from the constraints themselves.

We can immediately solve for $\lambda_1, k, \hat{k}$ by inspection — by setting

$$f = \hat{f} = k + \hat{k} = 0 \ . \tag{56a}$$

Then, by (53b) and (55c),

$$\lambda_1 = 0, \ k = -\hat{k} = -\frac{\kappa}{2} \ . \tag{56b}$$

Eqs.(54) and (55b) then imply,

$$\partial_\mp g_\pm = \partial_\mp \hat{g}_\pm = 0 \ , \tag{57a}$$

which together with (53a),(55a) gives:

$$\beta_-/\beta_+ = -g_-(u^-)/g_+(u^+) \ , \tag{57b}$$

$$\hat{\beta}_-/\hat{\beta}_+ = -\hat{g}_-(u^-)/\hat{g}_+(u^+) \ . \tag{57c}$$

But by eqs.(B.8),(56):

$$\beta_+ = \beta_+(u^+), \ \beta_- = \beta_-(u^-) \tag{57d}$$

$$\hat{\beta}_+ = \hat{\beta}_+(u^+), \ \hat{\beta}_- = \hat{\beta}_-(u^-) \tag{57e}$$

Therefore eqs.(57b)-(57c) determine $g_\pm, \hat{g}_\pm$ up to four numerical constants*.

$\star$ Once $g_\pm$ and $\hat{g}_\pm$ are known, the multipliers $\lambda_2^\pm, \hat{\lambda}_2^\pm$ are known from eqs.(53c) and (55d).

From this point on, the analysis of appendix B, part (II) applies, since we are on shell— except that the constant r_1 is not determined, as mentioned above, and must be set to 1 by hand. We know that eq.(56a) must follow from the equations of motion of the action S_2, but have not proven it directly, although it appears to us plausible that eqs.(49)-(55) indeed imply (56a).

9. Quantization

The action formulation of the last section is a promising point of departure for quantizing the self-gravitating string. To do so, a Hamiltonian should be derived from the worldsheet action S_2. Then, Unless a particular orthonormal parametrization (such as the light-cone gauge in standard string theory) can be found where all the constraints are easily soluble, these constraints must be imposed as conditions on physical states — as is done in the covariant formulation of standard strings†.

Another possible starting point for quantizing the theory, is the classical Liouville-like equation (24). It is, of course, likely that other conformal matter must be added on the worldsheet to render the quantization consistent— this is certainly the case for $\kappa = 0$.

What could such a string theory mean? Even though it is a first quantized string, it automatically includes the gravitational interactions amongst different portions of the string. These interactions do not include graviton exchange, since there are no gravitons in three-dimensional Einstein gravity‡. Therefore, scattering amplitudes constructed in this theory would still be tree diagrams, from the point of view of unitarity. Yet, they would include effects to all orders in κ — which means, in non-geometrized units, all orders in G. Thus, we expect the *first quantized* self-gravitating string to include some string field-theoretic effects — per-

† Incidentally, the light-cone orthonormal gauge *cannot* be chosen here, since in three dimensions this gauge exists only when $\partial^2 x = 0$.

‡ Nor do they include any of the quantum physics of pure $3d$ gravity

turbative, as well as nonperturbative[§]. This model could therefore be a laboratory for investigating string-nonperturbative physics in the continuum.

10. Conclusions

We have investigated a family of classical spacetimes in $2+1$ dimensions, produced by a Nambu-Goto self-gravitating string. Due to the special properties of three-dimensional gravity, the metric is completely described as a Minkowski space with *two* identified worldsheets. The geometry of our spacetime is expressed as a flat region of Minkowski space, with the two worldsheets identified. The equations of motion of the worldsheet were found, and reduced to a Liouville-like equation for the induced worldsheet Liouville mode. The flat limit and the zero-tension limit were worked out for the case of open string with massive endpoints. We have shown how to expand solutions in the geometrized string tension. For small string tension, spacetime was found to have the causal structure found by Gott. The two-worldsheet formalism was recast, using auxiliary fields, as an action principle on the worldsheet. The new action is quadratic (with quadratic constraints) and perhaps amenable to consistent quantization. We suggest that the *first* quantized self-gravitating string could be a step towards understanding nonperturbative string field theory in a continuum setting.

Further work is in progress, mainly to find the possible causal and global structures of our classical spacetimes[¶], and to ascertain whether the quantization scheme proposed here can be implemented.

We have concentrated mostly on the case of open string with massive endpoints. In a quantized theory, one should work either with standard open strings (i.e., lightlike ends) or with closed strings. The treatments differ only in the boundary conditions; we have indicated the required changes.

§ However, if the quantization program can be carried out, it will certainly be simpler to extract κ- perturbative corrections to flat (standard) string theory, than to solve the κ-nonperturbative theory.

¶ Including the physics of self-intersecting strings.

Appendix A: Conventions and Units.

We use greek letters for three-dimensional world indices, which run from 0 to 2, and the vector component notation $a^\mu = (a^0, a^1, a^2)$; lower-case latin indices $i, j, ...$ denote world (target-space) spatial indices, whereas $a, b, ...$ are sometimes used to denote worldsheet indices. The Minkowski metric is $\eta_{\mu\nu} = (-1, 1, 1)$, and the totally antisymmetric ϵ-symbol is defined by $\epsilon_{012} = 1$.

The Minkowski scalar product of two three-vectors is denoted $a \cdot b = a^\mu b_\mu$. We also define a vector product $a \times b$, thus:

$$(a \times b)_\mu \equiv \epsilon_{\mu\alpha\beta} a^\alpha b^\beta \ .$$

This vector product obeys the relation

$$(a \times b) \times c = a(b \cdot c) - b(a \cdot c) \ , \qquad (A.1)$$

which has the opposite sign from the corresponding Euclidean relation.

We denote by ∂^2 the worldsheet d'Alembertian:

$$\partial^2 = \partial_+ \partial_-$$

In general relativity, we employ geometrized units $8\pi G = 1$, where G is Newton's constant in $2+1$ spacetime dimensions, and conform to the curvature-sign conventions of Weinberg[15].

Appendix B: Differential Geometry and Equations of Motion.

This appendix has three parts. In part (I) we derive some useful differential-geometric results *off-shell* — that is, we only assume orthonormality and metric-continuity, but do *not* yet assume the equations of motion. This part is thus particularly useful for the action formulation.

In part (II), we impose the vector equations of motion, show that the *scalar* equation of motion, eq.(22), is then only a numerical constraint, and impose it as well. We then show that the induced Liouville mode satisfies the differential equation, eq.(24).

Finally, part (III) is concerned with finding self-gravitating string solutions once the Liouville mode ϕ is known (see section 4 in text). Throughout this appendix, we ignore the boundary conditions.

(I) Off Shell Results: Consider a spacetime, constructed as described in section 2 (open or closed string). Disregarding the boundary conditions, we are left only with the orthonormality condition (eq.(5)) and the metric-continuity condition (eq.(6)). By applying $\partial_\mp$ to eqs.(5) and using eqs.(8a),(8c) we find,

$$\partial^2 x = kEn, \;\; \partial^2 \hat{x} = \hat{k} E \hat{n} \qquad (B.1)$$

where k,$\hat{k}$ are unknown scalar functions on the wordsheet. We have (eq.(8b)),

$$n \cdot \partial_\pm x = 0 \; . \qquad (B.2)$$

Applying $\partial_\pm$,

$$\partial_\pm^2 x \cdot n = -\partial_\pm n \cdot \partial_\pm x, \qquad (B.3)$$

whereas applying $\partial_\mp$ to (B.2) yields

$$\partial_\mp n \cdot \partial_\pm x = -kE \; . \qquad (B.4)$$

Eqs.(B.3),(B.4) and $n \cdot \partial_\pm n = 0$ (which holds since $n^2 = 1$), give all three components of $\partial_\pm n$ in the basis $\{\partial_+ x, \partial_- x, n\}$, and allow us to expand

$$\partial_\pm n = -k \partial_\pm x + \frac{1}{E} \beta_\pm \partial_\mp x \; , \qquad (B.5a)$$

where

$$\beta_\pm \equiv -(\partial_\pm^2 x) \cdot n \; . \qquad (B.5b)$$

The above derivations follow through for the variables of the top worldsheet as well, so

$$\partial_\pm \hat{n} = -\hat{k}\partial_\pm \hat{x} + \frac{1}{E}\hat{\beta}_\pm \partial_\mp \hat{x} \ , \tag{B.6a}$$

where

$$\hat{\beta}_\pm \equiv -(\partial_\pm^2 \hat{x}) \cdot \hat{n} \tag{B.6b}$$

Since the formulae for $\{\hat{x}, \hat{n}\}$ are in exact correspondence with those for $\{x, n\}$, we shall mostly work with the latter.

The integrability condition of eqs.(B.5a) for $\partial_\pm n$ is:

$$(\partial_- k)\partial_+ x - (\partial_+ k)\partial_- x = \partial_-[\frac{\beta_+}{E}\partial_- x] - \partial_+[\frac{\beta_-}{E}\partial_+ x] \ . \tag{B.7}$$

Dotting this with n produces an identity, by the second of eqs.(B.5); but dotting (B.7) with $\partial_\pm x$, and use of orthonormality, yields

$$\partial_\pm \beta_\mp = -E\partial_\mp k \tag{B.8a}$$

and similarly

$$\partial_\pm \hat{\beta}_\mp = -E\partial_\mp \hat{k} \tag{B.8b}$$

Next, we apply $\partial_\pm$ to eq.(6) and use eq.(B.1):

$$\partial_\pm^2 x \cdot \partial_\mp x = \partial_\pm^2 \hat{x} \cdot \partial_\mp \hat{x} = \partial_\pm E \tag{B.9}$$

This, together with $\partial_\pm^2 x \cdot \partial_\pm x = \partial_\pm^2 \hat{x} \cdot \partial_\pm \hat{x} = 0$ and eqs.(B.5b),(B.6b), gives the components of the four three-vectors $\partial_\pm^2 x, \partial_\pm^2 \hat{x}$ in the bases $\{\partial x, n\}$ and $\{\partial \hat{x}, \hat{n}\}$, respectively:

$$\partial_\pm^2 x = \frac{1}{E}(\partial_\pm E)\partial_\pm x - \beta_\pm n \tag{B.10a}$$

$$\partial_\pm^2 \hat{x} = \frac{1}{E}(\partial_\pm E)\partial_\pm \hat{x} - \hat{\beta}_\pm \hat{n} \tag{B.10b}$$

Now, apply $\partial_\mp$ to eqs.(B.10), and utilize (B.1): (recall $E < 0$)

$$\partial_\pm(kEn) = \partial^2\{\ln(-E)\}\partial_\pm x + (\partial_\pm \ln(-E))kEn - (\partial_\mp \beta_\pm)n - \beta_\pm \partial_\mp n \tag{B.11}$$

and a similar relation for $\hat{x}, \hat{n}, \hat{\beta}_\pm$. Dot (B.11) with $\partial_\mp x$, and use eq.(B.5a) for $\partial_\mp n \cdot \partial_\mp x$ and $\partial_\pm n \cdot \partial_\mp x$; the result is

$$E\partial^2 \ln(-E) = \beta_+\beta_- - k^2E^2 \ , \tag{B.12a}$$

and similarly

$$E\partial^2 \ln(-E) = \hat{\beta}_+\hat{\beta}_- - \hat{k}^2E^2 \ . \tag{B.12b}$$

Comparing the two eqs.(B.12), we find the relation

$$\beta_+\beta_- - k^2E^2 = \hat{\beta}_+\hat{\beta}_- - \hat{k}^2E^2 \ . \tag{B.13}$$

(II) The Equations of Motion: On shell, we have at our disposal also the vector equations of motion (17)-(18), and the scalar equation, eq.(22). We begin by using only the vector equations. Comparing with eq.(B.1),

$$k = -\frac{\kappa}{2}, \quad \hat{k} = \frac{\kappa}{2} \ . \tag{B.14}$$

Thus by eqs.(B.8)

$$\beta_+ = \beta_+(u^+), \quad \beta_- = \beta_-(u^-) \tag{B.15}$$

and similarly for $\hat{\beta}_\pm$; (B.13) therefore simplifies to

$$\beta_+(u^+)\beta_-(u^-) = \hat{\beta}_+(u^+)\hat{\beta}_-(u^-) \ , \tag{B.16}$$

which implies,

$$\hat{\beta}_+(u^+) = r_1\beta_+(u^+) \tag{B.17a}$$

$$\hat{\beta}_-(u^-) = \frac{1}{r_1}\beta_-(u^-) \tag{B.17b}$$

where $r_1 = const$. From the definitions (B.5b),(B.6b) and the scalar equation of motion (22), we now see that in fact $r_1 = 1$; and thanks to eqs.(B.17), we see that eq.(22), despite being a pair of equations among *functions*, is actually only a single numerical condition, once the vector equations of motion are used.

The functions $\beta_+(u^+), \beta_-(u^-)$ can be chosen freely. Let us restrict our attention to a region on S where $\beta_\pm$ have fixed signs, $s_\pm$. Perform a conformal (orthonormality preserving) coordinate transformation on the worldsheet, in order to set $\beta_\pm$ to the constants $s_\pm$; the new coordinates $\bar{u}^\pm$ are defined by

$$d\bar{u}^\pm = \sqrt{|\beta_\pm|}du^\pm \ , \tag{B.18}$$

and the new worldsheet conformal factor is

$$\bar{E} = E/\sqrt{|\beta_+\beta_-|} \ . \tag{B.19}$$

For notational convenience, we henceforth suppress the bar on all quantities. It must be kept in mind, though, that the residual worldsheet gauge freedom has now been used up. In the new coordinates, eqs.(B.12) become

$$E\partial^2 \ln(-E) = -s - \frac{\kappa^2}{4}E^2 \tag{B.20}$$

where $s = -s_+s_-$ is another sign. Recalling that $E < 0$, we define the Liouville mode ϕ of the classical, induced worldsheet metric, thus:

$$E \equiv -e^{\phi} \tag{$B.21$}$$

and we obtain from (B.20) the differential equation:

$$\partial^2\phi + se^{-\phi} = \frac{\kappa^2}{4}e^{\phi}\ . \tag{$B.22$}$$

(III) Solving for $x(u)$,$\hat{x}(u)$ in terms of $E(u)$: As above, we consider a region of S in which $\beta_\pm$ have fixed signs. Since reasonable choices for these functions vanish only at discrete values of $u^\pm$ (respectively), the method presented here can reproduce any generic solution over most of the worldsheet. We will present the method for $x(u)$, but again, the same procedure can be applied to $\hat{x}(u)$ provided κ is replaced with $-\kappa$.

As in part (II), we choose local orthonormal coordinates $u^\pm$ where $\beta_\pm = s_\pm$. Eq.(25a) then holds, namely

$$\partial_\pm x \times \partial_\pm^2 x = \pm s_\pm \partial_\pm x\ . \tag{$B.23$}$$

Defining the null three-vectors $a_\pm \equiv \pm s_\pm \partial_\pm x$, we find

$$a_\pm \times \partial_\pm a_\pm = a_\pm \tag{$B.24a$}$$

$$(a_\pm)^2 = 0\ . \tag{$B.24b$}$$

Form the cross-product of eq.(B.24a) with $\partial_\pm a_\pm$ and use (A.1):

$$(\partial_\pm a_\pm)^2 = 1 \tag{$B.24c$}$$

It is now a simple matter to solve eqs.(B.24). The general solution is

$$a^\mu_\pm = \frac{1}{\partial_\pm \gamma_\pm}(1, \cos\gamma_\pm, \sin\gamma_\pm) \tag{$B.25$}$$

where $\gamma_\pm$ are two unknown functions on the worldsheet.

Let us now impose the integrability condition

$$\partial_- a_+ = s\partial_+ a_- \ , \tag{B.26a}$$

which follows from the definition of $a_\pm$, and the vector equation of motion, which reads[⋆]

$$\partial_- a_+ = \frac{\kappa}{2} s_- a_+ \times a_- \ . \tag{B.26b}$$

Here $s = -s_- s_+$, as defined after eq.(B.20).

After some algebra, we obtain from (B.25)-(B.26) the following useful equations:

$$\frac{\kappa}{2} s_- [1 - \cos(\gamma_+ - \gamma_-)] = s(\partial_+\gamma_-)(\partial_+\gamma_+) = -(\partial_-\gamma_-)(\partial_-\gamma_+) \ . \tag{B.27}$$

On the other hand, the worldsheet conformal scale factor is, by definition,

$$E = \partial_+ x \cdot \partial_- x = s a_+ \cdot a_- = -s\frac{1}{(\partial_+\gamma_+)(\partial_-\gamma_-)}[1 - \cos(\gamma_+ - \gamma_-)] \tag{B.28}$$

Eqs.(B.27)-(B.28) give (25b); the rest of the procedure is described in section 4.

Appendix C: Constancy of End Masses — Geometric Proof.

We now prove that the geometrically defined endmasses for the open string (see section 3) are constants, that is, τ independent. The proof is for the P end, but carries over to Q.

The instantaneous mass m is the angle of the wedge between the top and bottom worldsheets, at P and in its instantaneous rest frame (the 'deficit angle').

⋆ The scalar equation (22) was already used up in showing that $\beta_\pm = \hat{\beta}_\pm$ (see part (II)).

Since the coordinates (τ, σ) are orthonormal and $\sigma = \ const$ at P, we have[†]

$$\sin m = \frac{|\frac{\partial x}{\partial \sigma} \times \frac{\partial \hat{x}}{\partial \sigma}|}{(\frac{\partial x}{\partial \sigma})^2} = \frac{|\frac{\partial x}{\partial \sigma} \times \frac{\partial \hat{x}}{\partial \sigma}|}{-(\frac{\partial x}{\partial \tau})^2} \qquad (C.1)$$

Thanks to orthonormality and the boundary condition, we have at P:

$$(\frac{\partial x}{\partial \tau})^2 = -(\frac{\partial x}{\partial \sigma})^2, \ (\frac{\partial \hat{x}}{\partial \tau})^2 = -(\frac{\partial \hat{x}}{\partial \sigma})^2 \qquad (C.2a)$$

$$\frac{\partial \hat{x}}{\partial \tau} = \frac{\partial x}{\partial \tau} \qquad (C.2b)$$

Thus we may define the following two unit spacelike vectors:

$$a \equiv \frac{1}{|\frac{\partial x}{\partial \tau}|}\frac{\partial x}{\partial \sigma}, \ b \equiv \frac{1}{|\frac{\partial x}{\partial \tau}|}\frac{\partial \hat{x}}{\partial \sigma}, \ a^2 = b^2 = 1 \qquad (C.3)$$

All equations from here on will be understood to hold at P. Clearly $(n \times \hat{n}) || \frac{\partial x}{\partial \tau}$, and

$$n^2 = \hat{n}^2 = 1, \ a \cdot n = b \cdot \hat{n} = 0, \ (n \times \hat{n}) \cdot a = (n \times \hat{n}) \cdot b = 0 \ . \qquad (C.4)$$

Next, invoke the scalar equations (22), from which follows[‡]

$$\frac{\partial x}{\partial \tau} \times \frac{\partial x}{\partial \sigma} \cdot \frac{\partial^2 x}{\partial \sigma \partial \tau} = \frac{\partial x}{\partial \tau} \times \frac{\partial \hat{x}}{\partial \sigma} \cdot \frac{\partial^2 \hat{x}}{\partial \sigma \partial \tau} \ , \qquad (C.5)$$

and thus

$$n \cdot \frac{\partial a}{\partial \tau} = \hat{n} \cdot \frac{\partial b}{\partial \tau} \qquad (C.6)$$

Since a,b are unit vectors, $a \cdot \partial a/\partial \tau = b \cdot \partial b/\partial \tau = 0$; this, together with (C.4) and

† We denote $|a| \equiv \sqrt{-a^2}$ for a timelike vector a. The angle m is positive, and bounded from above as discussed in section 5. Equation (C.1) requires a sign correction if $m > \pi$.

‡ By subtracting the two versions with $\pm$ signs one from the other.

(C.6), allows us to expand (with $\alpha, \beta, \gamma, \delta$ unknown functions of τ),

$$\frac{\partial a}{\partial \tau} = \alpha a \times n + \beta (a \cdot \hat{n}) n \tag{C.7a}$$

$$\frac{\partial b}{\partial \tau} = \gamma b \times \hat{n} + \delta (b \cdot n) \hat{n} \tag{C.7b}$$

$$\beta a \cdot \hat{n} = \delta b \cdot n \tag{C.7c}$$

We thus find:

$$\frac{\partial}{\partial \tau}(a \cdot b) = \beta (a \cdot \hat{n})(b \cdot n + a \cdot \hat{n}) \tag{C.8}$$

But $a, b, n, \hat{n}$ are all unit vectors in the two-dimensional spatial subspace of the instantaneous rest-frame at P. Thus, eq.(C.4) implies the vanishing of the right-hand side in eq.(C.8), and (C.1) then gives

$$\frac{\partial}{\partial \tau} m = 0 \ , \tag{C.9}$$

since $a \cdot b = \cos m(\tau)$. Thus the endmass at P is conserved, as claimed.

Appendix D: Flat Open String with End Masses.

In this appendix, we treat the three-dimensional classical open string in the flat limit (no gravity), with masses at the endpoints. We use non-geometric methods, and the results agree with the flat limit of the geometric formalism, described in section 6. We also describe the large endmass (or small string-tension) expansion for the flat case, and present as an example a simple infinite-mass configuration where the two endpoints pass each other with an impact parameter.

The flat equation of motion is the worldsheet wave equation,

$$\partial^2 x^\mu = 0 \ , \qquad (D.1)$$

with the general solution

$$x^\mu = A^\mu(\tau+\sigma) + \tilde{A}^\mu(\tau-\sigma) \ . \qquad (D.2a)$$

As usual, we choose an orthonormal gauge on the worldsheet so that P,Q are at $\sigma = 0, \pi$, respectively; as pointed out in the footnote following eq.(6), this still leaves the freedom to reparametrize $u^+ \to f(u^+)$, $u^- \to f(u^-)$, where $f(v) - v$ is periodic with period 2π.

In such a parametrization, eqs.(30b) hold, namely

$$(A')^2 = (\tilde{A}')^2 = 0. \qquad (D.2b)$$

In addition, the total momentum of the bulk of the string (without the ends) is $\kappa \int_0^\pi d\sigma \frac{\partial x^\mu}{\partial \tau}$, and the total momentum of the system*, including the ends, is:

$$p^\mu_{total} = \kappa \int_0^\pi d\sigma \frac{\partial x^\mu}{\partial \tau} + \sum_{P,Q} m \frac{dx^\mu}{ds} \ , \qquad (D.3)$$

where s is proper time at either endpoint. As in the text, we assume for simplicity that the endmasses are equal, $m^{(P)} = m^{(Q)} = m$.

$\star$ An equivalent expression for the bulk string momentum is the equal-time spatial integral $\int d^2\mathbf{x} T^{0\mu}$; this expression is, of course, valid in any worldsheet parametrization, though the expression (11) is not.

Imposing momentum conservation $dp_{total}/d\tau = 0$, we integrate by parts and use eq.(D.1); this yields the boundary conditions

$$\kappa \frac{\partial x}{\partial \sigma}(P) = m \frac{d}{d\tau}\left(\frac{dx(P)}{ds}\right) \tag{D.4a}$$

$$\kappa \frac{\partial x}{\partial \sigma}(Q) = -m \frac{d}{d\tau}\left(\frac{dx(Q)}{ds}\right) \tag{D.4b}$$

By eq.(D.2a), an element of proper time at P is given by

$$ds^2 = -2A'(\tau) \cdot \tilde{A}'(\tau) d\tau^2 \ , \tag{D.5}$$

with a similar relation holding at Q.

We define (see eq.(34b))

$$\rho \equiv m/\kappa \tag{D.6}$$

and use eq.(D.5) to rewrite the boundary conditions (D.4):

$$A'(\tau) - \tilde{A}'(\tau) = \rho \frac{d}{d\tau}\left\{\frac{1}{\sqrt{-2A'(\tau)\cdot\tilde{A}'(\tau)}}(A'(\tau) + \tilde{A}'(\tau))\right\}, \tag{D.7a}$$

$$A'(\tau + 2\pi) - \tilde{A}'(\tau) = -\rho \frac{d}{d\tau}\left\{\frac{1}{\sqrt{-2A'(\tau+2\pi)\cdot\tilde{A}'(\tau)}}(A'(\tau + 2\pi) + \tilde{A}'(\tau))\right\}. \tag{D.7b}$$

When $\rho = 0$, eqs.(D.7) imply that $A' = \tilde{A}'$ and that A' is periodic with period 2π, and the standard open string theory is recovered†.

† This limit should be taken carefully, since then $ds/d\tau = 0$ (lightlike endpoints), so it is not immediately clear that the r.h.s. of eqs.(D.7) vanishes in the limit.

It suffices to examine in detail the boundary condition at P, since the condition at Q is obtained from it by the replacements

$$\rho \to -\rho,\ A(\tau) \to A(\tau + 2\pi). \tag{D.8}$$

Eq.(D.7a) can be integrated, as both sides are τ derivatives; this introduces an unknown constant vector, c^μ, and the boundary condition at P assumes the form‡

$$\varphi(A' + \tilde{A}') = c - \frac{1}{\rho}(\tilde{A} - A)\ , \tag{D.9a}$$

where

$$\varphi^{-2} \equiv -2A' \cdot \tilde{A}'\ . \tag{D.9b}$$

The l.h.s. of (D.9a) is $dx(P)/ds$, the three-velocity at P. Thus, in the limit $\rho \to \infty$ (zero string tension or infinite endmasses), c^μ is the fixed velocity vector at P.

Due to eq.(D.9b), the square of the l.h.s. of (D.9a) is -1; in fact, (D.9a) *alone* implies $[\varphi(A' + \tilde{A}')] \cdot (A' + \tilde{A}') = 0$, which in turn implies that

$$[\varphi(A' + \tilde{A}')]^2 = \ const.$$

To solve for the dynamics of the string (eq.(D.2a)), eqs.(D.9) and the corresponding Q condition must be solved. For large ρ, this can be done by expansion in powers of $1/\rho$, as we now describe. We look for solutions for which A,$\tilde{A}$ are $O(\rho^0)$. Assuming

$$A' \cdot c < 0,$$

eqs.(D.9) give

$$\tilde{A}' = -A' - 2(A' \cdot c)c + O(1/\rho)\ , \tag{D.10a}$$

$$c^2 = -1 + O(1/\rho) \tag{D.10b}$$

((D.10b) follows from squaring eq.(D.9a) and using (D.9b)). Eq.(D.10a) ensures

‡ All arguments in eqs.(D.9) are τ.

that $(\tilde{A}')^2 = 0$ to lowest order in $1/\rho$, provided $(A')^2$ vanishes to that order.

By dotting (D.9a) with A' and using (D.2b),(D.9b) and eqs.(D.10), φ^{-1} is found to order $1/\rho$. Then we use (D.9a) again, and find the⋆ next-order solution for $\tilde{A}$ in terms of A:

$$\tilde{A}' = -A' - 2(A' \cdot c)c - \frac{4}{\rho}(A' \cdot c)A - \frac{4}{\rho}c[A' \cdot A + 2(A \cdot c)(A' \cdot c)] + O(\frac{1}{\rho^2}) \ , \quad (D.11a)$$

$$c^2 = -1 + O(\frac{1}{\rho^2}) \ , \qquad (D.11b)$$

where the integration constant for $\tilde{A}$ is so far only determined to $O(\rho^0)$:

$$\tilde{A} = -A - 2(A \cdot c)c + O(1/\rho) \ . \qquad (D.11c)$$

The $O(1/\rho)$ integration constant for $\tilde{A}$ can be determined, once we choose c^2 to order $O(1/\rho^2)$. It is easiest to simply choose $c^2 = -1$ to all orders.

$A(\tau)$ is so far unconstrained, except for $(A')^2 = 0$ and the assumed

$$A' \cdot c < 0 \ . \qquad (D.12)$$

Due to the former, and to $c^2 < 0$, it is sufficient to assume $A'^0 > 0$ to ensure eq.(D.12), since† $c^0 > 0$.

The boundary condition at Q is similarly solved, order by order; this gives *another* expression for $\tilde{A}(\tau)$, in terms of $A(\tau + 2\pi)$, $-1/\rho$ and d^μ (the constant vector arising in the integration of eq.(D.7b)). Comparison of the two expressions yields a vector relation between $A(\tau)$ and $A(\tau + 2\pi)$; this relation modifies the periodicity of A', and therefore of $\tilde{A}'$.

⋆ A,$\tilde{A}$ may be shifted by arbitrary constant vectors, provided c is adjusted so that (D.9a) still holds. Modulo this freedom, our $1/\rho$-perturbative solution is unique.

† Both $c^0 > 0$ and $c^2 < 0$ follow, for ρ sufficiently large, from eq.(D.9a), since dx/ds is timelike and $dx^0/ds > 0$.

To get a flavor for how this works, we restrict attention to the $\rho \to \infty$ limit itself. Physically, this corresponds either to two free masses with no string (an uninteresting case in the absence of gravity!), or two infinitely massive particles, with a string between them, moving past each other.

In this limit, eq.(D.11c) becomes

$$\tilde{A}(\tau) = -A(\tau) - 2(A(\tau) \cdot c)c \; , \qquad (D.13a)$$

and the corresponding boundary condition at Q is,

$$\tilde{A}(\tau) = -A(\tau + 2\pi) - 2(A(\tau + 2\pi) \cdot d)d \; . \qquad (D.13b)$$

Eliminating $\tilde{A}$, we find

$$A'(\tau) + 2(A'(\tau) \cdot c)c = A'(\tau + 2\pi) + 2(A'(\tau + 2\pi) \cdot d)d \qquad (D.14a)$$

But the operation $v \to v + 2c(c \cdot v)$ on vectors v is a reflection, and so is the corresponding operation with d. Thus, solving (D.14a) for $A'(\tau + 2\pi)$ gives

$$A'(\tau + 2\pi) = \Lambda^{(0)} \cdot A'(\tau) \qquad (D.14b)$$

where $\Lambda^{(0)}$ is a Lorentz transformation. In the center-of-mass frame of the two endmasses, let us choose the space axes such that

$$c^\mu = (\cosh\chi, 0, \sinh\chi), \; d^\mu = (\cosh\chi, 0, -\sinh\chi) \; . \qquad (D.15)$$

Let us further denote

$$\Lambda(\omega)^\mu{}_\nu \equiv \begin{pmatrix} \cosh\omega & 0 & -\sinh\omega \\ 0 & 1 & 0 \\ -\sinh\omega & 0 & \cosh\omega \end{pmatrix} \qquad (D.16a)$$

Then, from eqs.(D.14),

$$\Lambda^{(0)} = \Lambda(4\chi)\ . \qquad (D.16b)$$

The physical interpretation of the modified periodicity (D.14b) is simple: an end-point, being infinitely massive, totally reflects string waves in the rest frame of the mass. But the ends move with rapidities $\pm\chi$, so in the center-of-mass the waves undergoe a Doppler shift upon each reflection, by a boost of rapidity 2χ. Thus, by the time a wave reflects once from each boundary, it has undergone a boost of rapidity 4χ.

The general solution of the functional equation $(D.14b)$ is:

$$A'(\tau) = \Lambda(2\chi\tau/\pi)\cdot D(\tau)\ , \qquad (D.17)$$

where $D^\mu(\tau)$ is any lightlike vector which is a periodic function of τ (with period 2π). To guarantee eq.(D.12), it suffices to impose

$$D^0 > 0.$$

Once D is chosen, we have a leading-order solution, and eqs.(D.11) (with their couterparts at Q) can be used to find a functional equation for the $O(1/\rho)$ piece of $A(\tau)$; this can again be solved by means of periodic functions and hyperbolic functions of $2\chi\tau/\pi$, and so on to any desired order in the $O(1/\rho)$ expansion.

Returning to order $O(\rho^0)$, we conclude with a simple example of a string configuration of the type (D.17). Make the following simple choice for $D(\tau)$,

$$D^\mu(\tau) = (1, 1, 0) \ . \qquad (D.18)$$

We then find from eqs.(D.13) and (D.17), after making an arbitrary choice for $A(0)$, that

$$A^\mu(\tau) = (\frac{1}{\alpha}\sinh\alpha\tau, \tau, -\frac{1}{\alpha}\cosh\alpha\tau) \ , \qquad (D.19a)$$

$$\tilde{A}^\mu(\tau) = (\frac{1}{\alpha}\sinh\alpha(\tau+\pi), -\tau, \frac{1}{\alpha}\cosh\alpha(\tau+\pi)) \ , \qquad (D.19b)$$

where

$$\alpha \equiv 2\chi/\pi \ . \qquad (D.19c)$$

Substituting eqs.(D.19) in (D.2a), and denoting $x^\mu = (t, x, y)$, we obtain the string configuration in the COM frame:

$$y = -t\tanh[\frac{\alpha}{2}(x-\pi)] \qquad (D.20a)$$

with the ranges

$$-\infty < t < \infty, \ 0 \leq x \leq 2\pi \qquad (D.20b)$$

where the end P (Q) is at $x = 0$ ($x = 2\pi$). In this configuration (see Fig. 4), the instantaneous shape of the string is a section of a hyperbolic-tangent curve, symmetric about the string center $(x, y) = (\pi, 0)$; the section becomes the complete curve in the limit of infinite endpoint rapidities, and the relative scale of the x,y coordinates of the curve is linear in Minkowski time. The masses move in opposite directions, parallel to the y axis, and their separation at closest approach is along the x axis and equal[⋆] to 2π.

⋆ This separation can be scaled to any desired number, by rescaling the choice for D^μ (eq.(D.18)) by an arbitrary positive number.

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FIGURE CAPTIONS

Fig. 1: A local equal-time section of the worldsheet. Endpoints P,Q are connected by the string.

Fig. 2: A section of the $\{z\}$ coordinate patch. S is the worldsheet and T the transition surface (each has a top and bottom side). Arrows indicate indentifications.

Figs. 3: Fig. 2 for two free, static masses; (a) Spatial section open. (b) Spatial section closes due to excessive masses.

Fig. 4: The example flat-string configuration of Appendix D.

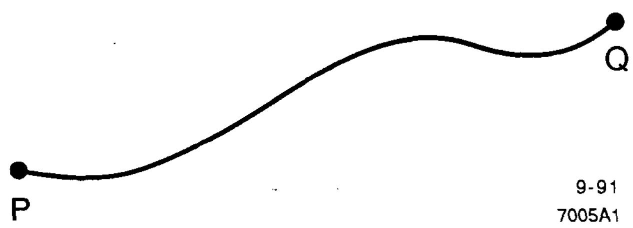


Fig. 1

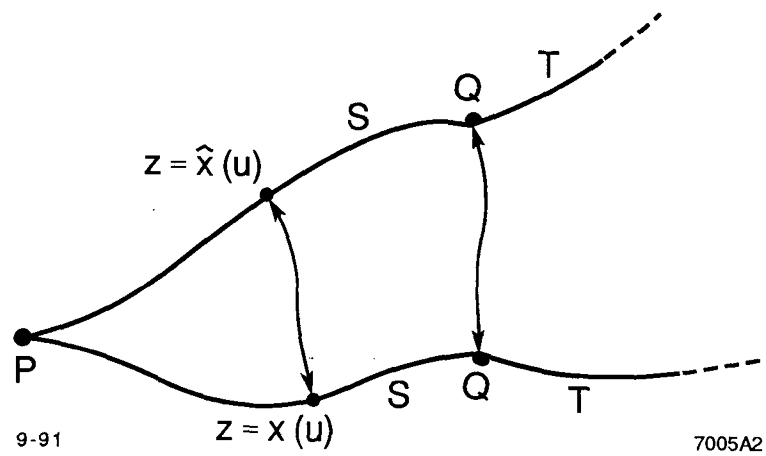
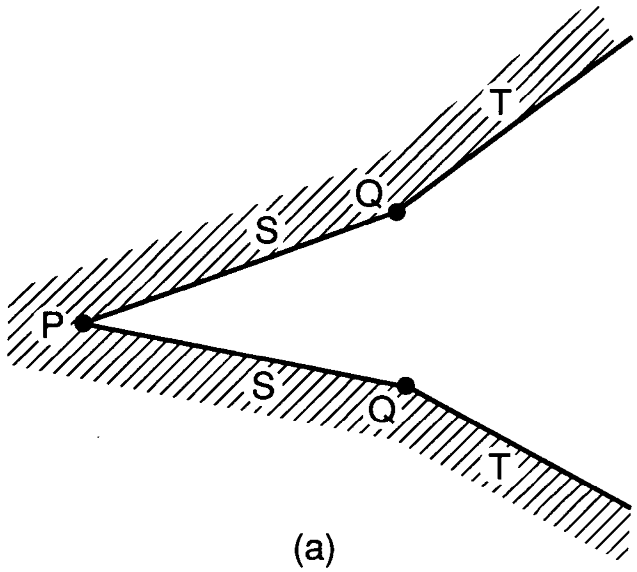


Fig.2



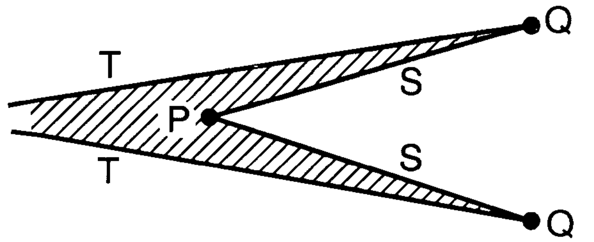


Fig. 3

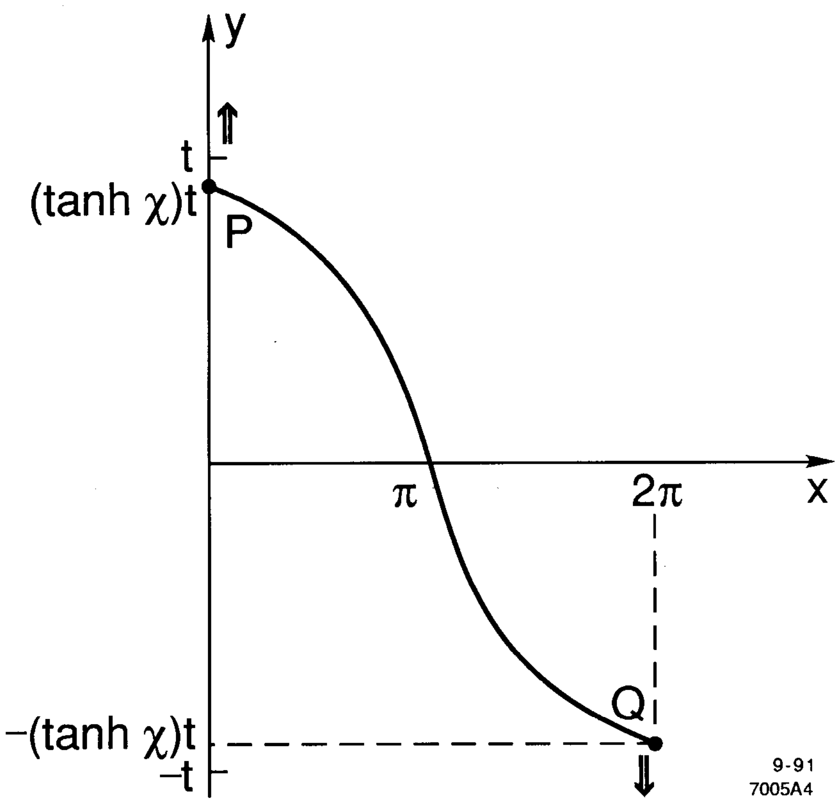


Fig. 4