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Quantum Cosmology on the Worldsheet*

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ABSTRACT

Two-dimensional quantum gravity coupled to conformally invariant matter with central charge $c > 25$ provides a toy model for quantum gravity in four dimensions. Two-dimensional quantum cosmology can thus be studied in terms of string theory in background fields. The large scale cosmological constant depends on non-linear dynamics in the string theory target space and does not appear to be suppressed by wormhole effects.

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1. Introduction

In this lecture some implications of string theory for Coleman's theory of the cosmological constant [1] will be discussed. String theory will not be considered in its role as a possible theory of everything but rather as a model of quantum gravity with the two-dimensional worldsheet playing the part of spacetime. Much of the material covered here has appeared previously in reference [2]. The starting point is the observation of Polchinski [3] that two-dimensional quantum gravity coupled to $D > 25$ scalar matter fields has all the ingredients to make Coleman's argument. The main requirements are:

- a sum over geometries which includes non-trivial topologies
- an action for the conformal mode which is unbounded from below
- a Euclidean saddle point consisting of a sphere with radius $\sim \frac{1}{\sqrt{\lambda}}$, where λ is the two-dimensional cosmological constant.

The leading order semi-classical approximation to the Euclidean action for such a spherical geometry is given by $S_E = \frac{D}{6}\log\lambda$ for large D†. The power law behavior in four dimensions is replaced by a logarithm in the two-dimensional theory. As a result the Baum-Hawking amplitude [4,5] becomes $\lambda^{-D/6}$, and after performing the sum over wormholes Coleman's 'exponential of exponential' [1] reduces to a single exponential $\exp\lambda^{-D/6}$. Nevertheless, if, as suggested by Coleman, this expression can be regarded as a probability distribution for the cosmological constant, then it implies the vanishing of λ in two dimensions.

Coleman's argument consists of two parts. The first says that the sum over wormhole topologies converts the constants of nature into probabilistic quantum variables governed by a wave-function on superspace. The second part assumes that the wave-function is proportional to the Euclidean path integral and that this

† Gravitational fluctuations are suppressed as $D \to \infty$. The existence of this semi-classical limit will be crucial for some of our arguments later on.

ill defined path integral can be evaluated by formally summing over saddle point configurations.

This is to be compared with the conclusions that string theory leads to. First of all, it appears to be correct that topology change makes the couplings of the two-dimensional worldsheet theory into quantum variables in target space. However, the second part of Coleman's argument is not supported by string theory. The wave-function for coupling constants appears to be controlled by phenomena which know nothing of the large scale structure of space-time and have no reason to prefer $\lambda = 0$.

The remainder of the paper is organized as follows. In section 2 we briefly review the formulation of two-dimensional quantum gravity in conformal gauge and establish the connection with string theory in background fields. In section 3 we study "cosmological" solutions and derive the Wheeler-DeWitt equation which governs the propagation of a one-dimensional universe in a background condensate of baby universes. In section 4 we examine the relation between the target space equations of motion and the renormalization group, and consider the evolution of couplings with scale. Section 5 deals with the question of the two-dimensional cosmological constant in this framework. We present an explicit calculation of the required string theory beta-function, using an appropriate renormalization procedure. Finally we conclude with a discussion of our results.

2. Two-dimensional Quantum Gravity and String Theory

Let us begin with a theory of quantum gravity, defined on a two-dimensional spacetime (σ^0, σ^1), involving a metric γ_{ab} and D scalar matter fields $X^i(\sigma^a)$. Most treatments have focused on the case $D \leq 1$ but for the purpose of modelling quantum cosmology it is appropriate to consider $D > 25$. The action is taken to be local and coordinate invariant, but can otherwise be quite general. It is also assumed that some covariant non-perturbative method of regularizing the theory exists. Unfortunately no such method is known at present for carrying out the continuum theory path integral in a manifestly covariant manner. Instead we shall have to rely on a prescription which is the analogue of old-fashioned methods of regularization and renormalization in gauge theories. Before the invention of gauge-invariant regulators, a procedure which worked was to regularize the theory in a non-covariant way and then compensate for the resulting non-invariance by allowing the Lagrangian to contain non-gauge-invariant terms, such as a photon mass. At the end of the day, the gauge symmetry is re-imposed through Ward identities, which place constraints on the values of the added terms. A particular version of this method for two-dimensional gravity follows:

- Gauge fixing: The first step is to remove the over-counting of metrics due to general coordinate invariance. For each worldsheet topology introduce some fixed fiducial, metric $\hat{\gamma}_{ab}$. Then choose coordinates such that the physical metric is conformal to the fiducial metric,

$$\gamma_{ab} = e^{\phi}\,\hat{\gamma}_{ab}\,. \tag{2.1}$$

  The original path integral is replaced by an integral over the matter fields $X^i(\sigma)$ and the one remaining degree of freedom of the metric $\phi(\sigma)$, which is called the Liouville field.

- Regularization: In order to define the gauge-fixed path integral the ultraviolet divergences of the theory need to be regularized. For example, a non-perturbative regulator can be introduced by discretizing the worldsheet. This discretization is to remain fixed and not be summed over as in matrix models. The regulator involves a shortest *fiducial* length defined by,

$$\hat{\gamma}_{ab}\,\epsilon^a\epsilon^b < \delta\,, \tag{2.2}$$

  where ϵ^a is the line element connecting the nearest lattice points and δ tends to zero as the cut-off is removed. A more covariant definition would refer the cut-off scale to the physical metric γ_{ab}, but then the regularization would depend on the Liouville field which is being integrated over. Thus we are obliged to use a non-covariant regularization procedure in order to have a concrete definition of the continuum theory path integral.

- Renormalization: Performing the path integral over short distance fluctuations of both the matter and gravitational fields generates various interaction terms, involving ϕ and X^i, in the effective Lagrangian. These terms will in general depend on the arbitrarily chosen fiducial metric $\hat{\gamma}_{ab}$ and therefore the effective theory will not be manifestly covariant. On the other hand, the original theory is assumed to be invariant under general coordinate transformations so no such dependence on $\hat{\gamma}_{ab}$ should occur. Thus we must impose upon the renormalized theory that the value of the path integral is not affected by the choice of fiducial metric. This can be achieved by, first of all, arranging the terms in the effective action to be covariant with respect to $\hat{\gamma}_{ab}$. This does not restrict the possible couplings but merely labels them according to their transformation properties under fiducial reparametrizations. The condition that the path integral does not depend on the determinant of $\hat{\gamma}$ requires that the beta-functions of all couplings in the theory vanish. This means that the gauge fixed theory must be an exact fixed point of the renormalization group in order to maintain the original general covariance.

To summarize: We start with a generally covariant theory of gravity coupled to scalar fields, X^i. In order to define the path integral we fix a gauge and regularize in a non-covariant manner. The resulting theory involves a scalar field, ϕ, in addition to the matter fields, and is in general quite complicated. The original covariance appears as a set of restrictions on the couplings, which include the requirement that all the beta-functions vanish. Notice that in this way of stating things ϕ and X^i are placed on equal footing. The Liouville field has been promoted to an additional target space dimension. This approach to the quantization of two-dimensional gravity has been advocated by a number of authors [3,6,7,8,9].

We are thus led to consider reparametrization invariant scalar field theory in two dimensions. The action can in general include terms with arbitrary functional dependence on ϕ and X^i, and with any number of derivatives acting on the fields. For convenience, let us define $X^0 = \frac{q}{2}\phi$ where $q^2 = \frac{D-25}{3}$. This rescaling leads to standard normalization for the Liouville kinetic term. The two-dimensional action can then be written,

$$S = \frac{1}{8\pi}\int d^2\sigma\,\sqrt{\hat{\gamma}}\,\Big\{T(X) + \hat{\gamma}^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) + 2\hat{R}\,\Phi(X) + \cdots\Big\}, \qquad (2.3)$$

with $\mu = 0, 1, \ldots, D$. We have written down the terms of scaling dimensions zero and two,* but there is an infinite sequence of possible couplings involving more derivatives on the X^μ and higher powers of the two-dimensional curvature $\hat{R}$.

This class of theories has been extensively investigated in string theory, where the action (2.3) describes strings in background fields in $D{+}1$ spacetime dimensions. The beta-function equations, implementing the conformal invariance of the two-dimensional theory, have the form of field equations in target-space for $T(X)$, $\Phi(X)$ and $G_{\mu\nu}(X)$ (tachyon, dilaton and graviton fields respectively), along with additional fields representing higher order couplings. These field equations describe the propagation and creation and annihilation of the particle-like eigenmodes of strings in spacetime, or more to the point of this paper, one-dimensional universes containing matter fields.† Because of the identification $X^0 = \frac{q}{2}\phi$, the role of time in target space is played by the two-dimensional scale. The tachyon field, $T(X)$, is of primary interest because it controls the two-dimensional cosmological constant. A cosmological term in the original classical action corresponds to a tachyon background which grows exponentially with increasing two-dimensional scale,

$$\int d^2\sigma\,\sqrt{\gamma}\,\lambda = \int d^2\sigma\,\sqrt{\hat{\gamma}}\,\lambda\, e^{\frac{2}{Q}X^0}\,. \qquad (2.4)$$

As we shall see, this remains qualitatively true in the quantum theory, as long as the tachyon background remains weak, but the rate of the exponential growth

* For simplicity, we have not included the anti-symmetric tensor field. Its presence would not qualitatively alter our conclusions.

† We will use the string theory names for the target-space fields, but the reader should keep in mind their cosmological interpretation.

is modified by quantum fluctuations. The exponentially growing background will eventually become strong and then non-linear effects in the target space theory can no longer be ignored. The two-dimensional cosmological constant will still be governed by the behavior of the tachyon background in the non-linear regime, but the connection between the two is more subtle.

The string theory equations of motion, obtained by setting beta-functions to zero, are derivable from a target-space action. For simplicity, we will work within a truncated theory, containing only the lowest order couplings, $T(X)$, $\Phi(X)$ and $G_{\mu\nu}(X)$. To leading order in derivatives, the target-space action for these fields is [10]

$$I=-\frac{1}{2g_0^2}\int d^{D+1}X\sqrt{G}\,e^{-2\Phi}\left\{\frac{25-D}{3}+R+4(\nabla\Phi)^2-(\nabla T)^2-2V(T)+\cdots\right\}, \quad (2.5)$$

where $V(T)=-T^2+\ldots$ is the tachyon effective potential. Since renormalization group beta-functions are not universal, the detailed form of $V(T)$ will depend on the regularization and renormalization prescription used. This is believed to correspond to field redefinition ambiguities in the target space equations. In fact all higher order terms in the tachyon beta-function can be arranged to involve target space derivatives, and therefore be removed from the potential leaving only $-T^2$ [11,12]. It should be stressed that using such a prescription in no way alters the fact that the target space equations are non-linear and are in general not exactly satisfied by a simple exponential tachyon background. The important question to ask is whether there exists a renormalization scheme in which the target space fields can be identified with Wheeler-DeWitt amplitudes of a one-dimensional universe. We will return to this point in section 5, where we propose an appropriate scheme and present a calculation of $V(T)$ to all orders in T.

3. Quantum Cosmology in Two Dimensions

The equations of motion which follow from the target space action (2.5) are

$$
\begin{aligned}
\nabla^2 T - 2\nabla\Phi\cdot\nabla T =&\ V'(T)\,, \\
\nabla^2\Phi - 2(\nabla\Phi)^2 =& -\frac{25-D}{6} + V(T)\,, \\
R_{\mu\nu} - \frac{1}{2}G_{\mu\nu}R =& -2\nabla_\mu\nabla_\nu\Phi + G_{\mu\nu}\nabla^2\Phi + \nabla_\mu T\nabla_\nu T - \frac{1}{2}G_{\mu\nu}(\nabla T)^2\,.
\end{aligned}
\tag{3.1}
$$

These equations have a simple solution, the so called linear dilaton background [13], which for $D > 25$ is given by

$$
\begin{aligned}
T &= 0\,, \\
G_{\mu\nu} &= \eta_{\mu\nu}\,, \\
\Phi &= -\frac{q}{2}X^0\,.
\end{aligned}
\tag{3.2}
$$

The target space is Lorentzian and it is the conformal mode, X^0, which is time-like. This means that the kinetic term of X^0 in (2.3) has the "wrong" sign and the Euclidean action of the two-dimensional theory is unbounded from below. This is analogous to the instability of the Euclidean path integral in four-dimensional gravity, which lies at the heart of Coleman's argument for the vanishing of the cosmological constant. On the other hand, it means that Euclidean two-dimensional gravity coupled to $D > 25$ matter is ill-defined and the renormalization group computation, which led to the target space equations (3.1), can only be viewed as a formal argument. Ideally the theory should be reformulated on a worldsheet of Lorentzian signature but it is unclear at present how to perform the steps involved in the quantization of such a theory (regularization, renormalization, *etc.*). In four-dimensional gravity people have sought to circumvent this problem by formally rotating the contour of path integration over the conformal factor into the complex plane to obtain a well defined integral [14]. While this formal procedure can also be applied in the two-dimensional theory, its validity has been called into question [15].

We will use the target space picture to define the two-dimensional theory for $D > 25$. The equations of motion (3.1), which were arrived at via a formal derivation based on a Euclidean worldsheet, lend themselves to an interpretation as a Lorentzian field theory of strings. Our assumption, which may be unwarranted, is that a consistent Lorentzian worldsheet formulation would lead to the same target space field theory. Since the equations are non-linear, singular geometries which describe splitting and joining strings will have to be included in the Lorentzian two-dimensional path integral. In addition, the path integral will receive contributions from universes being absorbed or emitted from the background, which also involves two-dimensional singularities. By contrast, in Euclidean space the metric can be chosen with no singularities. We only use Euclidean methods to compute renormalization group beta-functions, but our subsequent discussion of the two-dimensional cosmology takes place with Lorentzian signature.

An important difference between the worldsheet theory and four-dimensional gravity is that the gravitational coupling in two dimensions is dimensionless, so there is no proper Planck scale. However, as is well known, the strength of the string coupling depends on the dilaton field in target space. A key feature of the linear dilaton background (3.2) is that the string loop coupling constant is related to the two-dimensional scale,

$$g = g_0 \, e^{\Phi} = g_0 \, e^{\frac{Q}{2} X^0} \, . \tag{3.3}$$

We can only expect the effective field theory to be simple where this coupling is weak. For $D > 25$ the theory is strongly coupled for sufficiently small strings and target space quantum mechanics (string loops) are important in the ultraviolet on the worldsheet*. One can say that a Planck scale is spontaneously induced, and define it by the point at which $g_0 e^{\Phi} = 1$. The factor of g_0 can be absorbed by a constant shift of the dilaton. The effective Planck-scale is then set by q^{-1}. It

$\star$ In contrast with the $D \leq 1$ case, where the strength of quantum corrections tends to zero in the limit of metrically small strings.

depends on the number of scalar fields in the theory, and in particular, $D \to \infty$ is a semi-classical limit for gravitational fluctuations†. Another way to see that q^{-1} defines the Planck scale in this theory is to consider the relation between the classical conformal mode and the quantum variable, $\frac{q}{2}\phi = X^0$.

A particularly interesting cosmological system is given by an expanding universe which starts out at small scale. The question of initial conditions is complicated, just as it is in four-dimensional quantum cosmology, because the theory is strongly coupled early on. We will assume that the short distance physics can be summarized by some unknown initial state at the Planck-time, which then evolves in the weakly coupled theory. In a classical theory this means initial conditions on the target-space fields and in a quantum theory it corresponds to a wave-function in target-space.

A background tachyon field can be added to the linear dilaton solution (3.2). Its beta-function equation depends on the shape of the effective potential, $V(T)$, and is non-linear. For the moment we will assume that the background field is weak. The tachyon equation can then be linearized as follows,

$$-\partial_0^2 T + \partial_i^2 T - q\partial_0 T + 2T = 0\,, \tag{3.4}$$

and if we further assume that the tachyon background only depends on X^0, we find solutions

$$T(X^0) = \lambda\, e^{(-\frac{q}{2}\pm\sqrt{\frac{q^2}{4}+2})X^0}\,. \tag{3.5}$$

Such a homogeneous background configuration is the $D > 25$ analog of the $D < 1$ two-dimensional field theories discussed by David [16] and by Distler and Kawai [17].

One of the solutions decays in the weak coupling regime $X^0 \to \infty$ but the other one grows exponentially with scale. The system is unstable and is likely to

† For $D \leq 1$ a corresponding semi-classical limit is reached as the number of scalar fields is formally taken to $D \to -\infty$.

form a condensate of background tachyons. In the $D \to \infty$ semi-classical limit we recover the classical cosmological term (2.4) from the exponentially growing solution. At this point the quantum behavior is qualitatively the same as in the classical theory, but the classical scale factor, $e^{\frac{\phi}{2}} = e^{\frac{1}{q}X^0}$, has been renormalized to $e^{\frac{\alpha}{2}X^0}$, with $\alpha = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + 2}$.

In order to make contact with a more conventional Wheeler-DeWitt description of cosmology, let us consider fluctuations of some target space field in the exponentially growing tachyon background, $T_B(X^0) = \lambda\, e^{(-\frac{q}{2}+\sqrt{\frac{q^2}{4}+2})X^0}$. Take, for example, a tachyon with some non-zero space-like momentum k. This corresponds to a one-dimensional universe with some matter excitation. The target-space action (2.5) is not time-translation invariant. In order to describe physical fluctuations it is convenient to absorb the $e^{-2\Phi}$ pre-factor by a field redefinition, which has the form

$$U(X) = e^{-\Phi(X)}\, T(X) \tag{3.6}$$

for tachyons. A fluctuation $U_k(X) = U_k(X_0)\, e^{ik_iX^i}$ satisfies a linear equation,

$$\partial_0^2 U_k + (k^2 + V''(T_B) - \frac{q^2}{4})U_k = 0\,. \tag{3.7}$$

Near the top of the potential the tachyon background is well approximated by the exponential form (3.5) and we can drop the contribution of all but the leading terms of the potential $V(T)$ in the fluctuation equation, whereupon (3.7) becomes

$$\partial_0^2 U_k + (k^2 - 2 - \frac{q^2}{4})U_k + \tilde{\lambda} e^{(-\frac{q}{2}+\sqrt{\frac{q^2}{4}+2})X^0}\, U_k = 0, \tag{3.8}$$

where $\tilde{\lambda} = \lambda\, V'''(0)$. If we change variables from X^0 to the scale factor $a = e^{\frac{\alpha}{2}X^0}$ this takes a more conventional form,

$$\left\{\frac{\alpha^2}{4}(a\frac{\partial}{\partial a})^2 + (k^2 - 2 - \frac{q^2}{4}) + \tilde{\lambda} a^2\right\}U_k = 0. \tag{3.9}$$

Up to factor-ordering ambiguities, this is the Wheeler-DeWitt equation derived from the mini-superspace Lagrangian of two-dimensional gravity,

$$L = -(\frac{\dot{a}}{a})^2 - \frac{1}{\alpha^2}\left[k^2 - (2+\frac{q^2}{4}) + \tilde{\lambda}a^2\right]. \tag{3.10}$$

The three terms in square brackets are the matter, curvature and cosmological constant energy densities.

It seems that we have recovered a more or less conventional Wheeler-DeWitt description of large scale cosmology. In particular, the problem of the cosmological constant is the usual one. In order to obtain vanishing cosmological constant, the exponentially increasing solution for $T(X^0)$ must be fine-tuned to zero. In other words, the tachyon must be delicately balanced at the top of the potential. We are ignorant about the short distance physics, which is supposed to determine the initial state, so we have no way of gauging how likely it is to find the system balanced at the top of this potential. At any rate, such a fine-tuned initial state is not allowed in a quantum theory, because of the uncertainty principle.

However, this is not the whole story. Even if the tachyon background starts out near the top of the potential it will eventually roll into the region where the higher-order non-linear terms in the tachyon beta-function cannot be ignored. As we have already mentioned, different renormalization prescriptions in the two-dimensional theory will lead to different evolutions for the tachyon background. Since the $-T^2$ term in $V(T)$ is universal the different schemes will all agree near $T=0$, but away from the origin they can present very different pictures. For example, the question of whether $V(T)$ has a minimum is scheme-dependent. The key issue here is to identify the definition of the tachyon field most closely corresponding to Wheeler-DeWitt amplitudes in the two-dimensional cosmology. In section 5 we propose a candidate scheme for calculations and obtain the tachyon potential to all orders in T within that framework.

It should be emphasized that the non-linear effects that we are talking about do not disappear in the semi-classical limit $D \to \infty$. In particular, the splitting and joining events described by the non-linear terms of the target-space equations are unsuppressed even at large scales. This may seem surprising because the string coupling is becoming weak, with $e^{\Phi} = e^{-\frac{q}{2}X^0}$. Indeed, the canonical tachyon field $U(X^0)$ defined in (3.6) satisfies

$$[\nabla^2 + (2 + \frac{q^2}{4})]U = \frac{1}{8}e^{-\frac{q}{2}X^0}U^2 . \tag{3.11}$$

As we move toward the semi-classical limit $q \to \infty$, though, the tachyon mass squared increases as $\frac{q^2}{4}$, so that the unstable exponential growth of U compensates the decreasing coupling strength. The existence of string interactions, along with the tachyon instability, shows that the usual Liouville model described by an exponentially growing tachyon background is not the complete theory in the $D > 25$ case.

4. The Running of Coupling Constants

Before delving further into the two-dimensional cosmology, we would like to clarify the connection between the target space equations of motion and the renormalization group flow of couplings in the two-dimensional field theory.

The equations of motion for the target-space fields are that the beta-functions of all two-dimensional couplings vanish. From this one might conclude that the couplings seen by a two-dimensional observer would not run. This, however, is not the correct interpretation. We can think of the equations for the target-space fields as renormalization group equations with $\frac{\alpha}{2}X^0$ identified with the logarithm of the renormalization scale. The X^0 dependence of the coupling functions $T, \Phi, G_{\mu\nu} \ldots$ hence determines their evolution with scale. This connection may appear unfamiliar because the equations of motion (3.1) are second-order in X^0 derivatives, whereas the usual renormalization flows are controlled by first-order equations.[⋆] The higher-order nature of the flows is a special feature of theories containing gravity, where the scale itself is a dynamical variable.

⋆ The equations of motion (3.1) are of course only the leading order approximation to the exact beta-function equations, which include terms with an arbitrary number of derivatives.

The situation is similar to the issue of time evolution in the Wheeler-DeWitt formulation of quantum gravity. We begin with an equation $H_{WD}\,|\Psi\rangle = 0$ which seems to imply that no time evolution occurs. Reinterpreted, though, the equation tells us how the wave function of matter evolves with the expansion of the universe. The Wheeler-DeWitt equation, like the equations of motion for $T, \Phi, G_{\mu\nu}$, is second-order. The first-order Schrödinger equation is only recovered in a semi-classical limit in which gravitational fluctuations become unimportant [18]. In our two-dimensional theory, the semi-classical limit corresponds to taking $D \to \infty$ or equivalently $q \to \infty$. In this limit we will see how the target-space field equations reduce to the familiar renormalization group equations, and how gravitational corrections to the ordinary renormalization group beta-functions can be obtained systematically in a "large q" expansion.

We consider, as a simple example, the case of fluctuations about the linear dilaton background at the top of the tachyon potential with a *flat* target-space metric. A field A_n at the n^{th} mass level in string theory will contribute to the effective action a term

$$\frac{1}{2g_0^2}\int \sqrt{G}e^{-2\Phi}\left\{(\nabla A_n)^2 - 2(1-n)A_n^2 + \cdots\right\}\,. \tag{4.1}$$

Its equation of motion in a linear dilaton background is

$$\Big[\big(\frac{\partial}{\partial X^0}\big)^2 + q\,\frac{\partial}{\partial X^0} + k^2 - 2(1-n)\Big]\,A_n = 0\,. \tag{4.2}$$

As we saw previously, this equation has unstable solutions for $n = 0$, which describe the tachyon rolling off the top of its potential. Note, however, that the solutions for $n \geq 1$ are stable for all values of q. In other words, the dilaton, graviton and higher couplings do not become "tachyonic" for large D.

Now, recalling that the scale factor is $a = e^{\frac{\alpha}{2}X^0}$, we can rewrite (4.2) as

$$\Big[\frac{\alpha^2}{4}\,\big(a\frac{\partial}{\partial a}\big)^2 + \frac{\alpha q}{2}a\,\frac{\partial}{\partial a} + k^2 - 2(1-n)\Big]\,A_n = 0\,. \tag{4.3}$$

Thus when $q \to \infty$ we find a first-order equation,

$$a \frac{\partial}{\partial a} A_n = \left(-k^2 + 2(n-1)\right) A_n \,, \tag{4.4}$$

where we have used that $\alpha \to \frac{2}{q} + O(\frac{1}{q^3})$ for large q. This is the usual lowest order Callan-Symanzik equation for a coupling of bare dimension $2n$. In particular the field $h_{\mu\nu}$ has anomalous dimension $-k^2$ as expected.

The exact target space equations of motion will include complicated higher-derivative terms, which are difficult to compute explicitly, but in the semi-classical limit they will all be suppressed by powers of q^{-2} in the same way as the second-order term in (4.3). To see that, note first of all that the only effect of the linear dilaton background (3.2) on beta-function calculations is to shift the anomalous dimensions of vertices. For example, a tachyon with target space momentum k_μ has its dimension shifted from $d_k = 2 - k^2$ to $d_k = 2 + iqk_0 - k^2$, but this is the only place where an explicit factor of q enters into the tachyon beta-function. One can easily convince oneself of this by considering sigma model Feynman graphs [2]. As a result, all terms in the equations of motion with higher-order derivatives, with respect to the conformal mode, pick up factors of q^{-2}, when we express the equations in terms of the scale factor a. These terms will therefore all vanish in the $q \to \infty$ semi-classical limit and should be viewed as gravitational corrections to the renormalization group beta-functions computed on a flat worldsheet.

In fact, by using simple manipulations, we can rewrite the target space equations of motion as conventional renormalization group equations with gravitational corrections. This is easily illustrated for the example considered above. The right-hand side of (4.4) is the leading contribution to $\beta_n^0(A_i)$, the beta-function of A_n on a flat worldsheet. To obtain the leading order gravitational correction to β_n^0 we write the second-order equation (4.3) as

$$\left[\frac{1}{q^2}(a\frac{\partial}{\partial a}) + 1\right] a\frac{\partial}{\partial a} A_n = \beta_n^0(A_i) \,, \tag{4.5}$$

and solve for the corrected beta-function to the next order in $\frac{1}{q^2}$,

$$\begin{aligned}\beta_n(A_i) &\equiv a\frac{\partial}{\partial a}A_n \\ &= \Big(1-\frac{1}{q^2}a\frac{\partial}{\partial a}+\cdots\Big)\beta_n^0(A_i) \\ &= \beta_n^0-\frac{1}{q^2}\sum_m\frac{\partial\beta_n^0}{\partial A_m}\beta_m^0+O(\frac{1}{q^4})\,.\end{aligned}\tag{4.6}$$

We can use this trick to rewrite any higher-derivative term in the target space equations as a contribution to the renormalization group beta-functions, suppressed by some powers of q^{-2}. In this way a systematic large q expansion can be developed to compute gravitational corrections to beta-functions. The higher-order equation of motion for a given target space field has a number of solutions. For example, we have a choice of sign in the exponential tachyon background (3.5). Only one of these solutions reduces to the expected classical behavior in the limit of large q, and it is not hard to see that this semi-classical branch also provides a solution to the corresponding first-order renormalization group equation with gravitational corrections.

We have so far been considering two-dimensional cosmology with a trivial matter sector, consisting of several free fields. A more complicated theory, involving an interacting matter sector coupled to the conformal mode, provides more stringent tests of the above ideas. One can, for instance, study an asymptotically free sigma-model coupled to gravity. This case was considered in reference [2], taking three of the target-space dimensions compactified to a sphere of time-dependent radius $r(X^0)$, but leaving the remaining $D-3$ spatial coordinates flat. We will not go into the details here, but only discuss the qualitative behavior. The sigma-model coupling strength is $1/r$, and the X^0 dependence of $r(X^0)$ gives the running of the coupling with scale. This is easily checked by inserting a metric of the above form into the target space equations of motion (3.1). In the semi-classical limit the standard renormalization group flow,

$$a\frac{\partial}{\partial a}\left(\frac{1}{r}\right) = 2\left(\frac{1}{r}\right)^3 , \tag{4.7}$$

is reproduced. This calculation is valid for large r, where the sigma-model is weakly coupled, but breaks down as the system passes into the strongly coupled regime. However, since we know that the flat space sigma-model contains only massive particles [19], we may speculate that well below the induced mass scale, the sigma-model degrees of freedom decouple. This would correspond to the effective central charge of the matter becoming smaller at some point in the evolution of the universe.

All this has important consequences for the cosmological constant. The non-trivial sigma-model dynamics generates a two-dimensional vacuum energy, which manifests itself as a source term in the tachyon equation of motion. As explained before, it is the exponential growth of the tachyon field as it rolls off the top of the hill that gives rise to the cosmological constant term in the Wheeler-DeWitt equation (3.9). We might imagine that it would be possible to "fine tune" the initial conditions so that the tachyon stays balanced at the top, and the cosmological constant would thus vanish. In our simpler examples in which the target-space was flat, we saw that this could indeed be done. Now, however, the coupling of the sigma-model to the two-dimensional gravity will make it impossible. There are terms in the target space effective action which couple T and $G_{\mu\nu}$. They can be determined explicitly by beta-function calculations, but for our argument it will suffice to note that there must be some such term because string theory has a non-zero graviton-graviton-tachyon vertex. There will thus be an extra source term involving some power of the target space curvature in the tachyon equation of motion. As the three-sphere contracts, this will knock the tachyon from the top of the potential. We would therefore have to search for new fine-tuned initial conditions to make the tachyon end up balanced at the top of the potential at large scales. This need to account for the matter vacuum energy is just the familiar cosmological constant problem.

5. The Tachyon Beta-Function and the Cosmological Constant

In this section we return to the issue of a large-scale cosmological constant. We will follow the system as the tachyon background rolls off the top of its potential into the non-linear region and investigate whether observers in a two-dimensional universe, interacting with the background, would register a non-zero cosmological constant. In order to discuss the evolution of the tachyon background at large scales we need to compute its beta-function. A more or less standard perturbative approach is described in reference [2] and the leading terms are obtained there. Such calculations rapidly get quite involved and it is not tractable to compute the complete beta-function to all orders. In addition, the cosmological interpretation of the results is sensitive to the choice of perturbative renormalization prescription.

In the semi-classical limit the problem simplifies enormously. The general argument given in the previous section can be applied to the equation of motion for a homogeneous tachyon background. As $q \to \infty$ all terms with higher derivatives, with respect to the conformal mode, will be suppressed. Since all space-derivatives vanish for homogeneous backgrounds the equation becomes first-order in the semi-classical limit,

$$a\frac{\partial}{\partial a}T = -V'(T)\,, \tag{5.1}$$

and the dynamics is completely determined by the shape of the effective potential. The problem is reduced to finding the beta-function for tachyons with vanishing target space momentum, *i.e.* for a constant tachyon field. This might appear to be a trivial task since, according to (2.3), a constant tachyon background only contributes a c-number, $\frac{T}{8\pi}\int d^2\sigma\,\sqrt{\hat{\gamma}}$, to the two-dimensional action. This is too simple a view to take, for it does not take into account the effect of the regularization which is required to define the quantum theory. If, for example, the ultra-violet divergences are cut off using a hard sphere regulator, then the excluded volume introduces non-trivial effects even when T is constant.

The tachyon potential $V(T)$ can be obtained using a lattice method introduced in reference [2]. This regularization scheme is particularly suitable for cosmological applications because the renormalized couplings are directly identified with Wheeler-DeWitt amplitudes. Begin by introducing a square lattice on the fiducial coordinate space with lattice spacing ϵ. In each cell we define an amplitude Ψ on the boundary by integrating over the two-dimensional fields in the interior of that cell, fixing the values on the boundary. This defines an effective theory that lives on the lattice edges. The remaining integration over the boundary values of the fields yields the full path integral. The integrand of the effective theory is given by the product over all cells of the cell amplitudes. Schematically,

$$Z = \int \prod_{\text{cells}} \mathcal{D}\phi_{\text{boundary}} \Psi(\phi_{\text{boundary}}) . \tag{5.2}$$

Renormalization can be carried out by fixing the field values on some sub-lattice and integrating over the field on the remaining lattice edges.

The amplitudes Ψ are by construction Wheeler-DeWitt amplitudes. To introduce target-space fields we can expand $\Psi(\phi_{\text{boundary}})$ in terms of string modes. Let $\hat{X}$ be the zero-mode part of X on the boundary. Then

$$\Psi = (1 - T(\hat{X}) - G_{\mu\nu}(\hat{X})\tilde{a}^{\dagger}_{\mu}a^{\dagger}_{\nu}\cdots)\Psi_0 , \tag{5.3}$$

where Ψ_0 is the free theory amplitude. By requiring the long wavelength behavior to be independent of the cutoff we can define beta-functions for the target space fields $T, G_{\mu\nu} \ldots$. This is certainly not a convenient scheme for beta-function calculations in the presence of general couplings, but in the special case of a constant tachyon field, we can obtain the full answer. Then the partition function is simply

$$\begin{aligned} Z &= \int \prod_{\text{cells}} \mathcal{D}\phi_{\text{boundary}} \left(1 - T\right) \Psi_0(\phi_{\text{boundary}}) \\ &= \left(1 - T\right)^N Z(T = 0) , \end{aligned} \tag{5.4}$$

where the total number of cells N is proportional to $\frac{1}{\epsilon^2}$. The free energy is therefore $F = \epsilon^2 \log Z = \log(1-T)$. The running of the coupling T with ϵ is defined by requiring the partition function to be independent of the cutoff scale. This implies

$$\begin{aligned} 0 &= \frac{\partial}{\partial\epsilon}\frac{1}{\epsilon^2}F(T) \\ &= -\frac{2}{\epsilon^3}F(T) + \frac{1}{\epsilon^2}F'(T)\frac{\partial T}{\partial\epsilon}. \end{aligned} \tag{5.5}$$

We define the beta-function in the usual way,

$$\begin{aligned} \beta(T) &= \epsilon\frac{\partial T}{\partial\epsilon} \\ &= 2\frac{F(T)}{F'(T)} \\ &= -\,2(1-T)\log(1-T). \end{aligned} \tag{5.6}$$

This zero-momentum tachyon beta-function corresponds to the following potential,

$$V(T) = -T + \frac{1}{2}T^2 - (1-T)^2\log(1-T), \tag{5.7}$$

which has the form of an unstable tachyon potential near $T = 0$ and has a stationary point at $T = 1$, which is singular ($V'' \sim \infty$). The potential cannot be continued past the singularity but, as we shall see, the tachyon field never rolls beyond $T = 1$. To see how this works, insert (5.7) into the first-order equation of motion,

$$a\frac{\partial}{\partial a}T = -2(1-T)\log(1-T). \tag{5.8}$$

This is easily solved by writing $(1-T) = e^S$, so that

$$a\frac{\partial}{\partial a}S = 2S. \tag{5.9}$$

There is one integration constant which is determined by initial conditions on T,

$$T = 1 - e^{-\lambda a^2}. \tag{5.10}$$

For small T this solution reduces to the classical cosmological term (2.4) with cosmological constant λ. The equation for S is linear so we see that this example provides a realization of the fact that tachyon field redefinition can eliminate the non-linear terms in the equation of motion. However, the resulting field S is no longer proportional to the Wheeler-DeWitt amplitude in equation (3.6).

Now consider a two-dimensional universe containing some matter excitation interacting with the tachyon background (5.10). The fluctuation equation it satisfies is linear,

$$\begin{aligned} 0 =& a\frac{\partial}{\partial a}\tau + k^2\,\tau + V''(T)\,\tau \\ =& \left(a\frac{\partial}{\partial a} + k^2 - 2 + \tilde{\lambda}a^2\right)\tau\,. \end{aligned} \tag{5.11}$$

In the semi-classical limit this is equivalent to the Wheeler-DeWitt equation (3.9) with non-vanishing cosmological constant.

It is very important for the large-scale cosmology that the tachyon potential we obtained takes precisely the form (5.7). It is the singular behavior at $T = 1$, due to the logarithm, which allows a non-vanishing cosmological constant at large scale. It is quite striking that in spite of the apparently complicated non-linear evolution of the tachyon background T given by (5.10), the linear Wheeler-DeWitt equation obtained from (5.1) is precisely that of mini-superspace Liouville theory in the $q \to \infty$ limit [3,15,20,21]. It is interesting in this context to note that the tachyon background also satisfies a *linear* equation,

$$a\frac{\partial}{\partial a}(1-T) + \tilde{\lambda}\,a^2(1-T) = 0\,. \tag{5.12}$$

This suggests the alternate definition for the canonical tachyon field (3.6),

$$\tilde{U}(X) = e^{-\Phi(X)}(1 - T(X))\,. \tag{5.13}$$

In the large q limit the following linear second-order equation for $\tilde{U}$,

$$\frac{1}{q^2}\left(a\frac{\partial}{\partial a}\right)^2\tilde{U} - \frac{q^2}{4}\tilde{U} + \tilde{\lambda}a^2\tilde{U} = 0\,, \tag{5.14}$$

is equivalent to (5.12). Equation (5.14) differs from the $k = 0$ Wheeler-DeWitt equation (3.9) by the term due to the bare dimension of the tachyon. This is the equation that the $SL(2, \mathbf{C})$ vacuum of string theory satisfies [3], and it is natural to identify that state with the most symmetric or Hartle-Hawking state of a one-dimensional universe [3,22].

For any non-zero cosmological constant the tachyon background (5.10) approaches the minimum of its potential at $T = 1$ as the two-dimensional universe evolves to ever larger scale. It is unclear what conformal field theory, if any, corresponds to a tachyon field sitting at rest at $T = 1$, but we suspect it to be a rather trivial one. By the arguments of Kutasov and Seiberg [20,23] it cannot be a standard matter theory coupled to gravity. Apparently the $T = 1$ fixed point describes the asymptotic behavior of an expanding universe long after all relevant scales (*e.g.* the cosmological constant scale) have been passed. The only remaining degrees of freedom are conformal matter fields from which the scale of the metric decouples. The situation is analogous to that in QCD at very large distance scales where the only degrees of freedom are massless pions. Another closer example is provided by the $D = 0$ one-matrix model. In this case a non-zero cosmological constant corresponds to a matrix potential slightly off criticality. The model flows to the trivial Gaussian matrix model at large scales and the random surface interpretation breaks down.

6. Conclusions

Our results can be summarized as follows. Two-dimensional quantum cosmology can be formulated as a string theory with background fields. The dynamics of fields in the string theory target-space determines the values of coupling constants in the two-dimensional universe. In particular, the cosmological constant in two dimensions is governed by the background tachyon field, which satisfies a non-linear equation of motion. Nevertheless, the non-linear dynamics is such that, for generic initial conditions, a two-dimensional universe interacting with the background obeys a standard linear Wheeler-DeWitt equation with a non-zero cosmological constant.

If this picture is correct then there is nothing in the classical target space dynamics which favors a vanishing cosmological constant at large scales. The question remaining is whether the effects of wormhole topologies can change this conclusion. Including arbitrary worldsheet topologies in string theory turns the classical target-space field theory into a quantum field theory. As emphasized by Coleman, the couplings of the *worldsheet* field theory become quantum variables. However, the effective value of Planck's constant for the target-space theory is itself a field, and is given by $\hbar_{\text{eff}} \propto e^{2\Phi} = e^{-qX^0}$. Thus quantum corrections are expected to become negligible for large X^0, and hence only to influence small scales.

Now consider the quantum mechanics of a tachyon field depending only on X^0. Its Lagrangian is

$$\frac{V}{2g_0^2} e^{qX^0}(-\dot{T}^2 + 2T^2 + \ldots)\,, \tag{6.1}$$

where V is the volume of $(X^1, \ldots, X^D)$-space. If this volume is infinite, then quantum fluctuations are negligible and the tachyon evolves classically. If the volume is finite, then the tachyon, and therefore the two-dimensional cosmological constant, is a true quantum variable. To describe it, a quantum wave-function for the target space fields must be introduced. The form of this wave-function at some value of X^0 for which e^{-qX^0} is already small summarizes the effect of small wormholes. If we assume that this wave-function is of some generic form at $X^0 \sim 0$, then since the subsequent behaviour rapidly becomes classical, the only effect of wormholes is to provide a generic probability distribution for the initial conditions. Thus, we see no way in which target-space quantization can force the large-scale cosmological constant to zero.

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