EQUIVALENCE OF A NEW WAVE EQUATION
TO THE BREIT AND SALPETER EQUATIONS[*]

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ABSTRACT

It is proven that a recently derived simple configuration-space wave equation for two spin-1/2 particles is equivalent to the Breit equation and to the Salpeter equation in first order perturbation theory. The wave equation is based on a simple quasipotential approximation. The potential in the equation is given by the Blankenbecler-Sugar correction series. The proof holds for an arbitrary combination of scalar and vector interactions.

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Interest in 3-dimensional two-body bound state equations for spin-1/2 particles has revived in the last 15 years, due to the success of the $q\bar{q}$ model of mesons. But two important questions have not been answered yet [1,2]. (1) What equation is best? (2) What interaction is best?

This note is about equations. Two have dominated: the Breit equation [3] and Salpeter's reduction [4] of the Bethe-Salpeter equation [5]. As we shall explain below, neither of these "big two" equations can be solved numerically exactly in configuration space. They are only solved in first-order perturbation theory, which is trusted most for non-relativistic systems. Here we will prove the equivalence in first order perturbation theory of these two equations to a recently derived third equation [6] which has favorable configuration space behaviour and may be susceptible to an exact numerical solution.

Light mesons, where relativistic effects are high, are still treated by first-order perturbation theory [7]. A practical configuration-space equation which treats relativistic effects in some way beyond first-order perturbation theory is badly needed. A one-particle example of what we mean is the Dirac equation with a static source. For weak potentials the level splitting can be calculated satisfactorily either from the exact solution of the equation or from first-order perturbation theory. But for stronger potentials it is clearly better to solve the Dirac equation exactly—numerically if need be. Because of the simple form of the Dirac equation in configuration space, such a numerical solution is easy to carry out. For this reason the Dirac equation is accepted as a relativistic wave equation. We would like to have a solvable relativistic wave equation for two spin-1/2 particles.

We first briefly review the two dominant equations. In the CM system the Breit equation is [3]

$$\left[(\gamma^0 m + \gamma^0 \boldsymbol{\gamma} \cdot \boldsymbol{p}) + (\Gamma^0 M - \Gamma^0 \boldsymbol{\Gamma} \cdot \boldsymbol{p}) - E\right] \psi(\boldsymbol{r}) = -\gamma^0 \Gamma^0 W(r) \psi(\boldsymbol{r}) \ . \qquad (1)$$

Here $\gamma^0, \boldsymbol{\gamma}, m$ refer to one particle and $\Gamma^0, \boldsymbol{\Gamma}, M$ refer to the other. The total energy

is E.

It is generally accepted that the primary $q\bar{q}$ binding potential is a scalar plus a vector. Therefore in this note we take the interaction $W(r)$ to be an arbitrary combination of scalar and vector potentials:

$$W(r) = S(r) + \gamma^0\Gamma^0 V(r) . \tag{2}$$

We omit non-binding correction terms such as the Breit interaction, the second term in the Coulomb-gauge photon propagator, and possible corrections to the long-range scalar potential, as they can be treated adequately by first-order perturbation theory.

At first sight the Breit equation appears to be a wave equation with as simple a singularity structure as the Dirac equation, and it looks as easy to solve. Yet 60 years have passed without an analytic solution. It is well known [3] that in the perturbation solution to the Breit equation the second-order terms are bigger than the first-order terms instead of smaller. Presumably for that reason, the Breit equation is treated in practice as a prescription for first-order perturbation theory only.

Bethe-Salpeter equations with an interaction kernel which depends on the relative position only ("instantaneous kernel") were reduced by Salpeter down to three dimensions [4]. Keeping first-order perturbation terms, Salpeter's reduction is

$$[E - \omega_p - \Omega_p]\psi(\boldsymbol{p}) = \lambda_p^+\Lambda_p^+ \int \frac{d\boldsymbol{\ell}}{(2\pi)^3}\, W(\boldsymbol{k}^2)\gamma^0\Gamma^0\psi(\boldsymbol{\ell}), \quad \boldsymbol{k} = \boldsymbol{p} - \boldsymbol{\ell} . \tag{3}$$

Here $\lambda_p^+ \equiv \left[\omega_p + \gamma^0 m - \gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{p}\right]/2\omega_p$, $\Lambda_p^+ \equiv \left[\Omega_p + \Gamma^0 M + \Gamma^0\boldsymbol{\Gamma}\cdot\boldsymbol{p}\right]/2\Omega$ are Casimir energy-projection operators, with $\omega_p \equiv \sqrt{m^2 + \boldsymbol{p}^2}$, $\Omega_p \equiv \sqrt{M^2 + \boldsymbol{p}^2}$. The terms ω_p, Ω_p clearly preclude a numerical exact solution in configuration space. In practice these terms are expanded to order $\boldsymbol{p}^4$ and first-order perturbation theory in configuration space is used.

Recently a wave equation was derived which may be a candidate to be the first numerically exactly solvable simple wave equation for two spin-1/2 particles in configuration space. The equation is [6]

$$(-\nabla^2 + \beta^2)\psi(\mathbf{r}) = -\frac{1}{2E} \left[m - \frac{1}{i} \boldsymbol{\gamma} \cdot \nabla + \gamma^0 E_m \right] \left[M + \frac{1}{i} \boldsymbol{\Gamma} \cdot \nabla + \Gamma^0 E_M \right]$$
$$\times \left[W(r) + \frac{W^2(r)}{2E} \right] \psi(\mathbf{r}) \tag{4}$$

In Eq. (4) the total bound-state energy and the partial energy of each particle are parametrised as

$$E = \sqrt{m^2 - \beta^2} + \sqrt{M^2 - \beta^2} \;, \quad E_m = \sqrt{m^2 - \beta^2} \;, \quad E_M = \sqrt{M^2 - \beta^2} \;. \tag{5}$$

Equation (4) is derived from a simple quasipotential approximation [8,9] to the Bethe-Salpeter equation combined with a correction series in the error of the approximation first given by Blankenbecler and Sugar [10]. The term $W^2/2E$ is the large component of the leading Blankenbecler-Sugar correction, the only one that contributes in first-order perturbation theory. In Ref. [6] we found that Eq. (4) gives the energy levels for the hydrogen atom and positronium without the annihilation term correctly to lowest order in the fine and hyperfine structure, *i.e.* to order α^4. In that case W was given by $S = 0$, $V = -\alpha/r$, the Breit interaction was included, and the correction term $W^2/2E$ was $(\alpha^2/r^2)/2E$.

In the present note we strengthen the case for the wave equation (4) by proving that for any combination of scalar and vector potentials, *i.e.* for the general interaction $W(r)$ given by (2), Eq. (4) gives the same results as the "big two" equations (1) and (3) to first-order perturbation theory.

Since the Breit equation must be solved by first order perturbation theory, Casimir operators are typically used [4] and the result is again Eq. (3). Thus it is sufficient to show that Eqs. (3) and (4) are equivalent in first order perturbation theory.

To that order (3) reads in configuration space:

$$E\phi = \left\{ m + M + \frac{\boldsymbol{p}^2}{2\mu} + [V + S] \right\} \phi \qquad (6a)$$

$$-\left\{ \frac{1}{8} \left(\frac{1}{m^3} + \frac{1}{M^3} \right) \boldsymbol{p}^4 + \frac{1}{4} \left(\frac{1}{m^2} + \frac{1}{M^2} \right) \boldsymbol{p}^2 [V + S] \right\} \phi \qquad (6b)$$

$$+ \frac{1}{4} \left\{ \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}\, \boldsymbol{\sigma} \cdot \boldsymbol{\ell}}{m^2} + \frac{\boldsymbol{\Sigma} \cdot \boldsymbol{p}\, \boldsymbol{\Sigma} \cdot \boldsymbol{\ell}}{M^2} \right\} [V - S]\phi \qquad (6c)$$

in which ϕ is the 4-dimensional large component of the 16-component wavefunction ψ. The reduced mass is μ. The domains of the configuration space differential operators $\boldsymbol{p}, \boldsymbol{\ell}$ are described by the integral form, Eq. (3).

To expand our wave equation (4) Casimir operators are not required. For $E < m + M$, it was shown in Ref. [6] that the operators on the right hand side of (4) are non-singular and that each of them is a divisor of $(-\nabla^2 + \beta^2)$. Thus (4) is identical to

$$\left[m + \frac{1}{i} \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} - \gamma^0 E_m \right] \psi(\boldsymbol{r}) =$$
$$- \frac{1}{2E} \left[M + \frac{1}{i} \boldsymbol{\Gamma} \cdot \boldsymbol{\nabla} + \Gamma^0 E_M \right] \left[W(r) + \frac{W^2(r)}{2E} \right] \psi(\boldsymbol{r}) \qquad (7)$$

and is also identical to

$$\left[M - \frac{1}{i} \boldsymbol{\Gamma} \cdot \boldsymbol{\nabla} + \Gamma^0 E_M \right] \psi(\boldsymbol{r}) =$$
$$- \frac{1}{2E} \left[m - \frac{1}{i} \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + \gamma^0 E_m \right] \left[W(r) + \frac{W^2(r)}{2E} \right] \psi(\boldsymbol{r}) . \qquad (8)$$

The identity of Eqs. (4), (7) and (8) show that each spin-1/2 particle obeys a Dirac equation in the field of the other, as would be expected physically. The Dirac equation (7) gives a Pauli reduction (small component) $= (-\boldsymbol{\sigma} \cdot \boldsymbol{p}/2m)$ (large

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component) for its particle. A similar Pauli reduction for the other particle from the other Dirac equation (8) gives all components of ψ terms of the 4-dimensional large component ϕ.

To express the resultant equation for ϕ we need to define the eigenvalue of the zero-order equation:

$$\frac{-\beta_0^2}{2\mu}\,\phi_0 = (E_0 - m - M)\,\phi_0 = \left(\frac{\mathbf{p}^2}{2\mu} + [V + S]\right)\,\phi_0 \;. \tag{9}$$

From Eq. (5) up to first-order perturbation theory, replacing β^2 by β_0^2 in the perturbation term, we have

$$E = m + M - \frac{\beta^2}{2\mu} - \frac{1}{8}\left(\frac{1}{m^3} + \frac{1}{M^3}\right)\beta_0^4 \;.$$

Solving this equation for β^2 we can substitute for β^2 in Eq. (4). Also from (5) we have to the required order

$$\frac{(m + E_m)(M + E_M)}{2E} = 2\mu\left\{1 - \frac{\beta_0^2}{4}\left(\frac{1}{m} - \frac{1}{M}\right)^2\right\} \;.$$

Finally, $(m + M)$ may be substituted for E in the Blankenbecler-Sugar term. From all these, Eq. (4) gives an equation for ϕ, the large component of ψ:

$$E\phi = \left\{m + M + \frac{\mathbf{p}^2}{2\mu} + [V + S]\right\}\phi \tag{10a}$$

$$+ \left\{-\frac{1}{8}\left(\frac{1}{m^3} + \frac{1}{M^3}\right)\beta_0^4 - \frac{1}{4}\left(\frac{1}{m} - \frac{1}{M}\right)^2\beta_0^2[V + S] + \frac{[V + S]^2}{2(m + M)}\right\}\phi \tag{10b}$$

$$+ \frac{1}{4}\left\{\frac{\boldsymbol{\sigma}\cdot\mathbf{p}\,\boldsymbol{\sigma}\cdot\boldsymbol{\ell}}{m^2} + \frac{\boldsymbol{\Sigma}\cdot\mathbf{p}\,\boldsymbol{\Sigma}\cdot\boldsymbol{\ell}}{M^2}\right\}[V - S]\,\phi \;. \tag{10c}$$

We wish to show that Eqs. (6) and (10) give identical results. The zero-order binding potential in lines (6a) and (10a) is the same, and is the *sum* of the

vector and scalar potentials. Lines (6c) and (10c) are also identical, and represent the well-known spin-independent and spin-orbit terms where the scalar potential appears with *opposite* sign to the vector potential. To complete the proof it is only necessary to show that the expectation values of the lines (6b) and (10b) are identical no matter what V and S are.

The procedure is familiar. The expectation value of line (6b) is

$$-\left\langle \phi_0 \left| \frac{1}{8} \left(\frac{1}{m^3} + \frac{1}{M^3} \right) \boldsymbol{p}^4 + \frac{1}{4} \left(\frac{1}{m^2} + \frac{1}{M^2} \right) \boldsymbol{p}^2 [V + S] \right| \phi_0 \right\rangle . \qquad (11)$$

From Eq. (9) we have $\boldsymbol{p}^2 \phi_0 = -(\beta_0^2 + 2\mu[V + S])\phi_0$. Bearing in mind that in momentum space $\boldsymbol{p}^4 = \boldsymbol{p}^2 (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{\ell})\boldsymbol{\ell}^2$ (see Eq. (3)), the expectation value (11) is equivalent to

$$\left\langle \phi_0 \left| \; -\frac{1}{8} \left(\frac{1}{m^3} + \frac{1}{M^3} \right) (\beta_0^2 + 2\mu[V + S])^2 \right. \right.$$
$$\left. \left. + \frac{1}{4} \left(\frac{1}{m^2} + \frac{1}{M^2} \right) (\beta_0^2 + 2\mu[V + S])[V + S] \; \right| \phi_0 \right\rangle . \qquad (12)$$

It is elementary to show that the coefficients of 1, $\langle [V + S] \rangle$ and $\langle [V + S]^2 \rangle$ in (12) are all exactly the same as those of (10b). This completes the proof.

In conclusion, we have shown that the wave equation (4) gives the same results in first order perturbation theory as the "big two" equations (1) and (3) for any combination (2) of vector and scalar potentials. The point of the proof was to establish Eq. (4) as a legitimate alternative to the Breit and Salpeter equations, as a preliminary to solving the equation numerically in situations—*e.g.* light mesons— which are too relativistic for first order perturbation theory to handle.

Since Eq. (4) would have 32 components when written as a set of first-order differential equations, the numerical solution would presumably start from one of the 16-component Dirac equations (7) or (8), which are equivalent to (4). That work is beyond the scope of the present note.

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