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Borel-Summable Perturbation Series for Theories with Degenerate Minima*

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ABSTRACT

We shed new light on the large-order behavior of the perturbation theory for the quantum mechanics with degenerate minima. The dominant contribution at large-order of perturbation is identified as a bounce-like solution of an effective theory in Euclidean path-integral formalism. Based on this observation, we define an improved perturbation theory, which utilizes the symmetry of the theory. It is shown to yield a Borel-summable series.

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The perturbation theory for theories without degenerate minima is known to be asymptotic series:[1−3] The absolute value of the perturbative coefficient diverges as the order of perturbation diverges, but they alternate in sign. Due to this property, a finite-order perturbative result is reliable for small coupling constant. Further, one could safely apply several techniques, i.e., the Borel-transformation or Pade approximation, to obtain good estimate for the exact value. Especially, the Pade approximation is known to give excellent converging approximation for Borel-summable series.[4]

The situation is different when there are degenerate minima in the potential. The asymptotic estimates[5] show that the perturbation is in fact divergent; the perturbative coefficient diverges without alternating its sign.[6] This divergence is related with the existence of instanton in the theory: The perturbative series has a cut on the physical region of the coupling constant, which should be cancelled by instanton contribution. In this case, however, not much is known about convergence of re-summation methods. Thus the meaning of the results of the perturbation expansion of this type of theories is at best hard to comprehend. This difficulty is of rather serious consequence, since it also applies to some field theories, including QCD and Electroweak theory.[7,8]

In this letter, we present an analysis different from the previous works[5] and propose an improved perturbation theory, a means of resummation, which at large-order yields a Borel-summable series.

Let us take an Euclidean functional integral

$$\mathcal{Z}(g) \equiv \mathcal{N} \int \mathcal{D}\phi \; e^{-S[\phi,g]}, \tag{1}$$

under the periodic boundary condition in the imaginary time $\tau \in [0,\beta]$. (The exponent of the leading term for $\beta \to \infty$ gives the ground state energy.) The normalization factor $\mathcal{N}$ is defined by $\mathcal{Z}(0) = 1$. The action $S[\phi, g]$ is taken to be in the following,

$$S[\phi, g] = \int_0^\beta d\tau \left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{g^2}V(g\phi) \right), \quad V(\phi) = \frac{1}{2}\phi^2(1-\phi)^2. \tag{2}$$

Expanding in the coupling constant g, we define the n-th order integrand $F_n[\phi]$ by,

$$e^{-S[\phi,g]} = \sum_{n=0}^{\infty} g^n F_n[\phi]. \tag{3}$$

We can express $F_n[\phi]$ as an contour integral in the complex g-plane as,

$$\begin{aligned} F_n[\phi] &= \frac{1}{2\pi i}\oint \frac{dg}{g^{n+1}} e^{-S[\phi,g]} \\ &= e^{-S_f}\frac{1}{2\pi i}\oint \frac{dg}{g} e^{-(-S_3 g + \frac{1}{2}S_4 g^2 + n\log g)}, \end{aligned} \tag{4}$$

where S_f is the free action and

$$S_m = \int_0^\beta d\tau \phi^m \quad (m = 3, 4). \tag{5}$$

For $n \gg 1$, the saddle-point approximation is valid. There are two saddle points on the real axis of g for $S_3^2/4nS_4 \geq 1$ and off the real axis otherwise. Let us first take the latter case. We denote the two saddle points by $g_\pm = ge^{\pm i\theta}$, where g and θ are defined by

$$g \equiv \sqrt{\frac{n}{S_4}}, \quad \cos\theta \equiv \frac{S_3}{\sqrt{4nS_4}}. \tag{6}$$

We choose that $\theta \in [0, \pi]$, so that g_+ is always in the upper plane. At these points, we find that the real parts of $\tilde{S}''[\phi, g_\pm](\equiv \tilde{S}''_\pm)$ is positive. Therefore we choose the integration contour to go through the saddle points g_+ anti-parallel to the real axis and through g_- parallel to the real axis (see Fig.1). The gaussian approximation then yields

$$F_n[\phi] = \frac{2}{\sqrt{2\pi}}\,\mathrm{Im}\left[\frac{1}{\sqrt{S''_- g_-^2}} \frac{e^{-S[\phi,g_-]}}{g_-^n} \right]. \tag{7}$$

Note that g_- in the above is a functional of ϕ.

In order to find the maximum of the $F_n[\phi]$, we solve for the following equation,

$$\frac{\delta F_n[\phi]}{\delta\phi(\tau)} = \left.\frac{\partial F_n[\phi]}{\partial g_-}\right|_\phi \frac{\delta g_-}{\delta\phi(\tau)} + \left.\frac{\delta F_n[\phi]}{\delta\phi(\tau)}\right|_{g_-} = 0. \tag{8}$$

At the leading order of $\hbar$, however, the g_- derivative in the first term is zero simply because of the saddle-point condition. Thus we only need to take the second term, explicit derivative (with g_- fixed) with respect to ϕ. The resulting equations can be further simplified by scaling out the absolute value g by using the variable $\varphi(\tau) = g\phi(\tau)$. After some algebra, we find the following "effective" equation of motion,

$$-\frac{\partial^2\varphi}{\partial\tau^2} + \frac{\partial V_{\text{eff}}(\varphi)}{\partial\varphi} = 0, \tag{9}$$

where the effective potential $V_{\text{eff}}(\varphi)$ is defined by,

$$V_{\text{eff}}(\varphi) = \frac{1}{2}\varphi^2 - k_3\varphi^3 + \frac{1}{2}k_4\varphi^4, \tag{10}$$

with the effective coupling constants $k_{3,4}$,

$$k_3 \equiv \frac{\sin(\sigma-\theta)}{\sin\sigma}, \quad k_4 \equiv \frac{\sin(\sigma-2\theta)}{\sin\sigma}. \tag{11}$$

The angle σ in the above is the phase,

$$\sigma \equiv \arg\left[\frac{1}{\sqrt{S''_- g_-^2}} \frac{e^{-S[\phi,g_-]}}{g_-^n} \right] = n\left(\theta - \frac{1}{2}\sin(2\theta)\right) + \frac{\pi}{4} + \frac{\theta}{2}. \tag{12}$$

From the equations (6), we find that

$$\cos\theta = \frac{R_3}{2R_4}, \tag{13}$$

where

$$R_m \equiv \int d\tau \varphi^m. \tag{14}$$

Therefore the solutions can be now identified by first solving the equation of motion (9), and then by solving the self-consistent equation (13) for θ. By integrating the equation of motion (9) and using the equations given so far, we obtain the following expression of the value of $F_n[\phi]$ at a solution,

$$F_n[\phi] = \frac{1}{\sqrt{\pi n \sin\theta}} \left(\frac{n}{R_4}\right)^{n/2} e^{-\frac{n}{2}(1-\sin(2\theta)\cot\sigma)} \sin\sigma. \tag{15}$$

The solutions actually come in pairs. It is because for $\theta \to \pi - \theta$,

$$\begin{aligned} \sigma &\to (n+1)\pi - \sigma, \\ k_3 \to -k_3, &\quad k_4 \to k_4, \\ F_n[\phi] &\to (-1)^n F_n[\phi]. \end{aligned} \tag{16}$$

Therefore, for every solution (θ, ϕ), there exists another solution $(\pi-\theta, -\phi)$. This is a result of the obvious symmetry $F_n[-\phi] = (-1)^n F_n[\phi]$.

Let us now look for the solutions of the equations, (9)–(14). For large n, σ and $k_{3,4}$ are highly oscillating functions of θ. For small θ,

$$k_3 \sim 1 - \theta\cot\sigma, \quad k_4 \sim 1 - 2\theta\cot\sigma, \tag{17}$$

and

$$\sigma \sim \frac{2}{3}n\theta^3 + \frac{\pi}{4}. \tag{18}$$

Thus $k_{3,4}$ diverges at $\theta = \theta_c = (9\pi/8n)^{1/3}$. We shall look for the solution for $0 \le \theta \le \theta_c$. From (17), we find that the effective potential has zeros at $\varphi = 0, 1/k_4$

and 1. For large β, the solution of eqation of motion is a bounce solution which starts from $\varphi = 0$ and bounces back at $\min(1/k_4, 1)$. Using the "energy" conservation law, we can write down the functions $R_{3,4}$ as integrals

$$R_m = 2\int_0^{\min(1/k_4,1)} \frac{\varphi^m \, d\varphi}{\sqrt{\varphi^2(1-\varphi)(1-k_4\varphi)}}. \tag{19}$$

Since $\theta \ll 1$, the self-consistency condition to be solved is now

$$R_3(k_4) = 2R_4(k_4). \tag{20}$$

We have solved this equation numerically and have found that it has only one solution $k_4 \approx 1.740406$, where $R_4 \approx 0.143582$. The value of θ is then obtained as

$$\theta = \theta_c - \frac{8}{9\pi}\frac{1}{k_4 - 1}\theta_c^2 + O(\theta_c^3). \tag{21}$$

Thus we can justify the original assumption that $0 \le \theta \le \theta_c$. We denote this solution by ϕ_+ and its partner at $\pi - \theta$ by $\phi_-(= -\phi_+)$. We find that

$$\begin{aligned} F_n[\phi_+] &\approx C n^{-2/3} n^{n/2} A^{n/2}, \\ F_n[\phi_-] &\approx (-1)^n C n^{-2/3} n^{n/2} A^{n/2}, \end{aligned} \tag{22}$$

where C is a positive number of $O(1)$ and $A = e^{-k_4}/R_4 \approx 1.22194$.

Next let us look at the region $S_3^2/4nS_4 \ge 1$, when we have the following saddle-points on the real axis of g,

$$\bar{g}_\pm = \frac{S_3}{2S_4}\left(1 \pm \sqrt{1 - \frac{4nS_4}{S_3^2}}\right). \tag{23}$$

We notice that $\mathrm{sign}(\tilde{S}''(\bar{g}_\pm)) = \mp 1$. Therefore, we can choose the contour to go through $\bar{g}_-$in parallel to the imaginary axis. The condition for the maxima (8) then becomes an ordinary equation of motion with the effective coupling constant

$\bar{g}_-$, which is to be determined self-consistently. The (approximate) solution is then an instanton and anti-instanton pairs[5] well-separated by distance d. Since for large d,

$$S_3 \approx \frac{d}{g^3}, \quad S_4 \approx \frac{d}{g^4}, \tag{24}$$

the self-consistent equation for $\bar{g}_-$ is

$$\bar{g}_- \approx \frac{\bar{g}_-}{2}\left(1 - \sqrt{1 - \frac{4n\bar{g}_-^2}{d}}\right). \tag{25}$$

Since this equation cannot be satisfied by any real value of $\bar{g}_-$, we find that there is no maxima in this region of the functional space.

In the present context, we find that the calculation by Brezin et.al.[5] uses the saddle point $\bar{g}_+$. In order to obtain the perturbative coefficient

$$c_n = \mathcal{N}\int \mathcal{D}\phi F_n[\phi], \tag{26}$$

they first reduce the ϕ-integral to the integration over the d-integration. They then rotate the d-integration contour in the d-complex plane and then finally carry out the g integration. This is equivalent to doing the g-integration along the real axis through $\bar{g}_+$, and then doing d-integration along the imaginary axis. In fact, one can show that the self-consistent equation for $\bar{g}_+$ yields their value of the effective coupling constant, $g^2 = 2/3n$. While this is perfectly fine for obtaining c_n, this is useless for our purpose, since it yields a purely imaginary $F_n[\phi]$.

Since the leading behavior of the c_n is known, one could compare it with the value of $F_n[\phi]$ at the maxima (22). Since $F_n[\phi]$ is an odd functional of ϕ for odd n, c_n is obviously zero. For even n, the known result is $c_n \approx n^{n/2}$. Thus the most leading term agrees with (22).[9]

We shall now discuss the improved perturbation theory. The action (2) is symmetric under $\phi \to 1/g - \phi$. Therefore the functional integral (1) can be written as

$$\begin{aligned} \mathcal{Z}(g) &= \mathcal{N}'\left(\int_{\langle\phi\rangle<1/2g} + \int_{\langle\phi\rangle>1/2g}\right) d\phi\ e^{-S[\phi,g]} \\ &= 2\mathcal{N}'\int_{\langle\phi\rangle<1/2g} d\phi\ e^{-S[\phi,g]}. \end{aligned} \tag{27}$$

Namely, we choose to integrate only over the fundamental region and multiply the symmetry factor two on the result. (In this sense, this prescription is readily applicable to other theories with degeneracy, including field theories.) We shall fix the normalization factor later. By expanding the integrand with respect to the coupling constant g as before, we define the new perturbation series as follows

$$\mathcal{Z}(g) = \sum_{n=0}^{\infty} k_n(g) g^n, \tag{28}$$

where

$$k_n(g) = 2\mathcal{N}'\int_{\langle\phi\rangle<1/2g} d\phi\ F_n[\phi]. \tag{29}$$

For $g \to 0$,

$$\mathcal{Z}(0) = k_0(0) = 2\mathcal{N}'\int_{\langle\phi\rangle<\infty} d\phi\ e^{-S_f}. \tag{30}$$

Therefore we find that the normalization factors are related by $\mathcal{N} = 2\mathcal{N}'$. Because of this, we find the following relation between the coefficient c_n of the naive perturbation theory and the coefficient $k_n(g)$ of the proposed perturbation theory,

$$k_n(g) = c_n - \mathcal{N}\int_{\langle\phi\rangle>1/2g} d\phi\ F_n[\phi] \tag{31}$$

At the lower order of the perturbation, $k_n(g)$ differs from c_n by only a small amount, which is non-perturbative in g. This is because in (31) the second term is dominated by the boundary of the integration $\langle\phi\rangle = 1/(2g)$. This term can be estimated by minimizing the free action under the constraint $\langle\phi\rangle = 1/(2g)$. Using the Lagrange-multiplier method we find that the solution is $\phi = 1/(2g)$, which has the free action $\beta/(8g^2)$. Therefore, the dominant contribution form the second term is of $O((\beta/g^2)^n e^{-\beta/(8g^2)})$.

As n increases, the second term becomes significant. Earlier in this letter we have identified the major contribution $\phi_\pm$ to $F_n[\phi]$. The maxima have

$$\langle\phi_\pm\rangle = \pm c\frac{\sqrt{n}}{\beta}, \tag{32}$$

where c is a positive constant of $O(1)$. Therefore, as n exceeds $(\beta/2cg)^2$, ϕ_+ moves outside of the integration region and only ϕ_- remains as a major contribution to $k_n(g)$. For even n, this has an effect of reducing the value of $k_n(g)$ to half of c_n. For odd n, it yields negative result, which is about half of the appropriate analytic continuation of c_n. In other words, for $n \to \infty$, the most leading terms are

$$k_n \to \begin{cases} n^{n/2} & \text{for } n = \text{even}, \\ -n^{n/2} & \text{for } n = \text{odd}, \end{cases} \tag{33}$$

Thus our improved perturbation theory yields a Borel summable series.

In Fig.2, we give results of the ordinary perturbation theory and the improved perturbation theory for a simply toy model, where the time dependence is frozen. It is apparent that the improved perturbation theory gives result that oscillates around the exact value with increasing amplitude, just as any asymptotic series does, while the naive result simply diverges at higher orders.

In summary, we have estimated the n-th order functional $F_n[\phi]$ for $n \gg 1$. We have found a pair of configurations that maximizes $F_n[\phi]$, which are bounce solutions of height $\propto \pm O(\sqrt{n})$. Unlike the previous analysis by Brezin, et.al., these configurations do not follow the classical equation of motion, since two complex pairs of the saddle-points in the coupling-constant plane have contributions

of equal strength. We have proposed an improved perturbation theory, in which one integrates only over the fundamental region of the functional space that includes the perturbative vacuum one is expanding about. Using the knowledge of $F_n[\phi]$, we have found that the resulting series have at large even order half the coefficient of the naive perturbation coefficient and at odd order negative but of the same leading order asymptotically. Thus the improved perturbation series is Borel-summable and allows application of the various re-summation methods.

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FIGURE CAPTIONS

1. The integration contour for g-integration.
2. The result of the perturbation theories for the double-well model with time frozen at $g = 0.1$. The horizontal axis is the (highest) order of the perturbation. The dash-dotted line is the exact value, the dotted line the naive perturbation, and the solid line the improved perturbation.

REFERENCES

1. L. N. Lipatov, *Sov. Phys. JETP* **45** (1977) 216.
2. E. Brezin, J. C. Le Guillou and J. Zinn-Justin, *Phys. Rev.* **15** (1977) 1544, 1558.
3. The above and many of the relevant references can be found in; "Large-Order Behaviour of Perturbation Theory" (edited by J. C. Le Guillou and J. Zinn-Justin, North-Holland 1990).
4. S. Graffi, V. Grecchi, and B. Simon, *Phys. Lett.* **B32** (1970) 631.
5. F. Brezin, G. Parisi and J. Zinn-Justin, *Phys. Rev.* **D16** (1977) 408.
6. An related analysis of a Borel-summability appears in ; H. Kikuchi, Yukawa Institute Preprint, YITP/K-937 (July 1991).
7. A. V. Yung, Leningrad preprint, LENINGRAD-90-1617 (July 1990).
8. H. Aoyama and H. Kikuchi, To appear in Nucl. Phys. B. (1991)
9. The gaussian integration around the $\phi_\pm$ may be necessary to reproduce the next-leading term, $(\text{const})^n$. This problem is left for future study.

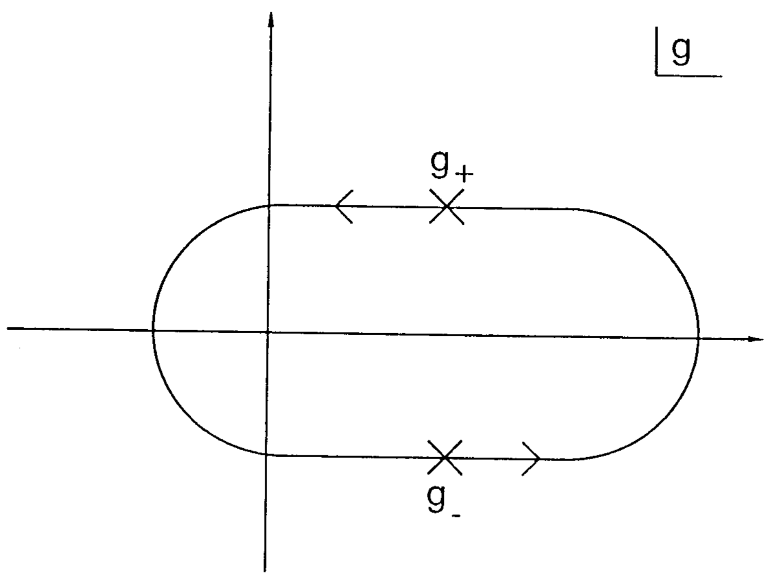

Fig.1

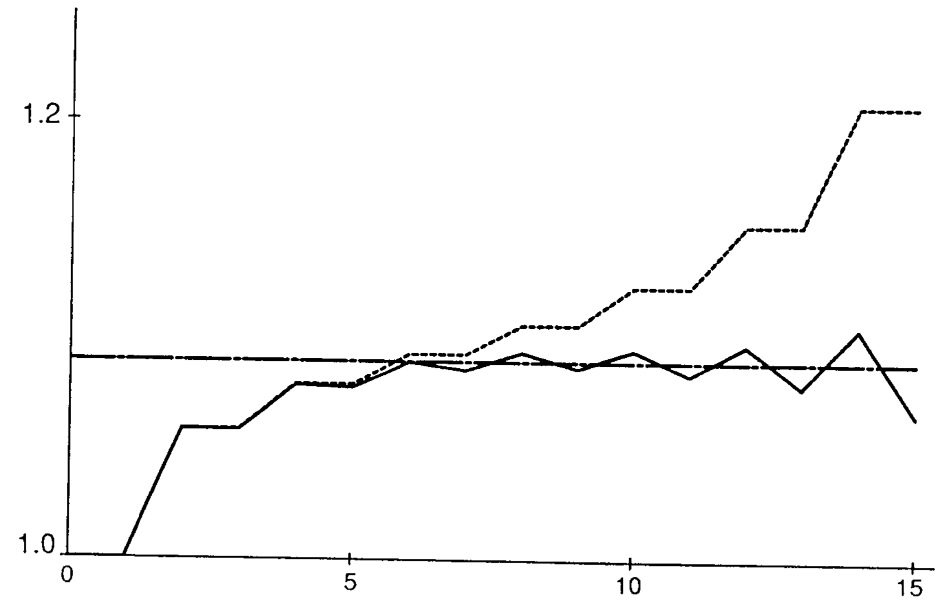

Fig.2