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Dependence of Yukawa Couplings on the Axionic Background Moduli of Z_N Orbifolds*

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Abstract

The metrical and axionic background moduli which determine a general symmetric Z_N orbifold model have to be chosen in such a way that the rotational twist leaves the underlying σ–model action invariant. A thorough analysis of this condition will be given. We notice that it plays a key role in the evaluation of the four–point correlation functions of ground states which belong to the lowest twisted sectors. Having fixed the normalization of these functions we factorize them w.r.t. the twisted intermediate channel. This method yields the moduli dependent part of the twisted sector Yukawa couplings of an orbifoldized heterotic string model. We then perform various discrete mappings (axionic shifts, duality) on the space of background moduli and recognize that the induced linear transformations of the Yukawa couplings are essentially independent of the choice of a specific background. If compensating unitary redefinitions of the twist fields are applied then orbifold models whose backgrounds are related by one of the above mappings cannot be distinguished. For many twist orders we arrive at an explicit form of the phase factors needed to redefine twist fields in order that a general discrete axionic shift can be undone. The requirement of duality invariance is sufficient to determine the moduli dependence of the Yukawa couplings. Hence one may even bypass the evaluation of instanton actions.

1 Introduction

The vacuum of the heterotic string theory [1] is a two–dimensional conformal field theory (2D CFT) [2, 3], which possesses all the necessary features to be a serious candidate for a unified theory of all interactions (including gravity) and all matter fields. The effective field theory turns out to be an $N = 1$ supergravity action if a Calabi–Yau manifold describes the internal compact space (in this case the Kähler metric gives rise to the holonomy group $SU(3)$). *Orbifold models* may be understood as certain geometrically degenerate limits of such a manifold. Some of these compactifications [4] indeed give rise to four–dimensional heterotic string vacua whose particle contents closely resembles that of the standard model [5, 6, 7].

For a detailed phenomenological analysis of such models a string theory computation of the effective action is mandatory. In order to find out how Yukawa couplings depend on the Kähler structure moduli of orbifold models, it suffices to merely consider three–point functions of *bosonic* twist fields σ. The Yukawa coupling is then recovered from a string theory computation by also including a (moduli dependent) normalization factor. However, to gain a full understanding one must also allow for non–trivial Wilson line configurations. There are two distinct ways to embed the spatial twist into the $\mathrm{E}_8 \times \mathrm{E}_8$ gauge algebra lattice: either a twist or a shift can be chosen. In the first case Wilson lines can be continuously deformed (i.e. they represent additional moduli) whereas in the second case their components are quantized [5, 8]. The requisite computation of Yukawa couplings for these embeddings has not yet been performed.

According to [9], [10] the scalar potential is entirely determined by

$$K + \ln W + \ln \bar{W} \tag{1.1}$$

where K is the Kähler potential and W is the holomorphic superpotential. Both these quantities are given in terms of the field theory counterparts M, A of the moduli vertices V_M and the associated charged matter vertices V_A from the underlying $N = 2$ supersymmetric CFT.

In the orbifold limit one has to distinguish between the fields M_{U}, A_{U} belonging to the untwisted sector U and additional fields M_{T}, A_{T} which originate from a twisted sector T. Both e^{-K} and W contain (besides other terms) contributions which are cubic in the superfields A_{U}, A_{T}, respectively. In particular, the strength of these trilinear interactions (the one belonging to W amounts to a *Yukawa coupling*) is governed by functions of the Kähler structure and complex structure moduli of the six-dimensional compactified target subspace. Moreover, K and W are intimately related in compactified heterotic string models. Because the corresponding moduli spaces are *restricted* Kähler manifolds, the set of Yukawa couplings suffices to uniquely determine K as well.

A more detailed discussion of the relationship between a heterotic string theory and its effective action can be found in [11], [12] where additional literature has been pointed out.

Other three-point functions which correspond to Yukawa couplings exclusively involving fields from the untwisted sector have been determined in [13]. The knowledge of orbifold correlation functions also allows for a thorough investigation of the symmetry properties of the CFT moduli space. As has been argued in [14] the form of the low-energy action is severely restricted provided that the non-perturbative effects of string theory do not spoil these background symmetries. Furthermore, apart from their interpretation as string vacua, a study of these CFTs is worthwhile from the point of view of 2D quantum field theory, since they constitute examples of exactly solvable *irrational* models for a generic choice of the background parameters.

The outline of this paper is as follows: In section 2 we present a detailed discussion of the moduli contained in the antisymmetric tensor B and the torus metric. We assume that the background B commutes with the twist operation Θ in order to have a consistent action formulation of the orbifold CFT. In fact, this restriction on the components of B proves to be indispensable for the calculation of the instanton contribution to the four-twist function (see section 3). This correlator is then used to derive the twisted sector string emission coupling via s–channel factorization.

The moduli dependent part of a twisted sector Yukawa coupling for general symmetric Z_N orbifolds is the subject of section 4. Discrete axionic shifts and the duality inversion of the background matrix modify these correlation functions. However by introducing unitary redefinitions of the twist fields w.r.t. the new background values any such change can be compensated. An analysis of these two types of symmetry operations is presented in section 5 and 6. We discuss the main results and present a list of some open questions in section 7.

2 Action description and moduli

The starting point for the construction of two–dimensional (2D) orbifold conformal field theories (CFTs) is the linear σ–model action [15]

$$S_{\mathrm{E}} = \frac{1}{2\pi} \int dz\, d\bar{z} \left\{ \bar{\partial} X^{\mu}(z,\bar{z})\, (G+B)_{\mu\nu}\, \partial X^{\nu}(z,\bar{z}) \right\}, \qquad (2.1)$$

where $B_{\mu\nu} = -B_{\nu\mu}$ denotes the components of the (constant) *antisymmetric background tensor* B. The d–dimensional *target manifold*, whose constant metric is conveniently chosen to be[1] $G = \frac{1}{2}\mathbf{1}_d$, is parametrized by the string *coordinate fields* $X^{\mu}(z,\bar{z})$.

To define a toroidal orbifold we invoke two sorts of closed string boundary conditions:

$$X(e^{2\pi i}z, e^{-2\pi i}\bar{z}) = \left\{ \begin{array}{rl} X(z,\bar{z}) + 2\pi e_j & (1 \leq j \leq d) \\ \theta_k X(z,\bar{z}) & (1 \leq k \leq t) \end{array} \right. . \qquad (2.2)$$

They will now be discussed in turn. The set of *winding vectors* $\{e_1, \ldots, e_d\}$ is assumed to form a basis of $\mathbb{R}^d$. Let us introduce the *basis matrix* e whose j–th column vector is e_j $(1 \leq j \leq d)$. Obviously one has to identify points of the target space if their difference $2\pi w$ is an element of the d–dimensional lattice $\Lambda_d := \{en; n \in \mathbb{Z}^d\}$. Thus the first restriction forces the string to propagate on the torus $T_d = \mathbb{R}^d/(2\pi\Lambda_d)$. The second condition features t

[1]This corresponds to the Regge slope parameter $\alpha' = 2$.

rotations θ_k which generate a finite *point group* P. We take it for granted that each θ_k acts as an isometry on the fundamental lattice Λ_d. By repeated composition of strings subject to (2.2) one finds that the complete set of boundary conditions amounts to the semidirect product $\mathrm{P} \ltimes \Lambda_d$ which is called the *space group* S. Being a subgroup of the Euclidean group in $\mathbb{R}^d$ its elements can be conveniently labeled (θ, w). Actually we might tolerate certain rotations θ even if they will not act as isometries of Λ_d. However it must again be possible to arrive at a (discrete) space group of the form $\mathrm{P} \times \hat{\Lambda}_d$ where the new lattice $\hat{\Lambda}_d$ comprises Λ_d.

In the sequel we will concentrate on *symmetric Z_N orbifolds* where a twist θ acts in the same way on the right– and left–moving parts[2] of $X(z,\bar{z})$. Since in this case P is generated by a rotation Θ of finite order N ($\Theta^N = \mathbf{1}_d$) we have

$$S = \left\{(\Theta^k, w) | k = 0, \ldots, N-1;\ w \in \Lambda_d\right\} \qquad . \tag{2.3}$$

As has been pointed out in [13], the action (2.1) is well–defined only if[3]

$$[B, \Theta] = 0 \qquad . \tag{2.4}$$

This strong condition will be solved below. Later (see section 3) we will recognize that it is both necessary and sufficient to permit the computation of the classical parts of twist field correlation functions.

We will first concern ourselves with the case of even d. Upon an orthogonal change of basis Θ becomes block–diagonal:

$$\Theta \rightarrow D := R^T \Theta R = \begin{pmatrix} \Theta(k_1) & & \\ & \ddots & \\ & & \Theta(k_{\frac{d}{2}}) \end{pmatrix} \tag{2.5}$$

where

[2]The two–dimensional wave equation $\partial\bar{\partial}X(z,\bar{z}) = 0$ follows from (2.1). Its solution can obviously be split according to $X(z,\bar{z}) = X_{\mathrm{R}}(z) + X_{\mathrm{L}}(\bar{z})$. Observe that the coupling via $B_{\mu\nu}$ is just a divergence whence it cannot affect the equation of motion.

[3]The corresponding condition $[G, \Theta] = 0$ is trivially fulfilled for our preferred choice $G = \frac{1}{2}\mathbf{1}$.

$$\Theta(k_j) = \begin{pmatrix} c_j & -s_j \\ s_j & c_j \end{pmatrix} ; \begin{cases} c_j + is_j := e^{2\pi i k_j} \\ Nk_j \in \{1, \ldots, N-1\} \end{cases} \quad . \tag{2.6}$$

Hence the possibility of having fixed subspaces under Θ will not be taken into account. Likewise we decompose B into 2×2 blocks (w.r.t. our new coordinate system):

$$\mathbf{B}_{ij} := \begin{pmatrix} B_{2i-1,2j-1} & B_{2i-1,2j} \\ B_{2i,2j-1} & B_{2i,2j} \end{pmatrix} \qquad (i, j \in \{1, \ldots, \frac{d}{2}\}) \quad . \tag{2.7}$$

The condition (2.4) now reads

$$\mathbf{B}_{ij}\Theta(k_j) - \Theta(k_i)\mathbf{B}_{ij} = 0 \tag{2.8}$$

(no summation over i, j). It is convenient to parametrize

$$\mathbf{B}_{ij} = \begin{pmatrix} q & r \\ s & t \end{pmatrix} \tag{2.9}$$

when the indices i, j are kept fixed.

A similar restriction is obtained for $\mathbf{B}_{ji} = -\mathbf{B}_{ij}^T$:

$$\mathbf{B}_{ij}\Theta(k_j)^T - \Theta(k_i)^T\mathbf{B}_{ij} = 0 \quad . \tag{2.10}$$

Upon forming both the sum and the difference of (2.8) and (2.10) we learn that

$$\begin{aligned} (c_j - c_i)\mathbf{B}_{ij} &= 0 \\ s_i\mathbf{B}_{ij} + s_j\epsilon\mathbf{B}_{ij}\epsilon &= 0 \, ; \qquad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad . \end{aligned} \tag{2.11}$$

Clearly, non–trivial solutions for $\mathbf{B}_{ij}$ require that $c_j = c_i$. We may then distinguish between three possible cases:

1. $s_i = s_j \neq 0 \Rightarrow \mathbf{B}_{ij} = \begin{pmatrix} q & r \\ -r & q \end{pmatrix} \quad (q = 0, \text{ if } i = j)$

2. $s_i = -s_j \neq 0 \Rightarrow \mathbf{B}_{ij} = \begin{pmatrix} q & r \\ r & -q \end{pmatrix}$

3. $s_i = \quad s_j = 0 \Rightarrow \mathbf{B}_{ij} = \begin{pmatrix} q & r \\ s & t \end{pmatrix} \quad (q = t = 0 \,\text{and}\, r = -s, \text{ if } i = j)$.

These restrictions on the 2×2 blocks of B will prove to be crucial in the analysis of the instanton contribution to the four–twist field correlation function (see section 3). But before taking up this issue we wish to count the number of independent background moduli in the case of Z_N orbifold models.

For this purpose we relate the backgrounds G, B w.r.t. the target space basis to their counterparts g, b in the lattice basis:

$$g := e^T G e \; ; \qquad b := e^T B e \quad . \tag{2.12}$$

Note that all the information about the torus metric is encoded in g whereas G might be an arbitrary non–singular symmetric matrix.

If only torus boundary conditions ($S \cong \Lambda_d$) are imposed g and b contain $\frac{1}{2}d(d+1)$ and $\frac{1}{2}d(d-1)$ independent moduli, respectively. These numbers are in general smaller for orbifold models with $S \supset \Lambda_d$.

We first analyse B. Let us denote by ν_k the number of blocks $\Theta(k)$ contained in D. The above discussion of (2.11) reveals that the blocks $\mathbf{B}_{ij}$ for which $\Theta(k_i) = \Theta(k_j) = -\mathbf{1}_2$ can be chosen at will. Considering that B is antisymmetric we are left with $\begin{pmatrix} d_{\frac{1}{2}} \\ 2 \end{pmatrix}$ independent B–moduli from the $d_{\frac{1}{2}}$–dimensional subspace $\mathcal{R}$ where D acts as a reflection ($d_{\frac{1}{2}} = 2\nu_{\frac{1}{2}}$).

The subspace where D acts blockwise via $\Theta(k)$, $\Theta(1-k)$ ($k \neq \frac{1}{2}$) provides $(\nu_k + \nu_{1-k})$ real parameters from blocks $\mathbf{B}_{jj}$ along the diagonal and $2\begin{pmatrix} \nu_k + \nu_{1-k} \\ 2 \end{pmatrix}$ parameters from off–diagonal blocks $\mathbf{B}_{ij}$. Thus we obtain another $(\nu_k + \nu_{1-k})^2$ moduli. As a consequence of (2.4) $\mathbf{B}_{ij}$ vanishes whenever $c_i \neq c_j$; therefore the total number of antisymmetric background moduli is

$$M_B = \begin{pmatrix} d_{\frac{1}{2}} \\ 2 \end{pmatrix} + \sum_{\rho \in J} (\nu_\rho + \nu_{1-\rho})^2 \; , \tag{2.13}$$

where J denotes the set of all rational numbers in the interval $]0, \frac{1}{2}[$.

Next we address the analogous problem for the background metric $g = \frac{1}{2}e^T e$. Let us consider continuous (invertible) deformations $U \in \mathrm{GL}(d, \mathbb{R})$ of a lattice basis e_0:

$$e_0 \mapsto e = U e_0 \tag{2.14}$$

such that the automorphic action of D on the lattice is not affected:

$$D e_0 = e_0 K \Rightarrow D e = e K \qquad (K \in \mathrm{SL}(d, \mathbb{Z})) \quad . \tag{2.15}$$

Hence we require

$$D U e_0 = D e = e K = U e_0 K = U D e_0 \quad , \tag{2.16}$$

which reduces to $[U, D] = 0$.

The invariant quantity of interest however is the lattice metric $g = \frac{1}{2}e^T e$. The above deformation causes $g_0 \mapsto g = \frac{1}{2}e_0^T U^T U e_0$; admittedly some combinations of continuous parameters U depends on might drop out in $U^T U$. Indeed if we perform a real polar decomposition $U = OS$ (where O is orthogonal and S is symmetric[4]) g is seen not to depend on O.

Hence we have to restrict ourselves to $U = S$ with $[S, D] = 0$. This resembles (2.4) were it not for the fact that S is symmetric. An entirely similar counting procedure leads to the number

$$M_g \;= \binom{d_{\frac{1}{2}} + 1}{2} + \sum_{\rho \in J} (\nu_\rho + \nu_{1-\rho})^2 \quad . \tag{2.17}$$

of lattice moduli. The number M_b of moduli which parametrize the invariant background b equals M_B since $b = e_0^T (U^T B U) e_0$. Here the reference basis e_0 is kept fixed and the antisymmetric matrix $(U^T B U)$ is again subject to (2.4).

Thus the total number of (real) background moduli is given by

[4]It is not difficult to determine solutions for the factors O, S. By an orthogonal transformation $\tilde{\Omega}$ the symmetric matrix $U^T U$ can be diagonalized: $U_{\mathrm{D}}^2 \equiv \tilde{\Omega}^T U^T U \tilde{\Omega} = \mathrm{diag}(u_1^2, \ldots, u_d^2)$ with $U_{\mathrm{D}} = \mathrm{diag}(u_1, \ldots, u_d)$ ($u_l \in \mathbb{R}$). We then define $S := \tilde{\Omega} U_{\mathrm{D}} \tilde{\Omega}^T$, and $O := U S^{-1}$ evidently is orthogonal.

$$M_g + M_b = (d_{\frac{1}{2}})^2 + 2\sum_{\rho\in J}(\nu_\rho + \nu_{1-\rho})^2 \qquad , \tag{2.18}$$

and $h_{2,1} := (M_g - M_b) \leq d$ coincides with the (real) dimension $d_{\frac{1}{2}}$ of the subspace $\mathcal{R}$ which is precisely the (Hodge) number of real complex structure moduli of a Z_N orbifold target space.

The case $d = 6$ which is of immediate physical interest is easily surveyed. We assume here that $\mathcal{R} = \emptyset$. This then leads to $M_g = M_b \in \{3, 5, 9\}$. In fact, Kähler manifolds with these (complex) dimensions which are parametrized by the set M_{U} of untwisted moduli fields have already been discovered before (see [11], [16]).

Finally we touch on the case of an odd dimension d. Since Θ is supposed not to leave any direction fixed the multiplicity $d_{\frac{1}{2}}$ of the subspace $\mathcal{R}$ has to be odd as well. We again adopt the basis where the twist takes the block–diagonal form (2.5) augmented by a diagonal entry -1 which represents the unpaired "last" direction of $\mathcal{R}$. As a rule both B_{mn} and the symmetric deformation matrix element S_{mn} must vanish if m denotes a dimension of $\mathcal{R}$ while n does not (and vice versa). Of course, if the dimensions m, n both belong to $\mathcal{R}$ these matrix elements can be arbitrarily chosen were it not for the (anti–)symmetry of (B) S. From these arguments we infer that the above expressions for M_b, M_g continue to hold for odd values of d.

3 The four–twist field correlation function

In order to invoke the boundary conditions (2.2) for the coordinate vector $X(z,\bar{z})$ one has to exploit its operator product expansion with the primary fields associated to highest weight states. In the untwisted sector ($k = 0$ in (2.3)) we choose the *physical* twist invariant vertex operators

$$\begin{aligned}
V_{\mathbf{P}}^{\mathrm{inv}}(z,\bar{z}) &= \frac{1}{\sqrt{N}}\sum_{l=0}^{N-1} :\exp[i(\Theta^l P_{\mathrm{L}})^T X_{\mathrm{L}}(\bar{z}) + i(\Theta^l P_{\mathrm{R}})^T X_{\mathrm{R}}(z)]: \\
P_{\mathrm{R}} &= p + (G-B)w \qquad (w \in \Lambda_d,\ p \in \Lambda_d^*) \\
P_{\mathrm{L}} &= p - (G+B)w
\end{aligned} \tag{3.1}$$

which describe the emission of untwisted strings carrying the Narain momentum [17] $\mathbf{P} = (P_{\mathrm{R}}, P_{\mathrm{L}})$.[5] Upon analytic continuation of $X(z,\bar{z})\ V_{\mathbf{P}}^{\mathrm{inv}}(0,0)$ about the origin (i.e., $z \mapsto e^{2\pi i}z$, $\bar{z} \mapsto e^{-2\pi i}\bar{z}$) the coordinate field receives the shift $2\pi\Theta^l w$ from the l–th exponential operator in (3.1). Of course this operator product is not twist invariant because $X(z,\bar{z})$ is degraded to an auxiliary field when the torus T_d is transformed into an orbifold.

The monodromy of $X(z,\bar{z})$ about the world sheet locations of (3.1) is therefore described by the subset $(\mathbf{1}, \mathcal{O}_w)$ of the space group S where

$$\mathcal{O}_w := \{\Theta^l w; 0 \leq l \leq N-1\} \tag{3.2}$$

denotes the Θ–*orbit* of the winding vector w. $(\mathbf{1}, \mathcal{O}_w)$ is stable under conjugation with arbitrary elements of S and thus forms a *conjugacy class.* Quite generally, the classical monodromy caused by some vertex operator must be phrased in terms of such classes. Indeed, if $s_1, s_2 \in S$ then the constraints s_1, $(s_2^{-1} s_1 s_2)$ have to be considered as being equivalent because the configurations X, $s_2 \cdot X$ cannot be distinguished within an orbifold model.

Analogously the first twisted sector of the orbifold's Hilbert space endows strings with boundary conditions of the type (Θ, λ) $(\lambda \in \Lambda_d)$. The conjugacy classes have the form $(\Theta, [f])$ where

$$[f] \equiv f + (\mathbf{1} - \Theta)\Lambda_d \tag{3.3}$$

represents a coset of lattice translations. There exist $N_1 = \det(\mathbf{1} - \Theta)$ ground state twist fields $\sigma_f^+(z,\bar{z})$ which impose elements of a conjugacy class $(\Theta, [f])$ on the coordinate mapping $X(z,\bar{z})$. The characteristic winding vector f can be chosen to be any representative of the quotient space $\Lambda_d/(\mathbf{1} - \Theta)\Lambda_d$. Whatever element designates a particular coset the twisted string solution possesses the fixed point $x = \dfrac{2\pi}{(\mathbf{1} - \Theta)} f \bmod 2\pi\Lambda_d$ under Θ.

To display the local monodromy of $X(z,\bar{z})$ in the vicinity of a twist field σ_f^+ we introduce complex coordinates (as well as their complex conjugates)

[5] The accompanying cocycle operators which are mandatory within the operator quantization are thoroughly discussed in [13].

$$Y_j := X_{2j-1} + iX_{2j} \; ; \qquad (1 \le j \le \frac{d}{2}) \tag{3.4}$$

w.r.t. a system which guarantees (2.5). It follows from (2.2) that

$$\partial X(e^{2\pi i}z, e^{-2\pi i}\bar{z}) = D\partial X(z,\bar{z}) \; ;$$

this quantum monodromy condition is assured by the following set of operator products:

$$\begin{aligned}
\partial Y^j(z_1,\bar{z}_1)\,\sigma_f^+(z_2,\bar{z}_2) &= z_{12}^{-(1-k_j)}\,\tau_f^{+j}(z_2,\bar{z}_2) + \dots \\
\partial \bar{Y}^j(z_1,\bar{z}_1)\,\sigma_f^+(z_2,\bar{z}_2) &= z_{12}^{-k_j}\qquad \tau_f^{\prime +j}(z_2,\bar{z}_2) + \dots \\
\bar{\partial} Y^j(z_1,\bar{z}_1)\,\sigma_f^+(z_2,\bar{z}_2) &= \bar{z}_{12}^{-k_j}\qquad \tilde{\tau}_f^{\prime +j}(z_2,\bar{z}_2) + \dots \\
\bar{\partial} \bar{Y}^j(z_1,\bar{z}_1)\,\sigma_f^+(z_2,\bar{z}_2) &= \bar{z}_{12}^{-(1-k_j)}\,\tilde{\tau}_f^{+j}(z_2,\bar{z}_2) + \dots \qquad (k_j \in]0,1[) \; .
\end{aligned} \tag{3.5}$$

Here excited twist fields $\tau^+, \dots$ appear as local operators on the right hand side. They of course represent unphysical fields of the string theory because their conformal spin is in no case an integral number. The conformal dimension $h_+ = \bar{h}_+$ of the twist field σ_f^+ can then be determined via the stress–energy method:

$$h_+ = \frac{1}{2}\sum_{j=1}^{d/2} k_j(1-k_j) \qquad . \tag{3.6}$$

Similarly the ground states σ_{-f}^- of the first *antitwisted* sector of the Hilbert space are in one–to–one correspondence with the conjugacy classes $(\Theta^{-1}, -[f])$. The fixed points of nearby string coordinate fields are again located at $x = \dfrac{2\pi}{(1-\Theta)}f$ (mod $2\pi\Lambda_d$). The state $\sigma_{-f}^-|0\rangle$ is the world sheet CPT conjugate of $\sigma_f^+|0\rangle$; consequently the conformal dimension of σ_{-f}^- coincides with the one given in (3.6).

The four–twist field correlation function

$$Z_{\{f_i\}}(x,\bar{x}) := \lim_{|z_\infty|\to\infty} |z_\infty|^{4h_+} \langle \sigma_{-f_1}^-(0,0)\sigma_{f_2}^+(x,\bar{x})\sigma_{-f_3}^-(1,1)\sigma_{f_4}^+(z_\infty,\bar{z}_\infty)\rangle \tag{3.7}$$

provides us with various three–point couplings of the orbifold CFT upon factorization. The winding vectors f_i are subject to the space group selection rule [18, 19]

$$f_{21} + f_{43} \in (\mathbf{1} - D)\Lambda_d; \; f_{ij} := f_i - f_j \qquad . \tag{3.8}$$

$Z_{\{f_i\}}$ is obtained by performing the path integration over all mappings X^μ from the spherical world sheet into $\mathbb{R}^d$ which are compatible with the monodromy conditions imposed by the four twist fields. Since the action functional is quadratic in the coordinates X^μ, $Z_{\{f_i\}}$ naturally splits into two factors: one contribution arises from classical instanton solutions in the path integral, the other accounts for the quantum fluctuations about this classical sector [18]:

$$Z_{\{f_i\}}(x,\bar{x}) = Z^{\mathrm{qu}}(x,\bar{x}) \; Z^{\mathrm{cl}}_{\{f_i\}}(x,\bar{x}) \qquad . \tag{3.9}$$

We will primarily be concerned with $Z^{\mathrm{cl}}_{\{f_i\}}$ since Z^{qu} is not sensitive to the background $(g+b)$ and has already been evaluated in [18] for $d=2$. Here we simply have to multiply the quantum correlations related to the individual planar subspaces in which D acts as $\Theta(k_j)$ (cf. (2.5)) in order to arrive at the d–dimensional generalization (d is again supposed to be even):

$$Z^{\mathrm{qu}}(x,\bar{x}) = \nu |x(1-x)|^{-4h_+} \prod_{j=1}^{d/2} \frac{1}{2|F_j(x)|^2 (\tau_j)_2} \tag{3.10}$$

$$\tau_j(x) = (\tau_j)_1 + i(\tau_j)_2 := \frac{iF_j(1-x)}{F_j(x)} \tag{3.11}$$

where $F_j(x)$ is a shorthand for the hypergeometric function $F(k_j, 1-k_j; 1; x)$. The normalization constant ν in Z^{qu} will be fixed at the end of this section.

The classical instanton part is given by

$$Z^{\mathrm{cl}}_{\{f_i\}}(x,\bar{x}) = \sum_{X^{\mathrm{cl}}} e^{-S_{\mathrm{E}}[\partial X^{\mathrm{cl}}, \bar{\partial} X^{\mathrm{cl}}]} \qquad , \tag{3.12}$$

where the X^{cl} are solutions of the classical equation of motion, $\partial\bar{\partial}X(z,\bar{z}) = 0$, subject to the monodromy conditions imposed by the four twist fields in (3.7).

To facilitate the evaluation of (3.12) we first split

$$
\begin{aligned}
S_{\mathrm{E}} &= S_0 + \sum_{j,l=1}^{d/2} S_{jl} \,, \\
S_0 &= \frac{1}{4\pi} \sum_{j=1}^{d/2} \int dz\, d\bar{z}\, \bar{\partial} \vec{X}_j^T \partial \vec{X}_j \,, \\
S_{jl} &= \frac{1}{2\pi} \int dz\, d\bar{z}\, \bar{\partial} \vec{X}_j^T \mathbf{B}_{jl} \partial \vec{X}_l \,; \qquad \vec{X}_j := \begin{pmatrix} X_{2j-1} \\ X_{2j} \end{pmatrix} \quad .
\end{aligned}
\tag{3.13}
$$

With the help of the parametrization (2.9) and after a change to complex coordinates Y_k we arrive at

$$
\begin{aligned}
S_0 &= \frac{1}{8\pi} \sum_{j=1}^{d/2} \int dz\, d\bar{z}\, \left(\bar{\partial} Y_j \partial \bar{Y}_j + \bar{\partial} \bar{Y}_j \partial Y_j \right) \\
S_{jl} &= \frac{1}{8\pi} \int dz\, d\bar{z} \Big\{ (q-t) \left(\bar{\partial} \bar{Y}_j \partial \bar{Y}_l + \bar{\partial} Y_j \partial Y_l \right) \\
&\qquad + (q+t) \left(\bar{\partial} \bar{Y}_j \partial Y_l + \bar{\partial} Y_j \partial \bar{Y}_l \right) \\
&\qquad + i\,(r+s) \left(\bar{\partial} \bar{Y}_j \partial \bar{Y}_l - \bar{\partial} Y_j \partial Y_l \right) \\
&\qquad + i\,(r-s) \left(\bar{\partial} Y_j \partial \bar{Y}_l - \bar{\partial} \bar{Y}_j \partial Y_l \right) \Big\} \quad .
\end{aligned}
\tag{3.14}
$$

The four *cut differentials* $\partial Y_j^{\mathrm{cl}}$, $\partial \bar{Y}_j^{\mathrm{cl}}$, $\bar{\partial} Y_j^{\mathrm{cl}}$, and $\bar{\partial} \bar{Y}_j^{\mathrm{cl}}$ of the j–th planar subspace[6] are completely determined by the local monodromy rules (3.5) apart from constant prefactors:

$$
\begin{aligned}
\partial Y_j^{\mathrm{cl}} = (\bar{\partial} \bar{Y}^{\mathrm{cl}})^* &= a_j \left\{ (z-0)(z-1) \right\}^{-k_j} \quad (z-x)^{-(1-k_j)} \\
\partial \bar{Y}_j^{\mathrm{cl}} = (\bar{\partial} Y^{\mathrm{cl}})^* &= d_j \left\{ (z-0)(z-1) \right\}^{-(1-k_j)} (z-x)^{-k_j} \quad .
\end{aligned}
\tag{3.15}
$$

In order to fix a_j, d_j we consider the zero net twist loop $\mathcal{C}_1$ ($\mathcal{C}_2$) which surrounds the world sheet positions 0, x (x, 1). The global monodromy of X around $\mathcal{C}_1$ ($\mathcal{C}_2$) is restricted to the (space group) product of the conjugacy class $(D, [f_2])$ with $(D^{-1}, -[f_1])$ or with $(D^{-1}, -[f_3])$ in the second case:

[6]Under the twist Y_j^{cl}, $\bar{Y}_j^{\mathrm{cl}}$ are multiplied by the phase factor $e^{2\pi i k_j}$, $e^{-2\pi i k_j}$, respectively.

$$
\begin{aligned}
\Delta_{C_1} Y_j^{\rm cl} &\equiv \oint_{C_1} dz\, \partial Y_j^{\rm cl} + \oint_{C_1} d\bar{z}\, \bar{\partial} Y_j^{\rm cl} = 2\pi u_j \in 2\pi [f_{21}]_j \\
\Delta_{C_2} Y_j^{\rm cl} &\equiv \oint_{C_2} dz\, \partial Y_j^{\rm cl} + \oint_{C_2} d\bar{z}\, \bar{\partial} Y_j^{\rm cl} = 2\pi v_j \in 2\pi [f_{23}]_j
\end{aligned}
\tag{3.16}
$$

where the subscript $j \in \{1, \ldots, \frac{d}{2}\}$ attached to a coset $[f]$ reminds us to rewrite its projection onto the j–th plane in terms of *complex* vectors as in (3.4).

The solution of the inhomogeneous system (3.16) of linear equations is given by (cf. [18])

$$
\begin{aligned}
a_j &= -\frac{i\bar{F}_j(\bar{x})}{I_j(x,\bar{x})} \left\{ v_j + \bar{\tau}_j \bar{\beta}_j u_j \right\} \\
d_j &= -\frac{i\bar{F}_j(\bar{x})}{I_j(x,\bar{x})} \left\{ \bar{v}_j + \bar{\tau}_j \beta_j \bar{u}_j \right\}
\end{aligned}
\tag{3.17}
$$

with

$$
\begin{aligned}
I_j(x,\bar{x}) &= F_j(x)\bar{F}_j(1-\bar{x}) + F_j(1-x)\bar{F}_j(\bar{x}) \\
\beta_j &= -i \exp(-i\pi k_j) \quad .
\end{aligned}
\tag{3.18}
$$

We have also used (3.11).

If the endpoint x of the branch cut is encircled clockwise the instanton differential $\partial X^{\rm cl}$ will be rotated by D (this explicitly follows from the form (3.15) of cut differentials w.r.t. the eigenbasis of Θ). It is important that the integrand (Lagrangian) in (3.13) be a single–valued function on $\mathbb{C}$. However, employing (3.17), it can be demonstrated that the d^2 different products which are both linear in the holomorphic and the antiholomorphic cut differentials (3.15) are *independent* functions on $\mathbb{C}\times\ [f_{21}]\ \times[f_{23}]$ where the coset factors allow to distinguish the various instantons (see (3.16)). Hence we have to guarantee

$$
[B, D] = 0 \tag{3.19}
$$

in order that the multitude of instanton actions is well–defined (cf. also

[13]). Recall that the ensuing restrictions for the blocks $\mathbf{B}_{ij}$ have already been worked out in section 2.

To determine $S_0(u,v)$ we employ the auxiliary formula (see [18], [20])

$$\int dz\, d\bar{z}\, |(z-0)(z-1)|^{-2k_j}\, |z-x|^{-2(1-k_j)} = \frac{2\pi^2}{\sin(\pi k_j)} I_j(x,\bar{x}) \quad . \tag{3.20}$$

(This integral is invariant under the substitution $k_j \mapsto (1-k_j)$ in the integrand.) We then derive (recall the results (3.17))

$$\begin{aligned} S_0(u,v) &= \sum_{j=1}^{d/2} \frac{\pi I_j(x,\bar{x})}{4\sin(\pi k_j)} \{|a_j|^2 + |d_j|^2\} \\ &= \sum_{j=1}^{d/2} \frac{\pi}{4\sin(\pi k_j)(\tau_j)_2} \{|v_j|^2 + |\tau_j|^2 |u_j|^2 + (\tau_j)_1 (\beta_j \bar{u}_j v_j + \bar{\beta}_j u_j \bar{v}_j)\} \\ &= \frac{\pi}{4} u^T \frac{H_1^2 + H_2^2}{H_2} u + \pi v^T \frac{1}{(\mathbf{1}-D^T) H_2 (\mathbf{1}-D)} v - \pi u^T \frac{H_1}{H_2} \frac{1}{\mathbf{1}-D} v \end{aligned} \tag{3.21}$$

with the d–dimensional diagonal matrix

$$H = H_1 + iH_2 := \operatorname{diag}\left(\frac{\tau_j}{\sin(\pi k_j)}; 1 \le j \le \frac{d}{2}\right) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad . \tag{3.22}$$

Observe that the last line of (3.21) again contains lattice vectors w.r.t. the real coordinate system (we have refrained from introducing new symbols to emphasize this difference). In particular, the factor $\frac{\beta_j}{\sin(\pi k_j)}$ turned into $\frac{-2}{\mathbf{1}_2 - \Theta(k_j)}$.

According to our findings in section 2 the contribution S_{jl} to the instanton action vanishes except for the following cases:

1. $k_j = k_l \neq \frac{1}{2}$: We have

$$S_{jl}(u,v) = \frac{\pi I_j(x,\bar{x})}{2\sin(\pi k_j)} [(q+ir)\bar{d}_j d_l + (q-ir)\bar{a}_j a_l] \tag{3.23}$$

since we are forced to set $q = t$ and $r = -s$.

2. $k_j = 1 - k_l \neq \frac{1}{2}$: Here

$$S_{jl}(u,v) = \frac{\pi I_j(x,\bar{x})}{2\sin(\pi k_j)}[(q - ir)\bar{d}_j a_l + (q + ir)\bar{a}_j d_l] \tag{3.24}$$

because the elements of $\mathbf{B}_{ij}$ must satisfy $q = -t$ and $r = s$.

3. $k_j = k_l = \frac{1}{2}$:

$$\begin{aligned} S_{jl}(u,v) &= \frac{\pi I_j(x,\bar{x})}{4\sin(\frac{\pi}{2})}[\ q(\bar{a}_j + \bar{d}_j)(a_l + d_l) + t(\bar{a}_j - \bar{d}_j)(a_l - d_l) \\ &\qquad -ir(\bar{a}_j + \bar{d}_j)(a_l - d_l) + is(\bar{a}_j - \bar{d}_j)(a_l + d_l)] \end{aligned} \tag{3.25}$$

since any choice of $\mathbf{B}_{jl}$ is compatible with $[D,B] = 0$.[7].

To proceed further we have to remember that S_{ij} and S_{ji} depend on the same set of modular parameters $\in \{q,r,s,t\}$. With the help of the auxiliary formula[8]

$$\bar{d}_j d_l - \bar{a}_l a_j = \frac{i}{I_j(x,\bar{x})}\left\{\bar{\beta}_j u_j \bar{v}_l - \beta_j \bar{u}_l v_j\right\} \tag{3.26}$$

we establish for $k_j = k_l \neq \frac{1}{2}$ $(\beta \equiv \beta_j = \beta_l)$ that

$$\begin{aligned} (S_{jl} + S_{lj})(u,v) &= \frac{i\pi}{2\sin(\pi k_j)}\Big\{q\left((u_j\overline{\beta v_l} + \bar{u}_j\beta v_l) - (j \leftrightarrow l)\right) \\ &\qquad +ir\left((u_j\overline{\beta v_l} - \bar{u}_j\beta v_l) + (j \leftrightarrow l)\right)\Big\} \\ &= -2\pi i\left(u_j^T\mathbf{B}_{jl}\frac{1}{\mathbf{1}_2 - \Theta(k_j)}v_l + u_l^T\mathbf{B}_{lj}\frac{1}{\mathbf{1}_2 - \Theta(k_j)}v_j\right\} \end{aligned} \tag{3.27}$$

where we resorted once more to a real coordinate system at the end. It is then fairly easy to derive the contribution $(S_{jl} + S_{lj})$ in the case where $k_j = 1 - k_l$. Observe first that $\beta_l = \bar{\beta}_j$ holds. By comparing (3.23) with (3.24) we recognize that it is sufficient to perform the following substitutions in (3.27): $r \mapsto -r$, $u_l \leftrightarrow \bar{u}_l$, $v_l \leftrightarrow \bar{v}_l$. As concerns the bottom line of (3.27) the complex conjugation of (complex) vectors carrying the label j amounts

[7]Of course for $j = l$ we must set $q = t = 0$ and $r = -s$ since B is antisymmetric.

[8]Observe that $(\tau_j)_2 = \dfrac{I_j(x,\bar{x})}{2|F_j(x)|^2}$.

to the insertion of an extra factor $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in front of their real basis counterparts. Together with $r \mapsto -r$ the blocks $\mathbf{B}_{jl}$, $\mathbf{B}_{lj}$ in (3.27) adopt then the form which is appropriate to $k_j = 1 - k_l$ (see section 2).

In addition the bottom line of (3.27) applies also when $k_j = k_l = \frac{1}{2}$. Here $\beta_j = -1$, $\frac{1}{\mathbf{1}_2 - \Theta(k_j)} = \frac{1}{2}\mathbf{1}_2$ and the various linear combinations of the coefficients d and a present in (3.25) will immediately produce the real components of the lattice vectors u, v (see (3.17)).

Finally we take one half of the sum over $j, l \in \{1, \dots, \frac{d}{2}\}$ of (3.27) and also include (3.21) to obtain

$$\begin{aligned} S_{\mathrm{E}}(u,v) &= \tfrac{\pi}{4} u^T \frac{H_1^2 + H_2^2}{H_2} u + \pi v^T \frac{1}{(\mathbf{1} - D^T) H_2 (\mathbf{1} - D)} v \\ &\quad - \pi u^T \left(\frac{H_1}{H_2} + 2iB \right) \frac{1}{\mathbf{1} - D} v \quad . \end{aligned} \tag{3.28}$$

Apart from the unknown prefactor ν the four–twist correlation function reads

$$Z_{\{f_i\}}(x,\bar{x}) = \nu |x(1-x)|^{-4h_+} \left[\prod_{j=1}^{d/2} \frac{1}{I_j(x,\bar{x})} \right] \sum_{\substack{u \in [f_{21}] \\ v \in [f_{23}]}} e^{-S_{\mathrm{E}}(u,v)} \quad . \tag{3.29}$$

Our choice for ν will be consistent with the normalization $\langle \sigma^+_{f_2}(x,\bar{x})\, \sigma^-_{-f_1}(0,0) \rangle = \delta_{[f_1],[f_2]}\, |x|^{-4h_+}$ of twist field two–point functions. The r.h.s. of this equation dominates $Z_{\{f_1,f_1,f_3,f_3\}}(x,\bar{x})$ as $|x| \mapsto 0$. To explore this limit we first have to Poisson resum the series over winding vectors $\in [f_{23}]$ in (3.29). To this end we identify

$$\varphi := -Bu; \quad \epsilon := \frac{1}{\mathbf{1} - D} f_{23} - \frac{1}{2} H_1 u; \quad A := H_2 \tag{3.30}$$

in the resummation identity

$$\begin{aligned} \sum_{w \in \Lambda_d} e^{-\pi (w+\epsilon)^T A^{-1} (w+\epsilon) + 2\pi i \varphi^T (w+\epsilon)} &= \\ \frac{\sqrt{\det A}}{\det e} \sum_{p \in \Lambda_d^*} & e^{-\pi (p+\varphi)^T A (p+\varphi) - 2\pi i p^T \epsilon} \quad . \end{aligned} \tag{3.31}$$

After a few elementary rearrangements we get

$$\begin{aligned}
Z_{\{f_i\}}(x,\bar{x}) &= \nu \frac{|x(1-x)|^{-4h_+}}{\det e} \left[\prod_{l=1}^{d/2} \frac{(\tau_2)_l}{I_l(x,\bar{x}) \sin(\pi k_l)} \right] \\
&\times \sum_{\substack{p\in\Lambda_d^* \\ v\in[f_{21}]}} e^{-2\pi i p^T \frac{1}{1-D} f_{23}} \left[\prod_{j=1}^{d/2} w_j(x)^{h_j} \, \bar{w}_j(\bar{x})^{\bar{h}_j} \right]
\end{aligned} \tag{3.32}$$

with

$$\begin{aligned}
h_j &= \frac{1}{2} \sum_{r=2j-1}^{2j} [(p + \frac{1}{2} v - Bv)_r]^2 \\
\bar{h}_j &= \frac{1}{2} \sum_{r=2j-1}^{2j} [(p - \frac{1}{2} v - Bv)_r]^2 \\
w_j(x) &= e^{i\pi \tau_j(x)/\sin(\pi k_j)} \quad .
\end{aligned} \tag{3.33}$$

Space group considerations in the operator product expansion

$$\begin{aligned}
\sigma^-_{-f_1}(0,0)\, \sigma^+_{f_2}(x,\bar{x}) &= \frac{1}{N} \sum_{p\in\Lambda_d^*} \sum_{v\in[f_{21}]} x^{-2h_+ + h}\, \bar{x}^{-2h_+ + \bar{h}} \\
&\quad \times C(f_2,f_1;p,v)\; V^{\rm inv}_{p,v}(0,0) + \ldots
\end{aligned} \tag{3.34}$$

inform us about the leading terms. Here $h = \frac{1}{2} P_{\rm R}^T P_{\rm R}$, $\bar{h} = \frac{1}{2} P_{\rm L}^T P_{\rm L}$ denote the conformal, anticonformal weight of a (string emission) vertex operator $V^{\rm inv}_{p,v}$ (see (3.1)).[9] The expansion coefficients

$$C(f_2,f_1;p,v) := \lim_{|z_\infty|\to\infty} |z_\infty|^{4h_+} \, \langle V^{\rm inv}_{-p,-v}(0,0) \sigma^-_{-f_1}(1,1) \sigma^+_{f_2}(z_\infty,\bar{z}_\infty) \rangle \tag{3.35}$$

will soon be determined.

Using (3.34), (3.35) we can express (3.7) in terms of the coupling coefficients $C(\ldots)$ in the limit $x,\bar{x} \mapsto 0$ (s–channel factorization):

[9] The factor $\frac{1}{N}$ serves to undo an N–fold overcounting due to $V^{\rm inv}_{p,v} = V^{\rm inv}_{Dp,Dv}$ if $(p,v) \neq (0,0)$. Notice that $V^{\rm inv}_{0,0} = \sqrt{N}$ differs from the ordinary identity operator.

$$Z_{\{f_i\}}(x,\bar{x}) \;=\; \frac{1}{N} \sum_{\substack{p\in\Lambda_d^* \\ v\in[f_{21}]}} x^{-2h_+ + h}\, \bar{x}^{-2h_+ + \bar{h}}\, C(f_2,f_1;p,v)\, C(f_4,f_3;-p,-v) \\ + \ldots \quad . \tag{3.36}$$

Next we get the asymptotic behaviour of the hypergeometric function F_j from [21]:

$$\begin{aligned} F_j(x) &\approx 1 \\ F_j(1-x) &\approx \tfrac{1}{\pi} \sin(\pi k_j)\,[\ln \delta(k_j) - \ln x] \qquad \text{as } x \mapsto 0 \,, \\ &\quad \text{where} \begin{cases} \delta(k_j) = \exp\left[2\Psi(1) - \Psi(k_j) - \Psi(1-k_j)\right] \\ \Psi(z) = \frac{d}{dz} \ln \Gamma(z) \end{cases} \quad . \end{aligned} \tag{3.37}$$

Therefore

$$\begin{aligned} Z_{\{f_i\}}(x,\bar{x}) \;=\; & \frac{\nu}{\det e} \left[\prod_{l=1}^{d/2} \frac{1}{2\sin(\pi k_l)} \right] \sum_{\substack{p\in\Lambda_d^* \\ v\in[f_{21}]}} x^{-2h_+ + h}\, \bar{x}^{-2h_+ + \bar{h}} \\ & \times e^{-2\pi i p^T \frac{1}{1-D} f_{23}} \prod_{j=1}^{d/2} \delta(k_j)^{-(h_j+\bar{h}_j)} + \ldots \quad . \end{aligned} \tag{3.38}$$

Comparing with (3.36) we can now read off the string emission coupling constant

$$C(f_2,f_1;p,v) = \frac{\sqrt{N\nu}}{\sqrt{\det e}} \left[\prod_{l=1}^{d/2} \frac{1}{2\sin(\pi k_l)} \right]^{\frac{1}{2}} \prod_{j=1}^{d/2} \delta(k_j)^{-\frac{1}{2}(h_j+\bar{h}_j)}\, e^{-\pi i p^T \frac{1}{1-D}(2f_2 - v)} \; . \tag{3.39}$$

(combine the selection rules $v \in [f_{21}]$ and $f_{21} + f_{43} \in [0]$.)

As announced the normalization constant can then be fixed:

$$C(f,-f;0,0) = \sqrt{N} \;\Rightarrow\; \nu = \det e \prod_{l=1}^{d/2} (2\sin(\pi k_j)) \quad . \tag{3.40}$$

Therefore

$$C(f_2, f_1; p, v) = \sqrt{N} \prod_{j=1}^{d/2} \delta(k_j)^{-\frac{1}{2}(h_j+\bar{h}_j)} \; e^{-\pi i p^T \frac{1}{1-D}(2f_2+v)} \delta_{0,(f_2-f_1-v)\mathrm{mod}(1-D)\Lambda_d} \, . \tag{3.41}$$

This reproduces the result familiar from our earlier approach which was based on the operator formalism [13].

It is fairly easy to extend the above considerations to the (physically less relevant) case of an orbifold compactification O_d for which the target space dimension d is *odd.* Let us introduce an artificial extra coordinate which is subject to orbicircle boundary conditions $\mathcal{C}$, i.e. $\mathrm{P} = \{-1, 1\}$. We thus obtain an *even* dimensional product orbifold $O_{d+1} = O_d \times \mathcal{C}$ — a case which we already know how to handle. Notice that the action (2.1) relevant for O_{d+1} merely is the sum of the actions which describe the O_d model and the orbicircle construction provided that we set to zero the $d_{\frac{1}{2}}$ antisymmetric tensor moduli which arise by the above extension. Hence we just have to divide the four–twist field correlation function (3.29) of the O_{d+1} model by the correlator for the orbicircle model in order to get the desired result for O_d. As has been pointed out in [18] the square of the four twist field orbicircle correlator coincides with the correlation $Z_{\{f_i\}}$ of a *two*–dimensional Z_2 orbifold model (Λ_2 has to be a square lattice) if a particular choice of windings f_i is made. With the help of these remarks the slight modifications which are necessary to adapt (3.29) to the case of odd values of d are rapidly worked out: we adopt a coordinate system where Θ again takes the form (2.5) except for an additional entry $D_{dd} = -1$. Then the range of the products in (3.29) and (3.40) is adjusted to $\{1, \ldots, \frac{d-1}{2}\}$. Moreover an extra factor $\sqrt{\frac{2}{I_0(x, \bar{x})}}$ has to be supplied which we define by setting $k_0 = \frac{1}{2}$. It simply accounts for the *quantum* correlation factor which is contributed by the oscillators associated to the d–th dimension. All other factors in (3.29) are meaningful for odd d as well and don't have to be altered.

4 Yukawa couplings

If the anharmonic ratios x, $\bar{x}$ approach z_∞, $\bar{z}_\infty$ the four–twist field correlation function $Z_{\{f_i\}}(x,\bar{x})$ factorizes w.r.t. the u–channel. The intermediate states belong to the second twisted sector as follows from the point group selection rule. We define the *orbit*

$$\mathcal{O}_{s;f} := \{\Theta^l f \ \mathrm{mod}(\mathbf{1}-\Theta^s)\Lambda_d | 0 \leq l < s\} \tag{4.1}$$

associated to f. While one simply has $\mathcal{O}_{1;f} = \{f\}$ for the first twisted sector, $\mathcal{O}_{s;f}$ generically contains several elements for $s > 1$. Obviously in the case of $|\mathcal{O}_{s;f}| > 1$ the twist field σ_f^s (which provides the boundary conditions $(\Theta^s, f + (1-\Theta^s)\Lambda_d)$ is not invariant under the twist Θ. Hence we must resort to the more general definition

$$\Sigma_f^{(s)} = \frac{1}{\sqrt{|\mathcal{O}_{s;f}|}} \sum_{u\in\mathcal{O}_{s;f}} \sigma_u^s \quad . \tag{4.2}$$

of a physical (Θ–invariant) twist field. Its conformal dimension is given by[10]

$$h_{(s)} = \frac{1}{2}\sum_{j=1}^{d/2}[sk_j](1-[sk_j]) \quad . \tag{4.3}$$

From the decomposition

$$\begin{aligned}
(\Theta,[f_a])\cdot(\Theta,[f_b]) &= \sum_{f_c\in\tilde{\mathcal{P}}_{ab}} \bigcup_{\nu=0}^{|\mathcal{O}_{f_c}|-1} (\Theta^2, \Theta^\nu f_c + (\mathbf{1}-\Theta^2)\Lambda_d) \\
&\quad \tilde{\mathcal{P}}_{ab} := \mathcal{P}_{ab}/\mathrm{P} \qquad (\mathrm{P}: \text{ point group}) \\
&\quad \mathcal{P}_{ab} = f_a + f_b + (\mathbf{1}-\Theta)\frac{\Lambda_d}{(\mathbf{1}+\Theta)\Lambda_d}
\end{aligned} \tag{4.4}$$

of the product of conjugacy classes associated to $\sigma_{f_a}^+$, $\sigma_{f_b}^+$ into the classes connected with the ground states $\Sigma_{f_c}^{++}$ we determine the leading terms of the OPE

[10]We presuppose that Θ^s does not possess the eigenvalue one which is tantamount to fixed planar subspaces.

$$\sigma^{+}_{f_a}(z,\bar z)\sigma^{+}_{f_b}(0,0) = |z|^{2(h_{++}-2h_{+})} \sum_{f_c\in\tilde{\mathcal{P}}_{ab}} Y_{f_a,f_b;f_c}\, \Sigma^{++}_{f_c}(0,0) + \dots \quad . \tag{4.5}$$

The expansion coefficient

$$Y_{f_a,f_b;f_c} = \lim_{|z_\infty|\to\infty} |z_\infty|^{4h_{++}} \langle \sigma^{+}_{f_a}(0,0)\sigma^{+}_{f_b}(1,1)\Sigma^{--}_{-f_c}(z_\infty,\bar z_\infty)\rangle \tag{4.6}$$

describes the moduli–dependent part of twisted sector Yukawa couplings in the effective action of the orbifold model (see our introduction).

The four–point function (3.7) must then behave according to

$$Z_{\{f_i\}}(x,\bar x) = |x|^{-2h_{++}} \sum_{r\in\tilde{\mathcal{P}}_{24}} Y_{f_2,f_4;r}\, Y^{*}_{f_1,f_3;r} + \dots \tag{4.7}$$

in the u–channel factorization limit.

On the other hand we can explicitly deduce this limit by inspecting (3.29) with ν inserted from (3.40):[11]

$$Z_{\{f_i\}}(x,\bar x) = \det e\, |x(1-x)|^{-4h_{+}} \left[\prod_{l=1}^{d/2} \frac{2\sin(\pi k_l)}{I_l(x,\bar x)}\right] \sum_{\substack{u\in[f_{21}]\\ v\in[f_{23}]}} e^{-S_{\mathrm{E}}(u,v)} \quad . \tag{4.8}$$

The instanton action can already be recast into a sum of two bilinear symmetric forms whose kernels are a pair of complex conjugate matrices:

$$\frac{S_{\mathrm{E}}(u,v)}{\pi} = w_{+}^{T} H_1\left(\frac{\mathcal{S}}{4} + \frac{iB}{T-T^{T}}\right)w_{+} + w_{-}^{T} H_1\left(\frac{\mathcal{S}}{4} - \frac{iB}{T-T^{T}}\right)w_{-} \tag{4.9}$$

where

$$\begin{aligned} w_{\pm} &= -\frac{1}{H_1}\left(\frac{\mathbf{1}-T}{\mathbf{1}-D}\right)v \mp \left\{ \begin{array}{c} Tu \\ u \end{array} \right. \\ \mathcal{S} &= \mathrm{diag}\left(\frac{1}{\sin(2\varphi_l)}\cdot \mathbf{1}_2 \,;\, 1\le l\le \tfrac{d}{2}\right) \\ T &= \mathrm{diag}\left(\Theta(\tfrac{\varphi_l}{\pi}) \,;\, 1\le l\le \tfrac{d}{2}\right) \quad \text{and} \quad \varphi_l = \arctan\frac{(\tau_l)_1}{(\tau_l)_2} \quad . \end{aligned} \tag{4.10}$$

[11] In the sequel we resort again to the particular basis where D represents the twist.

Because of the asymptotic relations [21]

$$\left.\begin{matrix} F_l(1-x) \\ F_l(x) \end{matrix}\right\} \approx \frac{\Gamma(|1-2k_l|)}{\Gamma^2(|\frac{1}{2}+\frac{1}{2}|1-2k_l|)} x^{-\min(k_l,1-k_l)} \cdot \left\{\begin{matrix} 1 \\ e^{i\pi\min(k_l,1-k_l)} \end{matrix}\right.$$
$$\text{as } |x| \to \infty \tag{4.11}$$

the ingredients of the instanton action (4.9) will simplify enormously:

$$\begin{array}{rcl} \tau_l &=& ie^{-i\pi\min(k_l,1-k_l)} \\ \varphi_l &=& \pi\min(k_l,1-k_l) \in (0,\pi] \\ H_1 &=& \mathbf{1} \\ \mathcal{S} &=& \operatorname{diag}\left(\frac{1}{|\sin(2\pi k_j)|}\cdot \mathbf{1}_2\ ;\ 1\le j\le \frac{d}{2}\right) \\ T &=& \operatorname{diag}\left(\Theta(\min(k_l,1-k_l))\ ;\ 1\le l\le\frac{d}{2}\right) \end{array} \quad . \tag{4.12}$$

The factorization limit can be readily evaluated:

$$Z_{\{f_i\}}(x,\bar{x}) \approx |x|^{-2h_{++}} \det e \left[\prod_{l=1}^{d/2} \mathcal{G}^2(2\pi k_l)\right] \sum_{\substack{u\in[f_{21}] \\ v\in[f_{23}]}} e^{-S_{\rm Y}(u-v;-B)-S_{\rm Y}(Du+v;B)}$$
$$+\ldots \qquad \text{as } x\to z_\infty,\ \bar{x}\to\bar{z}_\infty \quad . \tag{4.13}$$

We abbreviated

$$\begin{array}{rcl} \mathcal{G}(2\pi k_l) &:=& \frac{\Gamma^2(\frac{1}{2}+\frac{1}{2}|1-2k_l|)}{\Gamma(|1-2k_l|)}\sqrt{|\tan(\pi k_l)|} \\ S_{\rm Y}(\lambda;B) &:=& \lambda^T\left(\frac{\mathcal{S}}{4}+\frac{iB}{D-D^T}\right)\lambda \end{array} \quad . \tag{4.14}$$

The limit of the classical action $S_{\rm E}(u,v)$ turns out to be a sum of two instanton actions $S_{\rm Y}(u-v;-B)$, $S_{\rm Y}(Du+v;B)$ which are obtained from the path integral evaluation of $Y^*_{f_1,f_3;r_c}$ and $Y_{f_2,f_4;r_c}$, respectively (r_c labels the intermediate physical state of the second twisted sector). Indeed the global monodromy vectors which characterize the corresponding instanton solutions refer to zero net twist contours $\tilde{\mathcal{C}}_1$, $\tilde{\mathcal{C}}_2$ enclosing the world sheet positions of a pair of twist fields from either the $s=1$ or the $s=-1$ sector. These global shifts read

$$\tilde{t}_1 := u - v, \qquad \tilde{t}_2 := Du + v \tag{4.15}$$

because $\tilde{\mathcal{C}}_1 = \mathcal{C}_1 - \mathcal{C}_2$ and $\tilde{\mathcal{C}}_2 = D\mathcal{C}_1 + \mathcal{C}_2$ hold. (Observe that $\mathcal{C}_1$ changes into $D\mathcal{C}_1$ by analytic continuation to the next sheet, i.e. when the cut that extends between 0 and x on the complex world sheet is crossed once before turning onto the same path again.) However

$$\begin{pmatrix} \tilde{t}_1 \\ \tilde{t}_2 \end{pmatrix} \in W(f_{21}, f_{23}) := \left\{ \begin{pmatrix} f_{31} + (\mathbf{1} - D)(\lambda - \mu) \\ f_{23} + Df_{21} + (\mathbf{1} - D)(D\lambda + \mu) \end{pmatrix} ; \mu, \lambda \in \Lambda_d \right\}.$$

which does not allow for the desired factorization right away. But it is possible to rewrite $W(f_{21}, f_{23})$ as a finite union of pairs of cosets (this fact has been demonstrated in [22] before), namely

$$W(f_{21}, f_{23}) = \bigcup_{r \in \mathcal{P}_{24}} \left\{ \begin{pmatrix} (\mathbf{1} - D^2)(\mu - \frac{1}{\mathbf{1} - D} f_1) + r \\ (\mathbf{1} - D^2)(\lambda + \frac{1}{\mathbf{1} - D} f_2) - r \end{pmatrix} ; \mu, \lambda \in \Lambda_d \right\}, \tag{4.16}$$

where we employed the selection rule (3.8). The union in (4.16) reflects of course the various intermediate physical states of type Σ^{++} (cf. (4.5)).

We then reorganize $\sum_{r \in \mathcal{P}_{24}} = \sum_{\tilde{r} \in \tilde{\mathcal{P}}_{24}} \sum_{r \in \mathcal{O}_{2;\tilde{r}}}$ and observe the property

$$\sum_{\tilde{t} \in -r + (\mathbf{1}+D)[f_2]} e^{-S_Y(\tilde{t}, B)} = \sum_{\tilde{t} \in -Dr + (\mathbf{1}+D)[f_2]} e^{-S_Y(\tilde{t}, B)} \quad .$$

We may now factorize (4.13) into a finite sum of products of Yukawa couplings as suggested by (4.7). The moduli dependent part of the Yukawa coupling reads

$$Y_{f_a, f_b; f_c} = \frac{\sqrt{\det e}}{\sqrt{|\mathcal{O}_{2;f_c}|}} \left[\prod_{l=1}^{d/2} \mathcal{G}(2\pi k_l) \right] \sum_{t \in \mathcal{U}_{f_c, f_a}} e^{-\pi t^T (\frac{1}{4} S + i \frac{B}{D - D^T}) t} \quad . \tag{4.17}$$

where

$$\mathcal{U}_{f_c, f_a} := \bigcup_{k=0,1} \left\{ (\mathbf{1} - D^2)\Lambda_d - D^k f_c + (\mathbf{1} + D) f_a \right\} \quad . \tag{4.18}$$

For some purposes [23] a multiplication by a moduli dependent phase factor might prove useful. This option clearly is at one's disposal in view of (4.7).

The Boltzmann factor series appearing in (4.17) can even be directly evaluated thereby avoiding the detour past the correlation function $Z_{\{f_i\}}$. Again the conjugacy classes attached to the ground states $\sigma^+_{f_a}$, $\sigma^+_{f_b}$, and $\Sigma^{--}_{-f_c}$ dictate the form of the classical instanton solutions $X^{\rm cl}$ whose actions reproduce the exponents in (4.17). If both $\partial Y_l \neq 0$ and $\bar\partial Y_l \neq 0$ one would encounter an *infinite* contribution to the action from the l-th planar subspace. Therefore the above series is exclusively due to strictly *holomorphic* or *antiholomorphic* instantons. The remaining metric dependence present in the prefactor $\sqrt{\det e}$ can also be predicted within this direct approach: the four–point function $Z_{\{f_i\}}$ contains its square $\det e$ in the Lagrangian formulation (two–fold series over winding vectors) because upon switching to the Hamiltonian formulation (a Poisson transformation converts one of these series into a series over momentum vectors) this factor will disappear as demanded by the normalization condition $\langle \sigma^+_f(0,0)\sigma^-_{-f}(1,1)\rangle = 1$. Admittedly the (background independent) product in (4.17) which contains quantum correlation factors is out of reach when such a direct calculation is carried out.

It is worthwhile to display the dependence of (4.17) on the background tensor $(G+B)$ in a more concise way. First we replace

$$\begin{aligned} \mathcal{S} &= -\frac{2}{D-D^T} Z_0 \\ &\quad \text{where } Z_0 = \text{diag}\left(\epsilon\, \text{sgn}(\sin(2\pi k_l))\,;\, 1 \leq l \leq \tfrac{d}{2}\right) \quad . \end{aligned} \tag{4.19}$$

In addition we reinstate the metrical factor $(2G)$ in the first term of the action's bilinear form to put it on the same footing with the second term that is governed by B. As a last step the original lattice basis is revived w.r.t. which the twist is given by Θ (cf. (2.5)). This yields

$$Y_{f_a,f_b;f_c} = (\det(2g))^{\frac{1}{4}}\, \eta(\Theta; f_c) \sum_{t\in\mathcal{V}_{f_c,f_a}} e^{-\pi\lambda^T(\Theta-\Theta^T)(GZ-iB)\lambda} \quad . \tag{4.20}$$

where

$$
\begin{aligned}
\eta(\Theta; f_c) &:= \sqrt{|\mathcal{O}_{2;f_c}|}\left[\prod_{l=1}^{d/2} \mathcal{G}(2\pi k_l)\right] \\
\mathcal{V}_{f_c,f_a} &:= \left\{\Lambda_d - \frac{1}{1-\Theta^2} f_c + \frac{1}{1-\Theta} f_a\right\} \\
Z &:= R Z_0 R^T \quad .
\end{aligned} \tag{4.21}
$$

Notice that we continue to denote the background matrices by the familiar symbols. The potentially complicated matrix Z which multiplies G must be viewed as the counterpart of the imaginary unit which multiplies B in the action kernel. This claim is part of the following set of algebraic rules:

$$
\begin{aligned}
Z^2 &= -\mathbf{1} \\
GZ &= Z^* G \\
BZ &= Z^* B \qquad \text{with } Z^* := (Z^T)^{-1} \quad .
\end{aligned} \tag{4.22}
$$

5 Discrete shifts of the moduli of the antisymmetric background

The vectors of the $2d$–dimensional Narain lattice

$$
\Lambda_{\mathcal{N}} := \{(P_{\mathrm{R}}, P_{\mathrm{L}}); p \in \Lambda_d^*, v \in \Lambda_d\} \tag{5.1}
$$

appear in the construction of the vertex operators (3.1) from the untwisted sector. By means of the background tensor $(g+b)$ one is able to parametrize the set of Narain lattices occurring in bosonic string theories (apart from trivial rotations of the coordinate system). Obviously $\Lambda_{\mathcal{N}}$ is not affected by a shift $a = e^T \alpha e$ $(a^T = -a)$ of the antisymmetric background provided that $a_{kl} \in \mathbb{Z}$ since it can be compensated by the shift $p \mapsto p + \alpha w$ of the momentum vectors.

This discrete symmetry group can even be generated by a group of unitary transformations acting on the set of vertex operators. Not only will the conformal dimensions h, $\bar{h}$ stick to their original values but there is also no way to distinguish a pair of orbifold models with backgrounds $g+b$ and $g+b+\alpha$, respectively, by means of their correlation functions. Thus it suffices to just consider a compact fundamental region of the antisymmetric moduli space for classification purposes.

For instance, a proof of the equality

$$\langle(\sigma^-_{-f_1})'|(V^{\rm inv}_{\mathbf{P}})'|(\sigma^+_{f_2})'\rangle_{g+b+a} = \langle\sigma^-_{-f_1}|V^{\rm inv}_{\mathbf{P}}|\sigma^+_{f_2}\rangle_{g+b} \tag{5.2}$$

between the twisted sector string emission correlators in two models with different backgrounds (indicated by the subscripts) has been given in [13]. The prime superscripts clarify that the associated fields in the "shifted model" differ unitarily from the conventional fields at the shifted background $g+b+a$:

$$\begin{array}{rcl} (V^{\rm inv}_{\mathbf{P}})' & = & V^{\rm inv}_{\mathbf{P}} \\ (\sigma^+_f)' & = & U_f\,\sigma^+_f \\ (\sigma^-_{-f})' & = & \bar{U}_f\sigma^-_{-f} \end{array} \quad . \tag{5.3}$$

According to [22][12] the phase factors U_f have to fulfill

$$U_{f_1+f_2} = U_{f_1}U_{f_2}\exp\left(-2\pi i f_1^T\frac{1}{\mathbf{1}-\Theta}\alpha f_2\right) \qquad (f_1, f_2 \in \Lambda_d) \quad , \tag{5.4}$$

and

$$(U_f)^2 = \exp\left(-2\pi i f^T\frac{1}{\mathbf{1}-\Theta}\alpha f\right) \qquad (f \in \Lambda_d) \quad . \tag{5.5}$$

The first condition says that $U : \Lambda_d \mapsto S_1$ provides a projective representation of the abelian group Λ_d whereas the second condition already determines these phase factors up to a sign. The associativity of the composition law (5.4) is ensured because the phase factor which contains the shift α is bi-multiplicative. Notice also that this factor has to be symmetric under an interchange of f_1 and f_2. For this property to hold we have to restrict ourselves to *modular* shifts of the antisymmetric background, i.e. $[\alpha, \Theta] = 0$.

As promised in [13] we report here on our progress in solving (5.4) and (5.5). For particular winding vectors these constraints are not involved at

[12] Although the emphasis was laid there on the case of two–dimensional bosonic orbifolds the Verlinde–type method of finding the first twisted sector representation U of a discrete shift $b \mapsto b + \epsilon$ can obviously be extended to higher dimensional target spaces.

all. Consider e.g. the sublattice $2\Lambda_d \subset \Lambda_d$. Taking $f_1 = f_2$ in (5.4) and using (5.5) subsequently we find that

$$U_{2f} = \exp(-2\pi i f^T \frac{1}{\mathbf{1}-\Theta}(2\alpha) f) \quad . \tag{5.6}$$

This entails $U_0 = 1$ which merely is a convenient normalization. It was already opted for in [22]. However in order for the the generalization $2f \mapsto f$ in (5.6) not to clash with (5.4) the axionic shift a (w.r.t. the lattice frame) would have to contain even components throughout. Of course these particular shifts no longer form a system of generators.

Interestingly the transformation behaviour of the moduli dependent part $Y_{f_a,f_b;f_c}$ of the Yukawa couplings under the shift $B \mapsto B + \alpha$ will play a role when we try to explicitly determine the mapping U_f for some even order orbifold models. Above it was contended that the background shift gives rise to an equivalent model. Hence we expect

$$\langle (\sigma^+_{f_a})' (\sigma^+_{f_b})' (\Sigma^{--}_{-f_c})' \rangle_{g+b+a} = \langle \sigma^+_{f_a} \sigma^+_{f_b} \Sigma^{--}_{-f_c} \rangle_{g+b} \tag{5.7}$$

to hold where the set of unitary redefinitions (5.3) has meanwhile been extended to the second twisted sector:

$$(\Sigma^{++}_f)' = U^{++}_f \, \Sigma^{++}_f \quad . \tag{5.8}$$

Actually one does not completely succeed in justifying this simple transformation law by repeating the analysis which led to (5.3) (cf. [22]). There it was shown that $(\sigma^+_f)'$ contains an admixture of a conventional twist field σ^+_t only if $(f - t) \in (1 - \Theta)\Lambda_d$. Since $[f] = [t]$ in this case the representation matrix U indeed turns out to be diagonal. However the above condition is equally valid w.r.t. a (higher) s–th twisted sector of the theory whereas the winding vector set $\bigcup_{j=1}^{|\mathcal{O}_f|-1} (\Theta^j f + (1 - \Theta^s)\Lambda_d)$ gives the translation group part of a conjugacy class. Therefore off–diagonal admixtures to, say, $(\Sigma^{++}_f)'$ cannot a priori be excluded $((\mathbf{1} - \Theta)\Lambda_d \subset (\mathbf{1} - \Theta^s)\Lambda_d)$.

On the other hand we need not anticipate a generalization of (5.8) since the second twisted sector might also (for a moment) be looked upon as the *first* twisted sector of another orbifold model for which the shifts a are indeed represented diagonally (see ([22]) on the set of $\det(\mathbf{1}-\Theta^2)$ ground states. To evaluate (5.7) we recall that the discrete shift α is supposed to commute with the twist. Only then will the ratio of the three-point functions (4.20) w.r.t. the above pair of modular backgrounds boil down to a constant phase factor. We end up with

$$U_{f_a} U_{f_b} \bar{U}^{++}_{f_c} = \exp(-2\pi i l^T \Theta \alpha l) \; ; \quad l = \frac{1}{\mathbf{1}-\Theta} f_a - \frac{1}{\mathbf{1}-\Theta^2} f_c \quad . \tag{5.9}$$

To disentangle this condition we simply choose the configuration $f_a = 0$, $f_b = f_c$ (which fulfills the space group selection rule) and obtain

$$U^{++}_f = U_f \exp(i\pi f^T \frac{\Theta\alpha}{\mathbf{1}-\Theta^2} f) \tag{5.10}$$

after symmetrizing the bilinear form in (5.9). Actually this relation is tantamount to the previous one because the product $U_{f_a} U_{f_b} \bar{U}^{++}_{f_c}$ can be rapidly reduced with the help of (5.4) and (5.5) for any admissible set of fixed points ($U_{f_b} = U_{f_c - f_a} = \ldots$). This equivalence results from the invariance of (4.20) under a discrete translation in target space by the fixed point vector $\frac{1}{\mathbf{1}-\Theta}\lambda$ ($\lambda \in \Lambda_d$). Since $f_a \mapsto f_a + \lambda$ implies $f_b \mapsto f_b + \lambda$ as well as $f_c \mapsto f_c + (\mathbf{1}+\Theta)\lambda$ the series ranging over $\mathcal{V}_{f_c, f_a}$ (cf. (4.21)) is indeed not affected.

Evidently, given the diagonal representation matrix U, its counterpart U^{++} for the second twisted sector may be explicitly read off, thus bypassing the more strenuous effort of solving the implicit conditions (5.4), (5.5) with Θ^2 replacing Θ. We must however keep in mind that (5.9) is not applicable whenever Θ possesses an eigenvalue (-1). On the other hand one might also solve for U once a solution U^{++} is at our disposal.

Let us nonetheless inspect the set of implicit conditions for U before embarking on a closer examination of (5.10). Notice that we have to insist on

$$U_f = U_{f+(\mathbf{1}-\Theta)\lambda} \qquad (\lambda \in \Lambda_d) \tag{5.11}$$

for them to be well-defined (This condition is joined by $U_f^{++} = U_{\Theta f}^{++}$ in the second twisted sector). Expanding the right hand side according to (5.4) one arrives at the simpler (but equivalent) condition

$$U_f = U_{\Theta f} \quad . \tag{5.12}$$

We also deduce that $U_0 = 1$ and $U_f = U_{-f}$ will always hold (take special values for the above winding vectors f, h).

Our next aim is to show that U_f is an N-th root of unity where N is the order of the twist. For even N we reformulate this claim, relying on (5.5), to read

$$\exp(-2\pi i f^T \frac{N/2}{\mathbf{1}-\Theta}\alpha f) = 1 \quad . \tag{5.13}$$

Resorting to the basis where the twist has a block–diagonal form (2.5) this can be quickly demonstrated. It suffices to recast the twist into the form $D = \begin{pmatrix} -\mathbf{1}_{d/2} & \mathbf{0} \\ \mathbf{0} & D' \end{pmatrix}$ whence the modular shift of B is bound to acquire the form $\alpha = \begin{pmatrix} \alpha_- & \mathbf{0} \\ \mathbf{0} & \alpha' \end{pmatrix}$. Similarly we decompose winding vectors as $f = \begin{pmatrix} f_- \\ f' \end{pmatrix}$ such that the upper components f_- are subject to a reflection under D. Then

$$\begin{aligned} f^T \frac{N/2}{\mathbf{1}-\Theta}\alpha f &= (f_-^T, (f')^T) \begin{pmatrix} \frac{N}{4}\mathbf{1}_{d/2} & \mathbf{0} \\ \mathbf{0} & P(D') \end{pmatrix} \begin{pmatrix} \alpha_- & \mathbf{0} \\ \mathbf{0} & \alpha' \end{pmatrix} \begin{pmatrix} f_- \\ f' \end{pmatrix} \\ &= f^T P(D)\alpha f + \{\tfrac{N}{4} - P(-1)\}(f_-^T \alpha_- f_-) \end{aligned} \tag{5.14}$$

where $P(D') := (\mathbf{1} + D')\frac{N/2}{\mathbf{1}-(D')^2}$ is a polynomial in D' with *integer* coefficients. This fact follows from $(D')^2$ having the smaller order $\frac{N}{2}$ together with the identity

$$\frac{N}{\mathbf{1}-\Theta} = \sum_{n=1}^{N} n\Theta^{N-n} \tag{5.15}$$

which holds for any order N twist if fixed directions are absent. Of course the evaluation of P for arguments of a different type (namely D and $(-1) \in \mathbb{Z}$) which occurs in the second line of (5.14) merely requires their formal substitution into $P(D')$. We recognize the last line of (5.14) to be an integer because $(f_-^T \alpha_- f_-) = 0$.

Next we consider the cases where $N = 1 \bmod 2$. In the subsequent calculation we will repeatedly use (5.4), (5.12), and the fact that $\frac{\mathbf{1}-\Theta^k}{\mathbf{1}-\Theta}$ simply gives an integer polynomial in Θ:

$$
\begin{aligned}
1 &= U_{f+\Theta f+\ldots+\Theta^{N_1} f} = (U_f)^N \exp\left(-2\pi i \binom{N}{2} f^T \frac{\alpha}{\mathbf{1}-\Theta} f\right) \\
&= (U_f)^N (U_f^2)^{\binom{N}{2}} = (U_f)^{N^2} \qquad .
\end{aligned}
\tag{5.16}
$$

Our claim follows since (5.15) tells us that $(U_f)^2$ is an N–th root of unity (cf. (5.5)) and since the binomial coefficient $\begin{pmatrix} N \\ 2 \end{pmatrix}$ is divisible by (odd) N. This result permits us to determine

$$
U_f = \left(U_f^2\right)^{-\frac{N-1}{2}} = \left\{\exp(2\pi i f^T \frac{\alpha}{\mathbf{1}-\Theta} f)\right\}^{\frac{N-1}{2}} \qquad (N \text{ odd}) \qquad . \tag{5.17}
$$

It is then easily checked that the implicit relations (5.4) and (5.5) are both satisfied. Also the stability condition (5.12) is trivially fulfilled.

In view of the absence of explicit factors N in the ratio of Yukawa couplings featuring on the right hand side of (5.9) one is tempted to recast (5.17) to get rid of the manifest dependence on the twist order. By combining (5.15) with

$$
(\mathbf{1}+K)(\mathbf{1}+K^2+\ldots+(K^2)^{\frac{N-1}{2}}) = \mathbf{1} \qquad (N \text{ odd}) \tag{5.18}
$$

we obtain the key relation

$$
\frac{N}{\mathbf{1}-K} = \frac{1}{\mathbf{1}+K} \bmod 2 \qquad . \tag{5.19}
$$

Hence the desired elimination can be performed:

$$
U_f = \exp(-2\pi i f^T \frac{\Theta\alpha}{\mathbf{1}-\Theta^2} f) = \exp(-2\pi i f^T \frac{\alpha}{\mathbf{1}-\Theta^2} f) \qquad . \tag{5.20}
$$

(In the first expression a symmetric bilinear form is used.)

On the other hand the representation matrices U_f for the class of even order orbifolds are much harder to find. We isolate the difficult part of their analysis by splitting

$$U_f = \exp(-i\pi f^T \frac{\alpha}{\mathbf{1}-\Theta} f)\ \phi_f \qquad . \tag{5.21}$$

Due to the normalization prescribed by (5.5) we recognize that ϕ maps Λ_d into $\{-1, 1\}$. We call this function henceforth a *sign distribution* (on the lattice). It is found to be subject to

$$\begin{array}{rcl} \phi_{\Theta f} & = & \phi_f \\ \phi_{f+h} & = & \phi_f \phi_h (-1)^{f^T \alpha h} \qquad (f, h \in \Lambda_d) \end{array} \tag{5.22}$$

upon substituting (5.21) into the primary conditions. Again ϕ defines a projective and associative representation of the compactification lattice Λ_d. Thanks to the composition law in (5.22) the sign distribution is determined by its values $\phi_j = \phi_{e_j}$ for a set of basis vectors. If we expand $f = \sum_{j=1}^{d} e_j n_j$ and introduce a d–dimensional vector t such that $(-1)^{t_j} = \phi_j$ the sign distribution will read

$$\begin{array}{rcl} \phi_f & = & (-1)^{t^T n + n^T a^+ n} \\ & & \text{where } (a^+)_{jk} = \left\{ \begin{array}{ll} a_{jk} & \text{if } j < k \\ 0 & \text{else} \end{array} \right. \quad . \end{array} \tag{5.23}$$

In retrospect the composition law of (5.22) is clearly fulfilled whereas the accompanying condition of orbit stability can be reduced to

$$\begin{array}{rrcl} & \phi_{\Theta e_j} & = & \phi_{e_j} \qquad (j \in \{1, \ldots, d\}) \\ \text{or} & (t^T(\mathbf{1}-K))_j & = & (K^T a^+ K)_{jj} \bmod 2 \end{array} \tag{5.24}$$

where K expresses (as usual) the action of Θ in the lattice basis. Given these subsidiary conditions the number of acceptable sign distributions is in many cases considerably smaller than 2^d (exceptional case: $K = -\mathbf{1}$). Let us develop a tighter upper bound. The quotient χ of two sign distributions ϕ, ψ will satisfy

$$\begin{aligned} \chi_{\Theta f} &= \chi_f \\ \chi_{f+g} &= \chi_f\, \chi_g \end{aligned} \tag{5.25}$$

whence it gives a group homomorphism. The solutions can be parametrized in the form $\chi_f = (-1)^{x^T n}$ ($x_j \in \{-0, 1\}$). One also has to impose $1 = (-1)^{x^T(\mathbf{1}-K)n}$ otherwise U_f will not become a well–defined function of the cosets $[f]$. A priori there are 2^d different candidate vectors x. If stability comes into play there must exist a vector ξ with even components such that $x^T = \xi^T \frac{1}{\mathbf{1}-K}$ is true. However the replacement $\xi \mapsto \xi + 2(\mathbf{1} - K^T)m$ (with m an integer vector) only gives us back the homomorphism we have started with. Hence at most $\det(\mathbf{1} - K)$ different solutions of (5.25) may exist. Consider for example the sequence of Coxeter orbifolds based on the $(N-1)$–dimensional root lattices of SU_N. Here the order N of the Coxeter twist coincides with the number of first twisted sector ground states (which is the upper bound we just arrived at). In comparison with this the less sophisticated bound 2^{N-1} is too crude for large order N. We point out that

$$\chi_{en} = \exp(2\pi i \frac{N}{2} z^T \frac{1}{\mathbf{1}-K} n) \quad , \tag{5.26}$$

where z is an arbitrary integer vector, will always solve (5.25).

We briefly sketch two examples:

1. The Coxeter orbifold obtained from the root lattice of SU_6: Here K has the non–vanishing entries $K_{j+1,j} = 1$ with $j \in \{1, \ldots, 4\}$, and $K_{r5} = -1$ for $r \in \{1, \ldots, 5\}$. This particular form permits only two different choices for x given that stability amounts to $x_1 = \ldots = x_5 \bmod 2$. (Here $d = 5$, but the considerations of this section also apply to an odd–dimensional target space.) Both solutions can be arrived at by using (5.26).

2. The Z_6 orbifold with the complex basis $e = (1, e^{i\frac{\pi}{3}})$: The twist acts via $K = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ on this hexagonal lattice. We arrive at $x_1 = x_2 = 0 \bmod 2$ whence there is a unique sign distribution. Correspondingly (5.26) just returns the trivial solution.

We did not succeed in solving the consistency constraint (5.24) even for the simplest non–trivial order four case (if $N = 2$ then $\Theta = -\mathbf{1}$ and t can be arbitrarily chosen). One comes across further relations if K is replaced by any higher power in (5.24). Playing with these constraints a host of amusing identities (containing solely K and a^+) can be written down, e.g.

$$\sum_{n=1}^{N-1} \left((K^T)^n a^+ K^n\right)_{jj} = 0 \bmod 2 \quad . \tag{5.27}$$

At any rate (5.24) offers a simple test on the existence of the sign distribution ϕ for even order orbifolds. In view of the linear dependence on the background shift $\alpha = e^* a e^{-1}$ it sufficcs to perform the check for the finite number of generators of discrete shifts a.[13]

Solutions $U_f^{(s)}$ of the conditions which replace (5.4) and (5.5) in a (higher) s times twisted sector will immediately follow once the answer for $s = 1$ has been worked out. If the twist Θ possesses an odd order N then (5.17), (5.20) will apply (with Θ replaced by its s–th power) because all higher sector twist operations are of odd order too. Matters are more diversified if N is an even number. In those sectors for which Θ^s is of odd order we make use of the type (5.17) whereas for even orders we simply resort to (5.21) and merely adapt the twist. Note that the first sector sign distribution ϕ can be kept since $\phi_{\Theta^s f} = \phi_f$ is an elementary consequence of the stability condition cited in (5.22). We underline once more that these recipes will fail if fixed subspaces should be present in the higher twisted sector.

Now we return to (5.10) which stands for the equivalence of two orbifold models that are related by a discrete shift of the background B. We discuss again the two main orbifold classes separately. For odd N the left hand side of (5.10) follows with (5.20) if we observe that $f^T \alpha \{R(\Theta) + R(\Theta)^T\} f = 0$.

Consider then the alternative $N = 0 \bmod 2$. Here we have to allow for a sign distribution in the phase representations U_{f_a}, U_{f_b}. According to (5.10) the phase transformation in the second twisted sector has to obey

[13] It can be shown that $\{\alpha; a_{ij} \in \mathbb{Z}, [\alpha, \Theta] = 0\}$ is an M_b–dimensional lattice. Given a basis of the axionic moduli space a unit cell of this lattice can always be arrived at by the successive construction of several auxiliary bases.

$$U_f^{++} = \exp(-2\pi i f^T \frac{\mathbf{1}+\Theta^2}{\mathbf{1}-\Theta^2}\frac{\alpha}{4} f)\,\Phi_f \qquad . \tag{5.28}$$

Obviously these phases do not depend upon the choice of a representative within the same second twisted sector conjugacy class ($\phi_{\Theta f} = \phi_f$ holds by assumption). At first sight we are somewhat puzzled by the appearance of the first sector sign distribution in an expression for U_f^{++}. However this peculiarity turns into an asset when $\frac{N}{2}$ is odd! We just have to insert (5.17) (where we adjust $\Theta \mapsto \Theta^2$, $N \mapsto \frac{N}{2}$) and (5.20) into (5.10) in order to solve for the sign distribution ϕ_f. We arrive at

$$\phi_f = \exp(2\pi i \frac{N}{4} f^T \frac{\alpha}{\mathbf{1}-\Theta^2} f) \tag{5.29}$$

which satisfies both axioms (5.22). When verifying the composition rule the following identity is seen to play a decisive role:

$$\frac{N}{2}\frac{\mathbf{1}+K^2}{\mathbf{1}-K^2} = \frac{N}{2} + 2K^2 \frac{N/2}{\mathbf{1}-K^2} = \mathbf{1} \bmod 2 \qquad . \tag{5.30}$$

(here we relied once more on (5.15).) Moreover these axioms guarantee that ϕ maps Λ_d into $\{-1, 1\}$.

The appearance of the factor $\frac{N}{4}$ in (5.29) can be avoided by slightly adapting the contents of (5.18) and (5.19). We then find

$$\phi_f = \exp(i\pi f^T \frac{\alpha}{\mathbf{1}+\Theta^2} f) \qquad . \tag{5.31}$$

Notice that $\frac{1}{\mathbf{1}+K^2}$ is integer due to $\frac{N}{2}$ being an odd number.

Unfortunately (5.29) is ill–defined if Θ acts as a reflection on part of the target space. Otherwise the phase transformation factors induced by a discrete axionic shift a read

$$U_f = \exp\left(i\pi f^T (\frac{1}{\mathbf{1}+\Theta^2} - \frac{1}{\mathbf{1}-\Theta})\alpha f\right) \qquad . \tag{5.32}$$

We have not yet studied the case where N is a multiple of 4. The phase factors U_f, U_f^{++} in (5.10) are of the type (5.21). Solving for the sign distribution ψ_f included in U_f^{++} shows that it agrees with the first sector sign distribution

ϕ_f (which is not too revealing). We already know that ϕ_f is also appropriate for the second twisted sector. The converse is however not true in general because the stronger condition $\phi_f = \phi_{\Theta f}$ need not be satisfied by a generic ψ_f.

The above results can still be generalized provided the orbifold models meet certain restrictions. Let us factorize the twist order into $N = 2^y z$ where the exponent y must be maximal thereby rendering an odd factor z. We discard then those orbifolds from our sample whose twist operations have an eigenvalue $\exp(2\pi i \frac{N}{y} k)$. In particular twists which reflect planar subspaces cannot be dealt with. For the remaining orbifolds we can at once construct the sign distribution

$$\phi_f = \exp(i\pi f^T \frac{\alpha}{\mathbf{1} + \Theta^{\frac{N}{y}}} f) \quad . \tag{5.33}$$

Again (5.22) is satisfied.

It is feasible that more cases can be explicitly investigated as soon as the moduli dependent factor $\langle \Sigma^k_{f_a} \Sigma^l_{f_b} \Sigma^{-k-l}_{-f_c} \rangle$ of the general twisted sector Yukawa coupling has been correctly evaluated (for a recent attempt to evaluate these factors cf. [24]). Taking an entirely practical standpoint we must concede that the the general expressions which have been derived here apply only to the limited number of symmetric orbifold models with a six–dimensional target space. On the other hand a case by case study would have been rather laborious except for the geometrically simplest twist patterns (e.g. $d = 2$ or $\Theta = -\mathbf{1}$). We underline that the construction of the axionic shift representation matrices was enormously facilitated by taking *all* mathematically consistent models into account.

6 Duality (background inversion)

Another *stringy* symmetry operation exchanges the momentum and winding numbers of closed string solutions. Parallelly the modular background $(G, B; e)$ is mapped into $(G, -B; \tilde{e})$ with

$$\begin{aligned} \tilde{e} &= \frac{1}{G-B} e^* \\ \Rightarrow \tilde{e}^* &= (G+B)e \quad . \end{aligned} \tag{6.1}$$

This involutive mapping is called the *duality* transformation. The dual torus model is described by the lattice $\tilde{\Lambda}_d = \{\tilde{e}n; n \in \mathbb{Z}^d\}$ which again admits Θ as an automorphism.

According to (3.1) a left right asymmetric rotation of the Narain lattice vectors is induced which however does not affect the spectrum of conformal and anti–conformal weights of the associated vertices $V_{\mathbf{P}}^{\mathrm{inv}}$.

These mappings are a consequence of the field theoretic substitutions

$$\begin{aligned} X_{\mathrm{R}} \mapsto \tilde{X}_{\mathrm{R}} &= \frac{1}{G+B}(G-B)X_{\mathrm{R}} \\ X_{\mathrm{L}} \mapsto \tilde{X}_{\mathrm{L}} &= -X_{\mathrm{L}} \end{aligned} \tag{6.2}$$

which must be implemented in every functional depending on the left– and rightmoving coordinate fields. The initial studies [25, 26] of the duality symmetry dealt with toroidal boundary conditions which did not contain an antisymmetric coupling B. If the string motion is confined to a circle then its radius r provides the single modular parameter. We read off from (6.1) that the dual partner circle possesses the inverse radius $\frac{1}{r}$ in suitably rescaled units. The discrete symmetries of the background space relevant for $(16+d, d)$ dimensional Narain compactifications have been addressed in [27]. An extension of the duality transformation to the twisted sector of the orbicircle model can be found in [28].

The Kähler modulus of two–dimensional tori reads $\varrho = \det e(\hat{B} + iG)$ (here $B = \begin{pmatrix} 0 & \hat{B} \\ -\hat{B} & 0 \end{pmatrix}$). The transformations $\varrho \mapsto \varrho + 1$, $\varrho \mapsto -\frac{1}{\varrho}$ describe the generating shift of the single axionic modulus (treated in section 5) and the transition to a dual background, respectively. Together they generate the *modular* group $\mathrm{PSL}(2, \mathbb{Z})$ which acts on the complex upper half plane. This feature generalizes to higher dimensional (orbifold) target spaces (with other discrete symmetry groups $\mathcal{K}$ at work).

Next we have to extend the substitution law (6.2) to all sectors of an orbifold model. This task was solved in [22] for the subclass of two–dimensional symmetric Z_N orbifolds. We briefly sketch the basic steps. There are three kinds of generators of a *different* discrete symmetry group $\mathcal{L}$ which acts at *prescribed* background values ϱ: The correlation functions are invariant under

1. a discrete translation of $X(z,\bar{z})$ by a vector of the fixed point lattice $\frac{1}{\mathbf{1}-\Theta}\Lambda_d$,
2. opposite translations of $X_{\mathrm{L}}(\bar{z})$, $X_{\mathrm{R}}(z)$ by vectors taken from the rescaled dual fixed point lattice $\frac{1}{2G}\frac{1}{\mathbf{1}-\Theta}\Lambda^*$,
3. point reflections of $X(z,\bar{z})$.

Needless to say that each of these prescriptions also induces a nontrivial transformation in the twisted Hilbert space sectors.

Moreover it is possible to determine for any generator of $\mathcal{L}$ the unique operation on the Hilbert space of a partner model whose background parameters are modified due to the action of an element of $\mathcal{K}$ (be it an axionic shift or the duality inversion). One starts from the untwisted sector and completes this pairing by a straightforward extension to the twisted sectors. The representation matrix U which gives the twisted sector substitution rule under an axionic shift was introduced in section 5 (cf. (5.3)). It relates the above mentioned pair of elements of $\mathcal{L}$ by joining them to form a *commutative* diagram. Its analysis provides us with the conditions (5.4) and (5.5) which are to be solved for the unitary matrix U. A similar consideration applies to the representation matrix U^{++} for the second twisted sector. In this case there will generically not arise as many constraints as are needed to determine U^{++}. This deficiency can be attributed to the fact that a second twisted sector fixed point might not yet be fixed under Θ.

The duality mapping was determined by this method as well [22]. It amounts to the discrete Fourier transformations

$$\tilde{\sigma}_f^+ = \frac{1}{\sqrt{N_1}} \sum_{h \in F_1^*} \exp(-2\pi i f^T \frac{1}{\mathbf{1}-\Theta} h)\ \sigma^+_{\frac{1}{G-B}h}$$

$$\tilde{\Sigma}_f = \frac{1}{\sqrt{N_2}} \sum_{h \in (F_2^*)'} \sqrt{\frac{|\mathcal{O}_{2;h}|}{|\mathcal{O}_{2;f}|}} \sum_{n=0}^{|\mathcal{O}_f|-1} \exp\left(2\pi i (\Theta^n h)^T \frac{1}{\mathbf{1}-\Theta^2} f\right)\ \Sigma^{++}_{\frac{1}{G-B}h} \tag{6.3}$$

with the sums ranging over

$$(F_s^*)' = F_s^*/\mathrm{P}\,; \qquad F_s^* = \frac{\Lambda_d^*}{(\mathbf{1}-\Theta^s)\Lambda_d^*} \quad . \tag{6.4}$$

The quotient F_s^* contains $N_s := \det(\mathbf{1}-\Theta^s)$ coset representatives. Observe that $(F_1^*)' = F_1^*$ and that $\frac{1}{G-B}h$ is indeed a winding vector of the dual orbifold model as is evident from (6.1). Once the ordinary coordinates and twist field operators have been replaced by (6.2), (6.3), respectively, the correlation functions referring to the duality transformed background should coincide with their counterparts in the original orbifold model.

We draw two important conclusions. Modding out the background parameter space by the action of $\mathcal{K}$ we are led to a fundamental region as is well known for $PSL(2,\mathbb{Z})$ in the case of $d = 2$. Orbifold models which belong to its complement need no longer be considered for classification purposes. Their phenomenological aspects are fully accounted for by suitable "representant" models whose background tensor is an element of the fundamental region. Furthermore the low energy effective action inherits $\mathcal{K}$. Tight restrictions were discovered for the scalar potential of an effective four–dimensional supergravity action (see [14]).

Now we establish the invariance of the three–twist field correlation (4.20) if the complete set of transformations induced by a background inversion (6.1) is applied. This claim amounts to having

$$\langle \sigma^+_{f_a}\, \sigma^+_{f_b}\, \Sigma^{--}_{-f_c} \rangle_{g+b} = \langle \tilde{\sigma}^+_{f_a}\, \tilde{\sigma}^+_{f_b}\, \tilde{\Sigma}^{--}_{-f_c} \rangle_{(g+b)^{-1}} \quad . \tag{6.5}$$

Therefore we expect that the moduli dependent part of a twisted sector Yukawa coupling is identical to a particular linear combination of correlation functions of the same type which refer to the dual background $(g+b)^{-1}$ (this conclusion extends of course to any correlation function which is parametrized by the set of background moduli).

In view of the background dependence of the symmetric bilinear form

$$A(G,B;e) = e^T(\Theta - \Theta^T)(GZ - iB)e \tag{6.6}$$

which characterizes the Boltzmann factors occurring in (4.20) it is advisable to try once more a Poisson resummation to uncover this linear relationship. Moreover we may focus on the reduced set $\{Y_{0,f;f};\ f \in \Lambda_d\})$ of twist field correlators given their invariance under translations by arbitrary fixed point vectors $\frac{1}{\mathbf{1}-\Theta}\lambda$ (cf. our comment below (5.10)).

A short calculation based on (3.31) then yields

$$Y_{0,f;f}(G,B;e) = \frac{(\det(2g))^{\frac{1}{4}}\,\eta(\Theta;f)}{\sqrt{\det A}} \sum_{m\in\mathbb{Z}^d} e^{-\pi m^T A^{-1} m}\, e^{-2\pi i (e^* m)^T \frac{1}{\mathbf{1}-\Theta^2} f} \quad . \tag{6.7}$$

In order to proceed we rely on the auxiliary formula

$$\frac{1}{GZ - iB} = (G+B)\frac{1}{GZ+iB}(G-B) \tag{6.8}$$

which can be readily proven by rewriting it exclusively in terms of Z and $\frac{1}{G}B$ and then using (4.22) to simplify it further. This identity also entails

$$\det(GZ - iB) = \det(G-B)\, e^{i\xi} \qquad \text{with } e^{2i\xi} = \frac{\det(GZ - iB)}{\det(GZ + iB)} \quad . \tag{6.9}$$

We will now recast (6.7) explicitly in terms of the ordinary three–point correlators w.r.t. the dual background position:

$$
\begin{aligned}
Y_{0,f;f}(G,B;e) &= \frac{(\det(2\tilde{g}))^{\frac{1}{4}}}{\sqrt{N_2}} e^{-i\frac{\xi}{2}}\eta(\Theta;f) \\
&\times \sum_{w\in\tilde{\Lambda}_d} e^{-\pi w^T \frac{1}{\Theta^T-\Theta}(GZ+iB)w \,-\, 2\pi i((G-B)w)^T \frac{1}{\mathbf{1}-\Theta^2} f} \\
&= \frac{e^{-i\frac{\xi}{2}}}{\sqrt{N_2}} \sum_{\tilde{f}\in\tilde{\mathcal{F}}_2} \sqrt{\frac{|\mathcal{O}_{2;f}|}{|\mathcal{O}_{2;\tilde{f}}|}}\, e^{2\pi i h^T \frac{1}{\mathbf{1}-\Theta^2} f}\; Y_{0,\tilde{f};\tilde{f}}(G,-B;\tilde{e}) \\
&\quad \text{where} \quad \tilde{\mathcal{F}}_2 = \frac{\tilde{\Lambda}_d}{(\mathbf{1}-\Theta^2)\tilde{\Lambda}_d} \text{ and } h = (G-B)\tilde{f} \quad .
\end{aligned}
\tag{6.10}
$$

As a last step we had to perform a coset decomposition

$$
\tilde{\Lambda}_d = \bigcup_{\tilde{f}\in\tilde{\mathcal{F}}_2} \left\{ -\tilde{f} + (\mathbf{1}-\Theta^2)\tilde{\Lambda}_d \right\} \tag{6.11}
$$

on the range of the series variable w which permitted us to identify $|\tilde{\mathcal{F}}_2|$ contributions of three–twist field correlations w.r.t. the dual background $(g+b)^{-1}$.

We need not to be worried about the background dependent phase factor $e^{-i\frac{\xi}{2}}$ contained in the transformation rule (6.10). It disappears upon including the compensating phase $e^{i\frac{\xi}{4}}$ into the definition of $Y_{f_a,f_b;f_c}(G,B;e)$ which is evidently sanctioned by (4.7) (cf. also [23]). It remains to verify our claim (6.5) by replacing the dual twist fields $\tilde{\sigma}^+$, $\tilde{\Sigma}^{--}$ by the superpositions (6.3) of *ordinary* twist fields w.r.t. the background tensor $(g+b)^{-1}$. Since neither of these prescriptions nor (6.10) explicitly depends on the target space dimension d or on the background matrices g, b we are reminded of the analysis performed for the two–dimensional case in [22]. In order to compare both sides of (6.5) we may entirely rely on the discrete translation invariance of correlators which consist of the ordinary twist fields σ^+, Σ^{--} and on projection identities such as

$$
\frac{1}{N_1}\sum_{h\in\mathcal{F}_1^*} \exp(2\pi i h^T \frac{1}{\mathbf{1}-\Theta} f) = \begin{cases} 1 \text{ if } f\in(\mathbf{1}-\Theta)\Lambda_d \\ 0 \text{ else} \end{cases} \quad . \tag{6.12}
$$

(It converts a sum of discrete phase factors into a selection rule.) Both these ingredients are likewise independent of the details of a particular compactification.

In view of these facts the duality invariance of the three twist field correlation function is guaranteed for symmetric Z_N orbifold constructions of the bosonic string.

7 Discussion

For the class of symmetric Z_N orbifolds an explicit parametrization of the (axionic) antisymmetric background B was derived assuming that $[B, \Theta] = 0$. This condition was recognized to be crucial for a calculation of the complete set of instanton actions in the case of the four–twist field correlation function.

Via a factorization w.r.t. the "twisted channel" the moduli dependent part $\langle \sigma^{+}_{f_a} \sigma^{+}_{f_b} \Sigma^{--}_{-f_c} \rangle$ of a particular Yukawa coupling could be read off. The calculation of three–point functions involving any set of (higher sector) twist fields $\{\Sigma^{(k)}, \Sigma^{(l)}, \Sigma^{(-k-l)}\}$ in the presence of a purely metrical background g has been attempted in [24]. It remains to include the axionic moduli as well [29].

In order to undo the change of these couplings caused by discrete shifts $b \mapsto b + a$ ($a_{kl} \in \mathbb{Z}$, $[\alpha, \Theta] = 0$) we had to resort to new twist fields which differ by a phase factor from the ordinary ones. The existence of a consistent set of such phases hinges on the validity of the algebraic condition (5.24). One might also reverse the argumentation: If these background operations constitute true symmetries of our class of Z_N orbifold models then a stringy proof guarantees that we can solve (5.24). Likewise the inversion in background space (duality) was recognized not to affect the B dependent Yukawa couplings provided that Fourier transformed twist fields are employed as opposed to the ordinary ones. Again we would like to turn the tables. Suppose that Yukawa couplings are duality invariant (due to the proper substitutions of the twist fields). Then the linear relation (6.10) which involves correlation functions of the *ordinary* twist fields can be immediately seen to hold. Equipped with some elementary knowledge of the background dependence of an instanton action (which follows from (2.1)) and being aware that the global windings are restricted to $\mathcal{U}_{f_c,f_a}$ (see (4.18)) it is then possible to deduce the complete instanton action formula: the unknown factors must be chosen in such a way that (6.10) can be reconstructed via a Poisson transformation. Thus one can in fact bypass the awkward normalization of cut differentials and the subsequent complex plane integration of their norm squares (cf. section 2) if Yukawa couplings need to be determined only up to a numerical prefactor.

Although the class of orbifolds treated in this paper is quite general some important cases have been skipped:

- We avoided to discuss those backgrounds B for which $[B, \Theta] \neq 0$. In this case we cannot rely on a consistent action description of the orbifold CFT which therefore complicates the evaluation of correlation functions. An exception is the case of an antisymmetric background $B = B_0 + \Delta$ with $[B_0, \Theta] = 0$ whereas Δ which does *not* commute with Θ maps Λ_d into Λ_d^* (i.e., $e^T \Delta e \in \mathbb{Z}^{d\times d}$; cf. ([13]). It is quite simple to introduce this background in (3.32) by letting

$$p \rightarrow p' = p + \Delta v \qquad .$$

  Thereby the conformal weights h_j, $\bar{h}_j$ in (3.32) adapt to the shifted background Δ. The string emission coupling (3.39) is then found to acquire a dependence on Δ (cf. also [13]). A redefinition (5.3) of the twist fields will not enable us to eliminate this explicit dependence on Δ in view of the remark made below (5.5). Consequently only the antisymmetric background B_0 will enter the Yukawa couplings (4.17) since (3.32) has in fact not been modified by the above substitution.

  Actually for twist invariant vertex operators $V_{\mathbf{P}}^{\mathrm{inv}}$ to exist it suffices that the twist acts as an automorphism of the Narain lattice $\Lambda_{\mathcal{N}}$. To have this property the background matrix B must satisfy $e^T[B, \Theta]e \in \mathbb{Z}^{d\times d}$ which evidently is a weaker condition than (2.4). As has been shown in an appendix of [13] there even exist solutions b which cannot be cast into the form $b = \beta + \Delta$ where $\beta = e^T B_0 e$.

With the help of the operator formulation of an orbifold CFT [13] one might be able to deduce three–point functions even under such circumstances. Unfortunately, a calculation of correlation functions involving more than two twist fields cannot be handled in this way. Besides, the path integral approach is doomed to failure. A more promising method (suggested in [30]) could be to resort to a doubling of the Narain lattice $\Lambda_{\mathcal{N}}$ (which gives a complete specification of the background data). If this strategy yields a new double–sized antisymmetric background B which commutes with the enlarged twist then the results of this paper carry over and various orbifold correlation functions should be accessible. We report elsewhere [31] about this approach.

- The definition and the evaluation of correlation functions for *heterotic* orbifolds in the presence of *quantized* Wilson lines remains a challenging problem, which is of course closely related to the bosonic situation just mentioned before.

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