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String Emission from Twisted Sectors: Cocycle Operators and Modular Background Symmetries*

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ABSTRACT

We compute coupling constants for the emission of untwisted states within the class of even dimensional, bosonic orbifolds. In order to apply the operator formalism we first have to construct the untwisted and twisted sector cocycle operators which complete the zero mode part of vertex operators. Their existence is guaranteed by a consistency condition on the axionic background whose general solution will be determined. These results enable us to study various discrete background symmetries (duality, shifts of axionic components) which reshuffle the vertices. We then recognize that the process of the emission of strings from whichever twisted sector provides a modular invariant.

1 Introduction

String theories are up to now the best candidates of a unified fundamental theory of all matter fields and all interactions including gravity. Among them the phenomenologically most appealing one is the heterotic string [1]. The main obstacle for predicting low energy physics from this ten–dimensional string model is the huge number of possible four-dimensional vacua. We mention the well known examples of non–linear sigma models on Calabi–Yau manifolds [2], covariant lattice models [3]-[5], vacua obtained by the fermionic construction [6, 7], and orbifold models [8]-[12]. Each of these string vacua corresponds to an internal two–dimensional conformal field theory (2D CFT) [13, 14] with particular values of the holomorphic and antiholomorphic conformal anomaly and some additional properties like e.g. modular invariance.

In this paper we will concentrate on the quite promising class of symmetric Z_N orbifolds which have a clear geometrical interpretation. Moreover, since the underlying CFTs mainly consist of free fields, their analytical treatment is considerably simplified. These 2D CFTs happen to be exactly solvable, even if they are irrational, that is to say, if they possess an infinite number of primary fields w.r.t. the (anti–) chiral algebra of the theory.

Although quite a lot is known about the spectrum of orbifoldized heterotic models, the calculation of orbifold correlation functions [15]-[18] is somewhat involved and not yet understood, if the most general background consisting of an axionic coupling and Wilson lines embedded into the gauge degrees of freedom is admitted. Such M–point functions are required for any thorough investigation of the phenomenological implications of this class of models. Also a complete discussion of discrete symmetries in the moduli background space hinges on a detailed knowledge of the basic three–point interactions [19, 20].

To contribute to a full understanding of these correlation functions we will treat in this paper the simpler case of bosonic strings compactified on a symmetric Z_N orbifold whose target space is d–dimensional (d even). It is remarkable that the axionic background B need not commute with the twist action Θ. However, as has already been stated in [12], the commutator must take a particular form which is dictated by the automorphic action of the twist Θ on the Narain momentum lattice (see [21]).

Explicit calculations of orbifold correlation functions were first carried out in [15, 17] for $d \leq 2$; however only metrical moduli were taken into account.

As regards pure twist field correlations, their classical part is provided by instanton configurations, whereas their quantum part is obtained via the stress–energy tensor method. The case of non–vanishing axionic coupling B was subsequently solved in [22] for $d = 2$.

Now one observes that for a consistent action description, which is obligatory for a path integral treatment of orbifold correlations, B even has to commute with the twist Θ. This condition is trivially satisfied when $d = 2$ but it imposes too severe a restriction for $d > 2$. What comes to our rescue is the operator formalism, which allows us to establish these correlation functions even if $[B, \Theta] \neq 0$. Apart from a short comment in appendix A we will however not bother about those twists which possess *fixed* directions.

On–shell scattering amplitudes of strings in the tree level approximation are derived from correlation functions or rather vacuum expectation values of products of particular local operators on the complex plane. If these *vertex operators* belong to the untwisted sector, they can be expressed in terms of exponentials of free bosons and have to mutually commute (for this reason they provide representations of Kac–Moody algebras for particularly chosen toroidal target spaces). One must implement this property either by introducing extra *cocycle operators* or via a proper quantization of the available bosonic *zero modes*. The latter route was advocated in [23, 24].

Our plan is as follows: In section 2 we shortly review some important properties of the bosonic Narain model. Particular zero mode exponentials yield the cocycle operators which are needed for the string emission vertices. If they were not used we would not have the correct relative phases between different three–point functions. We begin section 3 by listing the basic geometrical concepts of a bosonic orbifold CFT. The condition which singles out a consistent axionic background is solved in general. Next, we turn to the operator quantization of the first twisted sector following essentially the earlier approach of [23, 24]. Finally twist invariant vertex operators are constructed. In section 4 we calculate the twisted sector string coupling constant for the emission of an untwisted from a twisted string by factorizing the product of two vertices in the twisting vacuum. This result completes the derivation of all three–point functions which consist of two fields from (oppositely) twisted sectors and a single vertex operator. Section 5 opens with a short account of the moduli space of Z_N orbifolds. The generators of background transformations are seen to redefine the zero modes of orbifold constructions associated to the new background moduli. This property enables us to prove that the correlation function for string emission from the first twisted sector stays invariant both under the duality and the axionic shift operations.

We discuss our results in section 6. Furthermore we have included several appendices. First we explicitly show how to solve the constraint on the axionic background for various toy models. In appendix B we calculate twisted sector two–point functions of the left– and right–moving parts of the bosonic coordinate fields. We then argue why the string coupling constant is not affected by a duality transformation (appendix C). A detailed treatment of string emission from higher twisted sectors is the subject of appendix D: We derive the three–point function for these processes and verify their invariance w.r.t. a duality transformation.

2 The bosonic Narain model

2.1 The spectrum

The simplest method to compactify the 26–dimensional closed bosonic string theory consists in curling up d of its spatial coordinates $X^\mu(\tau,\sigma)$ on a (flat) torus $T_d = \mathbb{R}^d/2\pi\Lambda_d$ which is specified by a d–dimensional lattice Λ_d. This construction implies the boundary conditions

$$X^\mu(\tau,\sigma+2\pi) = X^\mu(\tau,\sigma) + 2\pi w^\mu \; ; \quad w \in \Lambda_d\, ,\; 1 \leq \mu \leq d \quad . \tag{2.1}$$

Let us introduce a basis $\{e_i\,; i = 1,\ldots,d\}$ of Λ_d. The components of the e_i w.r.t. the coordinate metric are $e^\mu{}_i$ and may be considered as the elements of a *basis matrix* e. Any *winding vector* $w \in \Lambda_d$ can now be expanded as $w = e_i\, n^i$ $(n^i \in \mathbb{Z})$.

After analytic continuation of the world–sheet time $\tau \longrightarrow -i\tau$ followed by a conformal mapping we arrive at the complex world sheet coordinates

$$z = e^{\tau+i\sigma},\; \bar{z} = e^{\tau-i\sigma}. \tag{2.2}$$

Hence (2.1) now reads

$$X^\mu(e^{2\pi i}z, e^{-2\pi i}\bar{z}) = X^\mu(z,\bar{z}) + 2\pi w^\mu. \tag{2.3}$$

The following mode expansions for the right– and left–moving parts of the compactified coordinates satisfy these boundary conditions:

$$X_R^\mu(z) = x_R^\mu - i\left(\frac{1}{2G}P_R\right)^\mu \ln z + i\sum_{n\neq 0} z^{-n}\frac{\alpha_n^\mu}{n} \tag{2.4}$$

$$X_L^\mu(\bar z) = x_L^\mu - i\left(\frac{1}{2G}P_L\right)^\mu \ln \bar z + i\sum_{n\neq 0} \bar z^{-n}\frac{\tilde\alpha_n^\mu}{n} \quad . \tag{2.5}$$

The string's center of mass position at $|z| = 1$ coincides with

$$x^\mu \equiv x_R^\mu + x_L^\mu \quad , \tag{2.6}$$

provided that the right– and leftmoving momentum P_R and P_L are equal. On the other hand (2.3) imposes

$$(P_R)_\mu - (P_L)_\mu = 2\, G_{\mu\nu} w^\nu \quad , \tag{2.7}$$

whence the identification of (2.6) with the string's center of mass must be given up, if the string possesses nontrivial homotopy (i.e. $w \neq 0$). The symmetric nonsingular $d \times d$ matrix G will shortly be identified with the target space metric.

As has been pointed out by Narain [21], the primary fields

$$\begin{aligned} V_{\mathbf{P}}(z,\bar z) &= c_{\mathbf{P}} : \exp\{iP_R X_R(z) + iP_L X_L(\bar z)\} : \\ &\equiv c_{\mathbf{P}}\ E(P_R, P_L; z, \bar z) \end{aligned} \tag{2.8}$$

of the corresponding CFT are labeled by the elements of a $2d$–dimensional lattice $\Lambda_{\mathcal{N}}$ (*Narain lattice*)

$$\mathbf{P} \equiv (P_R; P_L) \in \Lambda_{\mathcal{N}}. \tag{2.9}$$

The colons in (2.8) denote normal ordering and $c_{\mathbf{P}}$ is known as a *cocycle operator*, which guarantees appropriate commutation relations of the $V_{\mathbf{P}}$. We will get to know its properties in the next subsection[1].

Modular invariance (on a worldsheet torus) restricts $\Lambda_{\mathcal{N}}$ to be even and selfdual w.r.t. a lorentzian scalar product

$$\langle \mathbf{P}, \mathbf{Q}\rangle := (P_R, Q_R) - (P_L, Q_L) := \frac{1}{2}P_R^T \frac{1}{G} Q_R - \frac{1}{2} P_L^T \frac{1}{G} Q_L \tag{2.10}$$

[1]Actually, it was explained in [23] and [24], that one can avoid the introduction of cocycle operators by a proper quantization of the zero modes involved. We will rely on this approach, when discussing the twisted sector representation of $V_{\mathbf{P}}(z, \bar z)$.

of signature (d, d).

We mention (cf. [25]), that such lattices with the general signature (r, s) exist if and only if $(r - s)/8 \in \mathbb{Z}$. Moreover, for given $r, s \neq 0$ they are unique up to $SO(r, s)$ Lorentz-transformations.

The conformal dimensions

$$\begin{aligned} h &= \tfrac{1}{4} P_R^T \frac{1}{G} P_R \quad , \\ \bar{h} &= \tfrac{1}{4} P_L^T \frac{1}{G} P_L \end{aligned} \tag{2.11}$$

of the primary fields (2.8) are read off from the operator product with the stress energy $T(z) = \partial X^T G \partial X$, $\bar{T}(\bar{z}) = \bar{\partial} X^T G \bar{\partial} X$, respectively. Adopting a coordinate basis, where G is proportional to $\mathbf{1}$, these weights become invariant under $SO(d)_R \times SO(d)_L$ transformations. As we will recognize in section 2.2, this symmetry is respected by all correlation functions if the cocycle operators are properly redefined. Therefore the toroidal compactification (2.1) leads to a family of CFTs, which are in one–to–one correspondence with the points of the coset manifold

$$\mathcal{M}_d = \frac{SO(d, d)}{SO(d)_R \times SO(d)_L} \quad . \tag{2.12}$$

Thus d^2 independent parameters suffice to describe the distinct bosonic Narain models. Strictly speaking this is not yet the whole story, since there also exist *discrete background symmetries* (e.g. target space duality) by which $\mathcal{M}_d$ has to be divided out, too. These possibilities will be explored in section 5.

In [26] Narain, Sarmadi and Witten arrive at a simple interpretation for the coordinates of $\mathcal{M}_d$ in terms of the components of the torus metric g and of an antisymmetric tensor field b. Indeed g and b are characterized by $\frac{1}{2} d(d \pm 1)$ independent parameters, and these numbers add up to d^2 as required. Both g and b are assumed not to depend on the coordinate fields X^μ, whence they are referred to as *constant background fields.*

The starting point of the analysis given in [26] is the free bosonic action[2]

[2]In this paper we will focus on the compact part of the target space. The inclusion of uncompactified bosonic fields is obvious. Here and in the following we frequently choose $G = \frac{1}{2}\mathbf{1}$ which corresponds to having $\alpha' = 2$ for the Regge slope parameter.

$$S = \frac{1}{\pi}\int d\tau d\sigma\, \partial_+ X^T(\tau,\sigma)(G+B)\partial_- X(\tau,\sigma). \qquad (2.13)$$

This expression is formulated w.r.t. a Minkowskian world sheet metric, where $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$. We then relate the target space background tensors G and B to their counterparts g and b in the lattice frame:

$$g := e^T G e = \quad g^T \qquad (2.14)$$

$$b := e^T B e = -b^T \quad . \qquad (2.15)$$

The right– and left–moving momenta P_R and P_L, which enter (2.11) are parametrized by

$$\begin{array}{rcl} P_R &=& p + (G-B)w \\ P_L &=& p - (G+B)w \end{array} \qquad (2.16)$$

where $w \in \Lambda_d$, and $p \in \Lambda_d^*$ denotes the *momentum* vector canonically conjugate to the zero mode $x = x_R + x_L$ (Λ_d^* is the lattice dual to Λ_d). We also assign the dual basis $\{e^{*i}; i = 1,\ldots,d\}$ to Λ_d^*, whose coordinate components $e^{*\ i}_{\mu}$ form the elements of the *dual basis matrix* e^*, which by definition is subject to

$$e^T e^* = \mathbf{1} \quad . \qquad (2.17)$$

Any $p \in \Lambda_d^*$ now decomposes as $p = e^{*i} m_i$ ($m_i \in \mathbb{Z}$). Then a basis for the (d,d)–dimensional Narain lattice $\Lambda_{\mathcal{N}}$ reads:

$$\begin{array}{rcl} k^i &=& (e^{*i}; e^{*i}) \\ \bar{k}_j &=& (e^{*k}(g-b)_{kj}; -e^{*k}(g+b)_{kj}) \end{array} \quad . \qquad (2.18)$$

Using the lorentzian scalar product (2.10) we find

$$\langle \mathbf{P}_1, \mathbf{P}_2 \rangle = p_1^T w_2 + p_2^T w_1 \quad . \qquad (2.19)$$

It follows, that the (d,d)–dimensional lattice introduced via (2.16) is even and selfdual, as required for $\Lambda_{\mathcal{N}}$.

Since the target space is a torus, (2.8) has to stay invariant, if the coordinates $X(z,\bar{z})$ are shifted by $2\pi v$ ($v \in \Lambda_d$):

$$\begin{array}{rcl} X_R &\longmapsto& X_R + 2\pi v_R\,, \\ X_L &\longmapsto& X_L + 2\pi v_L\ ; \quad v_R + v_L = v \end{array} \quad . \qquad (2.20)$$

Because this requirement amounts to

$$P_R^T v_R + P_L^T v_L = p^T v + w^T B v + w^T G (v_R - v_L) \in \mathbb{Z} \qquad (2.21)$$

for any pair $(w,p) \in \Lambda_d \times \Lambda_d^*$ we are able to determine v_R and v_L separately. The simplest solution reads

$$v_{R/L} = \frac{1}{2}(v \mp \frac{1}{G} B v) \quad . \qquad (2.22)$$

It clearly becomes left–right asymmetric should the axionic background B not vanish.

Furthermore it is obvious that the whole spectrum and, as we will argue in the next subsection, all correlation functions remain invariant under nonsingular linear coordinate transformations

$$e \longmapsto Fe, \qquad e^* \longmapsto F^* e^* \qquad (F^* := (F^T)^{-1}) \qquad (2.23)$$

$$G \longmapsto F^* G F^{-1}, \; B \longmapsto F^* B F^{-1} \quad . \qquad (2.24)$$

Apparently, with this freedom one may always turn to $2G = \mathbf{1}$.

2.2 Correlation functions and cocycle operators

The primary fields of the bosonic Narain model for a generic background are $\partial_z X$, $\partial_{\bar{z}} X$ and $V_{\mathbf{P}}$ with $\mathbf{P} \in \Lambda_{\mathcal{N}}$. The fundamental operator product expansion (OPE) involves two free bosonic (coordinate) fields:

$$X^\mu(z_1, \bar{z}_1)\, X^\nu(z_2, \bar{z}_2) = -(\frac{1}{2G})^{\mu\nu} \ln |z_{12}|^2 + :X^\mu(z_1, \bar{z}_1) X^\nu(z_2, \bar{z}_2): \quad . \qquad (2.25)$$

It is computed by relying on a Fock space interpretation of the mode expansions (2.4) and (2.5). Thus one promotes all Laurent coefficients to *operators* (marked by a hat) with commutation relations

$$[\hat{\alpha}_m^\mu, \hat{\alpha}_n^\nu] = \left[\hat{\bar{\alpha}}_m^\mu, \hat{\bar{\alpha}}_n^\nu\right] = (\frac{1}{2G})^{\mu\nu}\, m\, \delta_{m+n,0} \quad . \qquad (2.26)$$

Likewise the zero modes are quantized, too:

$$\left[\hat{x}_R, \hat{P}_R^T\right] = \left[\hat{x}_L, \hat{P}_L^T\right] = i \cdot \mathbf{1} \qquad . \tag{2.27}$$

(The definition $[\gamma, \delta^T]_{ij} := [\gamma_i, \delta_j]$ of a commutator matrix between two vectors of operators γ, δ permits us to drop all indices.)

The *ground state* $|0\rangle$ in the momentum space representation by definition neither carries momentum nor winding and is annihilated by all positive frequency mode operators:

$$\begin{aligned} \hat{P}_{R\mu}|0\rangle &= \hat{P}_{L\mu}|0\rangle &= 0 \\ \hat{\alpha}_n^\mu|0\rangle &= \hat{\tilde{\alpha}}_n^\mu|0\rangle &= 0\,, \quad n > 0 \qquad . \end{aligned} \tag{2.28}$$

According to (2.16) $\hat{P}_R$ and $\hat{P}_L$ can be expressed in terms of the canonical momentum and winding operators

$$\begin{aligned} \hat{P}_R &:= \hat{p} + (G - B)\hat{w} \\ \hat{P}_L &:= \hat{p} - (G + B)\hat{w} \qquad . \end{aligned} \tag{2.29}$$

It is worth analysing the zero mode structure in greater depth, especially since we could not yet decompose $\hat{x}_R$ and $\hat{x}_L$ into geometrical zero modes. We simply overlooked that there is one more besides $\hat{x}$. Now, we infer from (2.27) that there exists a linear transformation

$$\begin{pmatrix} \hat{P}_R \\ \hat{P}_L \end{pmatrix} = K \begin{pmatrix} \hat{p} \\ \hat{w} \end{pmatrix}, \quad \begin{pmatrix} \hat{x}_R \\ \hat{x}_L \end{pmatrix} = K' \begin{pmatrix} \hat{x} \\ \hat{q} \end{pmatrix}, \tag{2.30}$$

which leaves the commutator matrices invariant. We abbreviated

$$K = \begin{pmatrix} \mathbf{1} & (G - B) \\ \mathbf{1} & -(G + B) \end{pmatrix} \tag{2.31}$$

and introduced the operator $\hat{q}$ which is necessarily subject to

$$[\hat{q}, \hat{w}^T] = i \cdot \mathbf{1} \qquad . \tag{2.32}$$

We infer that $K' = (K^T)^{-1}$ and consequently one finds

$$\hat{x}_{R/L} = \frac{1}{2}(\mathbf{1} \mp \frac{1}{G}B)\hat{x} \pm \frac{1}{2G}\hat{q} \qquad . \tag{2.33}$$

in agreement with (2.6) and the left–right asymmetric shifts (2.22) of X_R, X_L. The (normal ordered) zero mode part of $E(P_R, P_L; z, \bar{z})$ is then given by

$$e^{i(p^T\hat{x}+w^T\hat{q})}\; z^{P_R^T\hat{P}_R}\; \bar{z}^{\,P_L^T\hat{P}_L} \qquad . \tag{2.34}$$

As far as the cocycle operator part $c_{\mathbf{P}}(\hat{\mathbf{P}})$ of $V_{\mathbf{P}}$ is concerned, we assume it to be bimultiplicative in the Narain momentum $\mathbf{P}$ of the vertex and the Narain momentum operator $\hat{\mathbf{P}}$. Then the field algebra will close under operator product expansions (OPEs):[3]

$$\begin{aligned} V_{\mathbf{P}_1}(z_1,\bar{z}_1) \quad & V_{\mathbf{P}_2}(z_2,\bar{z}_2) = \\ &= c_{\mathbf{P}_2}(-\mathbf{P}_1)\, c_{\mathbf{P}_1+\mathbf{P}_2}(\hat{\mathbf{P}})\; E(P_{1R},P_{1L};z_1,\bar{z}_1)\, E(P_{2R},P_{2L};z_2,\bar{z}_2) \\ &= z_{12}^{P_{1R}^T P_{2R}}\; \bar{z}_{12}^{P_{1L}^T P_{2L}}\; c_{\mathbf{P}_2}(-\mathbf{P}_1)\; V_{\mathbf{P}_1+\mathbf{P}_2}(z_2,\bar{z}_2) + \ldots \qquad . \end{aligned} \tag{2.35}$$

While performing the rearrangements we relied on the conservation law of Narain momentum which is a consequence of (2.34); hence only a single conformal block can appear on the right hand side of (2.35).

Since the above vertex operators carry integer spins $s_i = h_i - \bar{h}_i$, they are bosonic fields of the 2D CFT and so they have to commute. Therefore under a transposition $\mathbf{P}_1 \leftrightarrow \mathbf{P}_2$, $(z_1,\bar{z}_1) \leftrightarrow (z_2,\bar{z}_2)$ the left hand side of (2.35) is not affected. This gives rise to the cocycle consistency relation

$$c_{\mathbf{P}_2}(-\mathbf{P}_1) = e^{i\pi\langle \mathbf{P}_1,\mathbf{P}_2\rangle}\; c_{\mathbf{P}_1}(-\mathbf{P}_2) \qquad . \tag{2.36}$$

a particular (simple) solution of which is given by

$$c_{\mathbf{P}}(\hat{\mathbf{P}}) = e^{i\pi p^T\hat{w}}. \tag{2.37}$$

It takes its values from $\{-1,1\}$. However it is not difficult to completely analyze the condition which arises for the quotient of any pair of consistent cocycle operators from (2.36). Exploiting some of the freedom in solving (2.36), permits us to obtain suitably normalized two–point functions.

In order to evaluate a general M–point function we start with the simple gaussian correlator

$$\langle \prod_{k=1}^{M} E(P_{kR},P_{kL};z_k,\bar{z}_k)\rangle \;=\; \prod_{i<j}^{M} z_{ij}^{P_{iR}^T P_{jR}}\, \bar{z}_{ij}^{P_{iL}^T P_{jL}}\;\; \delta_{\mathbf{0},\sum_{k=1}^{N}\mathbf{P}_k} \tag{2.38}$$

[3]We abbreviate (as usual) $z_{ij} := z_i - z_j$, $\bar{z}_{ij} = \bar{z}_i - \bar{z}_j$.

for radially ordered products of normal ordered exponentials of free fields. Taking the cocycle operators into account then yields

$$\langle \prod_{k=1}^{M} V_{\mathbf{P}_k}(z_k,\bar{z}_k) \rangle = \prod_{i<j}^{M} c_{\mathbf{P}_j}(-\mathbf{P}_i)\; z_{ij}^{P_{iR}^T P_{jR}}\; \bar{z}_{ij}^{P_{iL}^T P_{jL}}\; \delta_{\mathbf{0},\sum_{k=1}^{N}\mathbf{P}_k} \quad . \tag{2.39}$$

(Notice that the scaling exponents again refer to the choice of $2G = \mathbf{1}$.)

Further correlation functions can be derived from this by repeatedly applying partial derivatives w.r.t. the punctures z_k. For the two–point function one gets

$$\langle V_{\mathbf{P}}(z_1,\bar{z}_1)V_{-\mathbf{P}}(z_2,\bar{z}_2) \rangle = c_{\mathbf{P}}(\mathbf{P})\, z_{12}^{-P_R^2}\; \bar{z}_{12}^{-P_L^2} \quad . \tag{2.40}$$

Following the usual conventions we would prefer $c_{\mathbf{P}}(\mathbf{P}) = 1$. Consequently we modify (2.37), replacing it by

$$c_{\mathbf{P}}(\hat{\mathbf{P}}) = e^{\frac{1}{2}\pi i(p^T\hat{w} - w^T\hat{p})}, \tag{2.41}$$

which is yet another solution of (2.36). Clearly, a redefinition of $V_{\mathbf{P}}$ by a local c–number phase factor $\phi(\mathbf{P})$ cannot spoil the commutation properties with other vertex operators at all. Such a vertex "renormalization" affects however the expansion coefficients in (2.35). If we absorb $\phi(\mathbf{P})$ into the original cocycle operator, then the bimultiplicative property of the latter will be ruined in general. We might for instance repalce (2.37) by

$$c_{\mathbf{P}}(\hat{\mathbf{P}}) = e^{i\pi p^T\hat{w}}\; e^{-i\frac{\pi}{2}p^T w} \quad , \tag{2.42}$$

as suggested in [24]. In addition, the same author already showed how to incorporate (2.42) into the naive zero mode exponential (2.34) with the help of the Baker–Campbell–Hausdorff formula (BCH formula for short). This will effectively cause an operator shift of $\hat{x}$, namely $\hat{x} \mapsto \hat{x} + \pi\hat{w}$. Notice that (2.42) again guarantees a properly normalized two–point function.

As announced in section 2.1, we infer from (2.39) and (2.41), that all interactions of the bosonic Narain model are indeed invariant w.r.t. a change of the coordinate system, if an invariant cocycle operator is employed at the same time (as in (2.41)).

Moreover we recognize that this M–point correlation function is *form invariant* under $SO(d)_R \times SO(d)_L$ transformations $\vartheta = (\vartheta_R, \vartheta_L)$. Whereas the scaling exponents are manifestly invariant the cocycle operator phases will always contain products in their exponent which mix the left– and right–mover's momenta. To have an example, we display explicitly the dependence of (2.37) on its Narain momentum arguments:

$$c_{\mathbf{P}}(\mathbf{Q}) = \exp[\frac{i\pi}{2}(Q_R - Q_L)^T(P_R + P_L + B(P_R - P_L))] \quad . \tag{2.43}$$

The presence of left–right mixing products forces us to resort to the new cocycle operator

$$c_{\mathbf{P}}(\hat{\mathbf{P}};\vartheta) \ := \ c_{\vartheta^{-1}\mathbf{P}}(\vartheta^{-1}\hat{\mathbf{P}}) \tag{2.44}$$

when a left–right asymmetric rotation ϑ takes place (cf. also section 2 in [22]).

3 Twisted Narain models

3.1 Twisting the Narain lattice

Toroidal compactification of the string target space is by far the simplest way to obtain lower dimensional string vacua. But starting from the heterotic string theory [1], one faces the clear disadvantage of having a $N = 4$ space–time supersymmetry in the four–dimensional effective action. A construction by which this obstacle can be circumvented is described in [8]. Instead of dividing $\mathbb{R}^d$ by a *group* Λ_d of discrete translations one extends Λ_d to a finite subgroup S_d of the Euclidean group of $\mathbb{R}^d$, known as the *space group.* Its elements are of the form $g = (\Theta, w)$ where Θ is taken from a suitable finite subgroup of $SO(d)$ and $w \in \Lambda_d$. g acts on the coordinates X^μ of $\mathbb{R}^d$ by[4]

$$gX := \Theta X + 2\pi w \quad . \tag{3.1}$$

[4]The representation of the *twist* w.r.t the coordinate system will be denoted by Θ throughout. Its integer-valued representations Q, M w.r.t. to the lattice bases e, e^* are introduced below.

Modding out by S_d we obtain a (toroidal) *orbifold*

$$\Omega_d := 2\pi \left(\frac{\mathbb{R}^d}{S_d}\right) = \frac{T_d}{P} \qquad \left(P := \frac{S_d}{\Lambda_d}\right) \quad . \tag{3.2}$$

If P does not act freely on T_d, this quotient space is no longer a manifold, since it will then contain conical singularities located at the fixed points of Θ's action on T_d.

Thus gX and X are identified for all $g \in S_d$. In the case of a closed string with target space Ω_d, the coordinate fields always comply with a boundary condition

$$X(\tau, \sigma + 2\pi) = gX(\tau, \sigma). \tag{3.3}$$

One encounters three different types of closed strings on orbifolds:

(i) Strings which are already closed before $\mathbb{R}^d$ is compactified

(ii) *Winding* strings which are not yet closed in $\mathbb{R}^d$ but on T_d

(iii) *Twisted* strings which close only on Ω_d.

The first two cases amount to the *untwisted sector* whereas the last case yields the *twisted sectors* of the CFT which are further distinguished by the point group entry Θ.

One might wonder, if it suffices to exclusively consider primaries of the untwisted sector of the orbifold, which result from the vertices (2.8) of the Narain model by a projection onto twist–invariant fields. From a careful analysis of the partition function one learns however, that the twisted sectors should be included [8, 12], if (worldsheet) modular invariance is to survive the orbifold construction.

Throughout this paper only *symmetric* orbifolds with point group $P = Z_N$ are to be considered, that is to say, the twist treats $X_R(z)$, $X_L(\bar{z})$ in exactly the same way. We mostly assume that $\det(\Theta - \mathbf{1}) \neq 0$. The situation where fixed tori appear together with a *general* axionic background B is closely connected to those heterotic orbifolds whose twist is embedded via a shift in the gauge group's lattice and which possess a non–trivial Wilson line background. Their analysis is deferred to [27].

For Θ to be an isometry of T_d, it must act as an automorphism of Λ_d:

$$\Theta^{\mu}{}_{\nu} e^{\nu}{}_{i} = e^{\mu}{}_{j} Q^{j}{}_{i}\,; \qquad Q \in SL(d, \mathbb{Z}) \quad . \tag{3.4}$$

It is worthwhile to introduce complex coordinates

$$\begin{array}{rcl} \mathcal{X}^{\mu}: & = & \frac{1}{\sqrt{2}}(X^{2\mu-1} + iX^{2\mu}) \\ \bar{\mathcal{X}}^{\mu}: & = & \frac{1}{\sqrt{2}}(X^{2\mu-1} - iX^{2\mu})\,; \qquad \mu \in \{1, \ldots, d/2\} \quad , \end{array} \tag{3.5}$$

after having block–diagonalized the twist Θ:[5]

$$\Theta \mapsto R^T \Theta R = \begin{pmatrix} \Theta_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \Theta_{d/2} \end{pmatrix}; \quad \Theta_{\mu} = \begin{pmatrix} \cos\varphi_{\mu} & -\sin\varphi_{\mu} \\ \sin\varphi_{\mu} & \cos\varphi_{\mu} \end{pmatrix} \quad . \tag{3.6}$$

Obviously the planar rotation angles $\varphi_{\mu} := 2\pi k_{\mu}$ are quantized: $Nk_{\mu} \in \{1, \ldots, N-1\}$.

The closed string condition (3.3) with $g = (\Theta, f)$ then reads (abbreviate $\omega = \exp(2\pi i/N)$)

$$\begin{array}{rcl} \mathcal{X}^{\mu}(e^{2\pi i}z, e^{-2\pi i}\bar{z}) & = & \omega^{Nk_{\mu}}\, \mathcal{X}^{\mu}(z, \bar{z}) + 2\pi f^{\mu} \\ \bar{\mathcal{X}}^{\mu}(e^{2\pi i}z, e^{-2\pi i}\bar{z}) & = & \omega^{-Nk_{\mu}} \bar{\mathcal{X}}^{\mu}(z, \bar{z}) + 2\pi \bar{f}^{\mu} \quad . \end{array} \tag{3.7}$$

(We kept the symbol f for the complex translation vector as well.) It follows that the solution $\mathcal{X}^{\mu}$ of the free wave equation $\partial_z \partial_{\bar{z}} \mathcal{X}^{\mu}(z, \bar{z}) = 0$ and its complex conjugate $\bar{\mathcal{X}}^{\mu}$ possess Laurent expansions

$$\mathcal{X}^{\mu}(z, \bar{z}) = x_f^{\mu} + i \sum_{n=-\infty}^{\infty} \left[\frac{\alpha^{\mu}_{n-k_{\mu}}}{n - k_{\mu}}\, z^{-(n-k_{\mu})} + \frac{\tilde{\alpha}^{\mu}_{n+k_{\mu}}}{n + k_{\mu}}\, \bar{z}^{-(n+k_{\mu})} \right] , \tag{3.8}$$

$$\bar{\mathcal{X}}^{\mu}(z, \bar{z}) = \bar{x}_f^{\mu} + i \sum_{n=-\infty}^{\infty} \left[\frac{\bar{\alpha}^{\mu}_{n+k_{\mu}}}{n + k_{\mu}}\, z^{-(n+k_{\mu})} + \frac{\bar{\tilde{\alpha}}^{\mu}_{n-k_{\mu}}}{n - k_{\mu}}\, \bar{z}^{-(n-k_{\mu})} \right] , \tag{3.9}$$

where x_f denotes a complex fixed point under Θ on the torus. Within the real coordinate system x_f must satisfy

[5]Notice, that this coordinate transformation does not destroy the convention $2G = \mathbf{1}$, since each element of $O(d)$ can be block–diagonalized by a basis change $R \in SO(d)$.

$$(\mathbf{1}-\Theta)x_f = 2\pi f \in 2\pi\Lambda_d \qquad . \tag{3.10}$$

To be more precise, the fixed point actually serves as a representative of a *coset*, since on T_d we do not distinguish between x_f and $x_f + 2\pi w$ ($w \in \Lambda_d$):

$$x_f \in \frac{2\pi}{\mathbf{1}-\Theta}[f]; \quad [f] := f + (\mathbf{1}-\Theta)\Lambda_d \qquad . \tag{3.11}$$

In order to obtain twist invariant vertex operators one simply has to take the average of the exponentials (2.8) over an entire Θ–orbit (cf. section 3.2). Such a sum can only be meaningful, if Θ extends to an automorphism of $\Lambda_{\mathcal{N}}$:

$$\Theta\mathbf{P} \equiv (\Theta P_R\,;\Theta P_L) \in \Lambda_{\mathcal{N}} \tag{3.12}$$

This severely restricts the background couplings G and B [12]. The action of Θ on the basis vectors (2.18) of $\Lambda_{\mathcal{N}}$ must take the form

$$\begin{aligned} \Theta k^i &= k^j M_j{}^i + \bar{k}_j \tilde{Q}^{ji} \\ \Theta \bar{k}_i &= k^j \tilde{M}_{ji} + \bar{k}_j Q^j{}_i \qquad , \end{aligned} \tag{3.13}$$

where M, $\tilde{M}$, Q and $\tilde{Q}$ are integer–valued quadratic matrices. In the case of a left–right symmetric twist $\tilde{Q}$ has to vanish, since the $\bar{k}_j$ are left–right asymmetric, whereas the k^i are symmetric.

Furthermore we recognize, that left–right symmetric orbifolds can always be looked upon as twisted toroidal compactifications, the twist action being given by

$$(G^{-1}\Theta G)e = eQ \tag{3.14}$$

$$\Theta e^* = e^* M. \tag{3.15}$$

Using the explicit expressions (2.18) one easily derives the restrictions

$$Mg = gQ \qquad \text{where } M = (Q^{-1})^T \tag{3.16}$$

$$\tilde{M} = bQ - Mb \tag{3.17}$$

on the background couplings g and b. (3.16) follows of course directly from (3.14).

In order to solve the condition on b it is advisable to recast it slightly:

$$Q^T bQ - b \;=\; R \in M(d \times d; \mathbb{Z}) \tag{3.18}$$

Here the integer, antisymmetric matrix $R = Q^T \tilde{M}$ plays the role of an inhomogeneous term in a system of linear equations for the $\binom{d}{2}$ a priori independent components of the antisymmetric matrix b.

We list a number of properties of the solution set. Clearly, every integer axionic tensor b solves (3.18). If R vanishes then the solutions form a continuous linear vector space $\mathcal{S}$. For a given $R \neq 0$ we either find no solution or there is an affine solution space of the form $b_R + \mathcal{S}$ in which b_R may stand for any special solution. Since the system of equations for the components b_{ij} contains exclusively integer coefficients we can always find a solution b with only rational components, should there exist any solution at all.

To further elucidate the latter statement we even construct such a solution explicitly. From (3.18) it follows (with the help of $Q^N = \mathbf{1}$) that

$$R + Q^T RQ + \ldots + (Q^T)^{N-1} RQ^{N-1} = 0 \tag{3.19}$$

which is a necessary condition for the solution space to be non–empty. It suggests to probe for a solution relying on the ansatz

$$b_R = \sum_{j=0}^{N-2} \lambda_j (Q^T)^j RQ^j \quad . \tag{3.20}$$

Indeed by choosing

$$\lambda_j = \frac{j+1}{N} - 1 \tag{3.21}$$

we can fulfill (3.18). Now, we recognize that (3.19) already is a sufficient condition for having a solution b_R. As is evident from (3.21), there always exist rational representatives b_R whose components are multiples of N^{-1}.

However, we have so far been unable to devise general criteria which guarantee the existence of *integer* representatives b_R such that

$$b = b_R + b_0 \; ; \qquad b_0 \in \mathcal{S} \tag{3.22}$$

comes about. In the coordinate basis this is then reflected by the analogous decomposition

$$B = \Delta + B_0 \tag{3.23}$$

where $\Delta = e^* b_R e^{*T}$ and $B_0 = e^* b_0 e^{*T}$ commutes with the twist Θ. The discussion will be continued in appendix A where we study several examples in order to find out which cases permit this particular splitting at all. In the forthcoming sections we confine our interest to those orbifolds which indeed happen to have an integer background representative b_R.

A complementary way to handle (3.18) opens, if we choose a (complex) basis matrix δ which diagonalizes Q. Then

$$Q\delta \; = \; \delta D \tag{3.24}$$

in which the diagonal matrix D contains the eigenvalues $D_{rr} = \omega^{t_r}$ of Θ. Denoting then the δ–basis counterparts of b and R by β and ϱ, respectively, we obtain the relation

$$\beta_{rs}(\omega^{t_s - t_r} - 1) = \varrho_{rs} \qquad . \tag{3.25}$$

Obviously $\varrho_{rs} = 0$ whenever $t_r = t_s$. This constraint is just the condition (3.19) given above. In addition, we notice that the associated component β_{rs} describes a *modulus* (i.e. a continuous degree of freedom in the background space).

3.2 The Hilbert space of states

String theories or rather the associated CFTs can be quantized in two different ways: Canonically or by path integrals. The latter approach prerequisites an action to define the functional integration measure, which in our context is given by (2.13). The classical equations of motion are obtained by varying this action w.r.t. the fields X^μ and demanding this variation to vanish. Proceeding in this way one encounters surface integrals, which must vanish separately for all possible boundary conditions of the fields.

Using $g = (\Theta, 0)$ the variation of (2.13) gives rise to the surface term

$$\int_{\tau_0}^{\tau_1} d\tau \; \partial_\tau X^T(\tau, 0) \left[\Theta^T B \Theta - B\right] \delta X(\tau, 0) \qquad , \tag{3.26}$$

which vanishes for arbitrary variations δX if and only if $[\Theta, B] = 0$. Furthermore the action (2.13) is twist invariant only if this condition is obeyed.

This purely *classical* constraint on B, which has to be satisfied in the path integral approach, is much more restrictive than (3.17). Hence, to have the most general B at our disposal, the *operator approach* has to be selected. It treats the set of conformal fields as an algebra which closes under operator product expansions [13]. There is no more any necessity to refer to an action.

As has been stated in section 3.1, the string Fock space encompasses different sectors, which are interrelated by the modular group and through interactions. We will now revisit the canonical quantization in these sectors, since it is a prerequisite for the construction of cocycle operators.

3.2.1 The untwisted sector

The primary fields $V_{\mathbf{P}}^{\text{inv}}(z,\bar{z})$ of the untwisted sector are expressed in terms of the vertices (2.8) of exponential type[6] such that they do not change under $z \mapsto e^{2\pi i}z$, $\bar{z} \mapsto e^{-2\pi i}\bar{z}$ whatever boundary condition (3.3) is applied. In section 2.2 we have already seen that lattice translations leave (2.8) invariant. Furthermore, the superposition

$$V_{\mathbf{P}}^{\text{inv}}(z,\bar{z}) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} c_{\Theta^k \mathbf{P}}(\hat{\mathbf{P}})\, E(\Theta^k P_R, \Theta^k P_L; z, \bar{z}) \tag{3.27}$$

is a Θ–singlet (i.e., it is invariant under $X_{R/L} \mapsto \Theta X_{R/L}$, $\hat{P}_{R/L} \mapsto \Theta \hat{P}_{R/L}$) provided that the cocycle operator satisfies

$$c_{\Theta \mathbf{P}}(\Theta \hat{\mathbf{P}}) = c_{\mathbf{P}}(\hat{\mathbf{P}}) \quad . \tag{3.28}$$

Given this property the operator algebra generated by (3.27) closes as well:

$$\begin{aligned} V_{\mathbf{P}_1}^{\text{inv}}(z_1,\bar{z}_1)\, V_{\mathbf{P}_2}^{\text{inv}}(z_2,\bar{z}_2) &= \frac{1}{\sqrt{N}} \sum_{m=1}^{N} z_{12}^{P_{2R}^T \Theta^m P_{1R}}\, \bar{z}_{12}^{P_{2L}^T \Theta^m P_{1L}} \\ &\times\; c_{\mathbf{P}_2}(-\Theta^m \mathbf{P}_1)\, V_{\Theta^m \mathbf{P}_1 + \mathbf{P}_2}^{\text{inv}}(z_2,\bar{z}_2) + \ldots \quad . \end{aligned} \tag{3.29}$$

Hence the Narain model fusion rule $[V_{\mathbf{P}_1}] \times [V_{\mathbf{P}_2}] = [V_{\mathbf{P}_1+\mathbf{P}_2}]$ is generalized to

[6] The partial derivatives of $\mathcal{X}^\mu$ and $\bar{\mathcal{X}}^\mu$ w.r.t. the world sheet punctures z, $\bar{z}$ cannot be extended to twist invariant operators, since $1 + \Theta + \ldots + \Theta^{N-1} = 0$.

$$[V_{\mathbf{P}_1}^{\mathrm{inv}}] \times [V_{\mathbf{P}_2}^{\mathrm{inv}}] = \sum_{m=1}^{N} [V_{\Theta^m \mathbf{P}_1 + \mathbf{P}_2}^{\mathrm{inv}}] \quad . \tag{3.30}$$

Correspondingly the selection rule is weaker than the one of the previous section.

The cocycle operator (2.41) violates the requirement (3.28), since Θ transforms $\hat{p}$ into $\hat{p}' := \Theta\hat{p} + [B, \Theta]\hat{w}$ and $\hat{w}$ into $\hat{w}' := \Theta\hat{w}$ as follows from (2.29). Yet given (3.23) we can construct an improved cocycle operator in conformity with (3.28):

$$c_{\mathbf{P}}(\hat{\mathbf{P}}) = \exp\left[\frac{i}{2}\pi\{w^T(\hat{p} - \Delta\hat{w}) - (p - \Delta w)^T\hat{w}\}\right] \quad . \tag{3.31}$$

Likewise (2.42) which was proposed in [24] satisfies (3.28) as soon as the "minimal" substitution $p \mapsto p - \Delta w$ has taken effect. In both cases we profited from the assumption that $e^T\Delta e$ be an integer matrix. Otherwise, we are forced to extend these cocycle operators such that they continue to solve (2.36) (cf. [27]).

3.2.2 Twisted sectors

For the time being we will concentrate on the first twisted sector; higher twisted sectors will be dealt with later.

In analogy to the spin field in the Ramond–Neveu–Schwarz model the non-trivial boundary conditions (3.7) are imposed by the bosonic *twist fields* $\sigma_f^+(z, \bar{z})$ in an orbifold CFT. They are associated to the *conjugacy classes* $(\Theta, [f])$ of the space group, i.e. they prescribe elements from this class as a global monodromy of the coordinates $X(z, \bar{z})$. Their quantum properties are determined by the basic expansions

$$\begin{aligned}
\partial\mathcal{X}^\mu(z_1, \bar{z}_1)\,\sigma_f^+(z_2, \bar{z}_2) &= z_{12}^{-(1-k_\mu)}\,\tau_f^{+\mu}(z_2, \bar{z}_2) + \dots \\
\partial\bar{\mathcal{X}}^\mu(z_1, \bar{z}_1)\,\sigma_f^+(z_2, \bar{z}_2) &= z_{12}^{-k_\mu}\quad \tau_f'^{+\mu}(z_2, \bar{z}_2) + \dots \\
\bar{\partial}\mathcal{X}^\mu(z_1, \bar{z}_1)\,\sigma_f^+(z_2, \bar{z}_2) &= \bar{z}_{12}^{-k_\mu}\quad \tilde{\tau}_f'^{+\mu}(z_2, \bar{z}_2) + \dots \\
\bar{\partial}\bar{\mathcal{X}}^\mu(z_1, \bar{z}_1)\,\sigma_f^+(z_2, \bar{z}_2) &= \bar{z}_{12}^{-(1-k_\mu)}\,\tilde{\tau}_f^{+\mu}(z_2, \bar{z}_2) + \dots \quad ,
\end{aligned} \tag{3.32}$$

where four *excited twist fields* $\tau_f^{+\mu}$, $\tau_f'^{+\mu}$, $\tilde{\tau}_f'^{+\mu}$ and $\tilde{\tau}_f^{+\mu}$ appear on the right hand side. Similar to (2.26) the Laurent expansion coefficients in (3.8) and

(3.9) satisfy canonical commutation relations:

$$[\hat{\alpha}^{\mu}_{m+k_\mu}, \hat{\alpha}^{\nu}_{n-k_\nu}] = [\hat{\tilde{\alpha}}^{\mu}_{m+k_\mu}, \hat{\tilde{\alpha}}^{\nu}_{n-k_\nu}] = (m + k_\mu)\delta^{\mu\nu}\delta_{m+n,0} \quad . \tag{3.33}$$

By definition, the *twisted sector ground states* $|\sigma^+_f\rangle := \sigma^+_f(0,0)|0\rangle$ are annihilated by all positive frequency mode operators:[7]

$$\begin{array}{rclcl} \hat{\alpha}^{\mu}_{n-k_\mu}|\sigma^+_f\rangle & = & \hat{\tilde{\alpha}}^{\mu}_{n-k_\mu}|\sigma^+_f\rangle & = 0\,; & n > 0 \\ \hat{\tilde{\alpha}}^{\mu}_{n+k_\mu}|\sigma^+_f\rangle & = & \hat{\alpha}^{\mu}_{n+k_\mu}|\sigma^+_f\rangle & = 0\,; & n \geq 0 \end{array} \tag{3.34}$$

These states are eigenstates of the unitary operators $\exp(ip^T\hat{x})$. Note that one cannot properly characterize $|\sigma^+_f\rangle$ as an eigenstate of $\hat{x}$ itself, because any meaningful operator depending on $\hat{x}$ must be stable under $\hat{x} \mapsto \hat{x} + 2\pi\lambda$ ($\lambda \in \Lambda_d$). This condition will in the sequel be referred to as *coset stability.*

The zero mode structure of this sector is somewhat intricate compared to the untwisted sector, owing to the vector

$$\vec{c} = \begin{pmatrix} (\mathbf{1} - \Theta)\hat{x} - 2\pi\hat{w} \\ \hat{p} \end{pmatrix} \approx 0 \tag{3.35}$$

of second class constraints, as is pointed out in [23]. In this case the quantization is provided by the *Dirac* method. Given two arbitrary operators $\mathcal{A}$, $\mathcal{B}$ their Dirac commutator is expressed as follows in terms of ordinary (canonical) commutators:

$$[\mathcal{A}, \mathcal{B}]_{\mathrm{D}} := [\mathcal{A}, \mathcal{B}] - [\mathcal{A}, \vec{c}^{\,T}]\, \frac{1}{[\vec{c}, \vec{c}^{\,T}]}\, [\vec{c}, \mathcal{B}] \quad . \tag{3.36}$$

It is then evident that Dirac commutators containing (components of) the constraint vector $\vec{c}$ must vanish. We also learn that these commutators will not be affected by an arbitrary linear (though invertible) redefinition of the constraints. In the sequel we will simply suppress the index D.

The zero mode commutators for the twisted sector can now be straightforwardly derived from (3.35)[8] (compare with [23]):

[7]According to (3.32), acting with the (fractional) oscillators $\hat{\alpha}^{\mu}_{-k_\mu}$, $\hat{\tilde{\alpha}}^{\mu}_{k_\mu-1}$, $\hat{\tilde{\alpha}}^{\mu}_{k_\mu-1}$ and $\hat{\tilde{\alpha}}^{\mu}_{-k_\mu}$ on $|\sigma^+_f\rangle$ results in the excited states $\tau^{+\mu}_f(0,0)|0\rangle$, $\tau'^{+\mu}_f(0,0)|0\rangle$, $\tilde{\tau}'^{+\mu}_f(0,0)|0\rangle$ and $\tilde{\tau}^{+\mu}_f(0,0)|0\rangle$, respectively.

[8]Strictly speaking the position–like operators $\hat{x}$ and $\hat{q}$ cannot serve as generators of a *continuous* symmetry group, since the target space of our theory is compact.

$$\begin{array}{rcl} [\hat{x}, \hat{p}^T] & = & 0 \\ [\hat{q}, \hat{w}^T] & = & i \cdot \mathbf{1} \\ [\hat{x}, \hat{q}^T] & = & -2\pi i \, \frac{1}{1-\Theta} \end{array} \quad . \tag{3.37}$$

Therefore, the former generator $\hat{p}$ of spatial translations gets effectively replaced by

$$\hat{r} := -\frac{1}{2\pi}(\mathbf{1} - \Theta^T)\hat{q} \quad , \tag{3.38}$$

since $[\hat{x}, \hat{r}^T] = i \cdot \mathbf{1}$ holds.

To be well defined, the spatial translation operator $e^{is^T\hat{r}}$ has to be single valued under $s \mapsto s + 2\pi w$ ($w \in \Lambda_d$). In other words, $\exp(2\pi i w^T \hat{r})$ must act as the identity operator.

For a while we concentrate on those axionic backgrounds $B_0 = e^* b_0 e^{*T}$ which commute with the twist Θ. It will be demonstrated in the next section (cf. (4.12)), that due to the algebra (3.37) the introduction of a genuine cocycle operator for a string emission vertex can be avoided. The generalization to axionic backgrounds of the type (3.23) will be provided afterwards.

The transformation from the dynamical variables $(\hat{x}, \hat{q})$ to $(\hat{x}_R, \hat{x}_L)$ does of course not depend on the sector being a twisted or an untwisted one (see (2.33)):

$$\hat{x}_{R/L} = \frac{1}{2}(\mathbf{1} \mp \frac{1}{G}B_0)\hat{x} \pm \frac{1}{2G}\hat{q} \quad . \tag{3.39}$$

The twisted sector zero mode part of $E(P_R, P_L; z, \bar{z})$ thereby turns out to be

$$e^{i(p^T\hat{x} + w^T\hat{q})} = e^{\pi i p^T \frac{1}{1-\Theta} w} \, e^{-2\pi i w^T \frac{1}{1-\Theta^T}\hat{r}} \, e^{ip^T\hat{x}}, \tag{3.40}$$

which is both twist invariant in the sense of (3.28) and stable on a coset $[f]$. Its local c-number phase factor is due to a (naive) application of the BCH–formula. The ground states $|\sigma_f^+\rangle$ are nontrivially affected by this operator:

$$e^{iw^T\hat{q}} \, |\sigma_f^+\rangle = e^{-2\pi i w^T \frac{1}{1-\Theta^T}\hat{r}} \, |\sigma_f^+\rangle = |\sigma_{f+w}^+\rangle \tag{3.41}$$

Hence the absorption of an untwisted string by a twisted one causes the fixed point $x_f = 2\pi(\mathbf{1} - \Theta)^{-1} f$ (the center of mass of the twisted string) to shift by $2\pi(\mathbf{1}-\Theta)^{-1}w$ where w is the winding vector of the absorbed closed untwisted string. The zero mode exponentials (3.40) form a representation of a Weyl–Heisenberg group in the orbifold's first twisted sector.

Vertex operators for the emission of untwisted states from twisted ones are again constructed according to (3.27). However, as we have just established, an extra cocycle operator is superfluous. Obviously for their twisted sector representation the mode expansions (3.8) and (3.9) have to be inserted. Furthermore the string coupling constant g' for this process differs from that of the untwisted sector and might even acquire a dependence on the Narain momentum (cf. [17] and section 4). Altogether the emission of an (unexcited) untwisted string from a twisted one is described by

$$V_{\mathbf{P}}^{\mathrm{inv}}(z,\bar{z}) = \frac{1}{\sqrt{N}}\sum_{k=1}^{N} g'(\Theta^k\mathbf{P})\, z^{-\frac{1}{2}P_R^2}\, \bar{z}^{-\frac{1}{2}P_L^2}\; E(\Theta^k P_R, \Theta^k P_L; z, \bar{z}) \qquad . \tag{3.42}$$

The new factors $z^{-\frac{1}{2}P_R^2}$ $(\bar{z}^{-\frac{1}{2}P_L^2})$ ensure that $V_{\mathbf{P}}^{\mathrm{inv}}$ remains a conformal field with dimension $h=\frac{1}{2}P_R^2$ $(\bar{h}=\frac{1}{2}P_L^2)$ under the respective Virasoro algebras. Finally, the requirement of twist invariance boils down to

$$g'(\Theta\mathbf{P}) = g'(\mathbf{P}) \qquad , \tag{3.43}$$

which is also urgently needed for the closure of the algebra (3.29) in the twisted sector representation.

4 The twisted sector string coupling

The type of non–trivial correlation functions which is by far the simplest to treat involves only two twist fields and an arbitrary number of exponential vertices. Their evaluation does not require an explicit construction of the twist field vertex operator, since the emission of an untwisted state does not impinge on the original twisted vacuum sector. In the operator approach one merely sandwiches the untwisted sector vertex operators between twisted incoming and outgoing *states*.

We assign the space group conjugacy classes $(\Theta,[f_2])$, $(\Theta^{-1},[-f_1])$ to $\sigma_{f_2}^{+}$, $\sigma_{-f_1}^{-}$. On the other hand the global monodromy about the worldsheet location of $V_{\mathbf{P}_i}^{\mathrm{inv}}$ is given by $(1,\mathcal{O}_{w_i})$ $(w_i = P_{Ri}-P_{Li})$ where

$$\mathcal{O}_v := \{\Theta^x v\,; 0 \le x \le N-1\} \tag{4.1}$$

is the *orbit* of the lattice vector v under the twist action. The *space group selection rule* [15, 17] then demands that the product of all classes which are associated to the fields involved contains the unit element $(1,0)$.

In order to determine the twisted sector string coupling $g'(\mathbf{P})$ we study

$$Z_{\mathbf{P}_1,\mathbf{P}_2}(x,\bar{x};y,\bar{y}) := \langle \sigma^-_{-f_1}| V_{\mathbf{P}_1}(x,\bar{x}) V_{\mathbf{P}_2}(y,\bar{y}) |\sigma^+_{f_2}\rangle \tag{4.2}$$

$$\equiv \lim_{z_\infty,\bar{z}_\infty \to \infty} |z_\infty|^{4h_\sigma} \langle \sigma^-_{-f_1}(z_\infty,\bar{z}_\infty) V_{\mathbf{P}_1}(x,\bar{x}) V_{\mathbf{P}_2}(y,\bar{y})\, \sigma^+_{f_2}(0,0)\rangle \quad ,$$

where[9]

$$V_{\mathbf{P}}(z,\bar{z}) = g'(\mathbf{P}) z^{-\frac{1}{2}P_R^2} \bar{z}^{-\frac{1}{2}P_L^2}\; E(P_R,P_L;z,\bar{z}) \quad . \tag{4.3}$$

The conformal and the anti–conformal dimension of the twist fields $\sigma^+_{f_2}$, $\sigma^-_{-f_1}$ are given by

$$h_\sigma = \bar{h}_\sigma = \frac{1}{2}\sum_{\mu=1}^{d/2} k'_\mu(1-k'_\mu)\,; \quad k'_\mu \equiv (k_\mu \bmod 1) \in \{\frac{1}{N},\ldots,1-\frac{1}{N}\}. \tag{4.4}$$

The space group selection rule for this amplitude reduces to

$$[0] = [f_2] + w_1 + w_2 - [f_1] \quad , \tag{4.5}$$

because of $\mathcal{O}_{\Theta v} = \mathcal{O}_v$.

Using (3.40) to account for the zero mode part of (4.3), it is straightforward to obtain

$$Z_{\mathbf{P}_1,\mathbf{P}_2}(x,\bar{x};y,\bar{y}) = g'(\mathbf{P}_1)\, g'(\mathbf{P}_2)\; e^{i\pi\{p_1^T \frac{1}{1-\Theta} w_1 + p_2^T \frac{1}{1-\Theta} w_2\}}$$

$$\times e^{2\pi i\{p_2^T \frac{1}{1-\Theta} f_2 + p_1^T \frac{1}{1-\Theta}(w_2+f_2)\}}\; x^{-\frac{1}{2}P_{1R}^2}\; y^{-\frac{1}{2}P_{2R}^2}\; \bar{x}^{-\frac{1}{2}P_{1L}^2}\; \bar{y}^{-\frac{1}{2}P_{2L}^2}$$

$$\times e^{-\mathcal{P}_{1R}^T \langle \bar{\mathcal{X}}_R(x) \mathcal{X}_R^T(y)\rangle_t \bar{\mathcal{P}}_{2R} \;-\; \bar{\mathcal{P}}_{1R}^T \langle \mathcal{X}_R(x) \bar{\mathcal{X}}_R^T(y)\rangle_t \mathcal{P}_{2R}}$$

[9]We have refrained here from using the lenghty physical twist invariant vertices (3.42). The final result (4.18) is not affected by this shortcut.

$$\times\, e^{-\mathcal{P}^T_{1L}\langle\bar{\mathcal{X}}_L(\bar{x})\mathcal{X}^T_L(\bar{y})\rangle_t\bar{\mathcal{P}}_{2L} - \bar{\mathcal{P}}^T_{1L}\langle\mathcal{X}_L(\bar{x})\bar{\mathcal{X}}^T_L(\bar{y})\rangle_t\mathcal{P}_{2L}} \quad . \tag{4.6}$$

We introduced the complex Narain momenta

$$\begin{aligned}\mathcal{P}^\mu_{R/L} &:= \tfrac{1}{\sqrt{2}}(P^{2\mu-1}_{R/L} + iP^{2\mu}_{R/L})\\ \bar{\mathcal{P}}^\mu_{R/L} &:= \tfrac{1}{\sqrt{2}}(P^{2\mu-1}_{R/L} - iP^{2\mu}_{R/L})\end{aligned} \tag{4.7}$$

and $\langle\ldots\rangle_t$ denotes a twisted sector correlation $\langle\sigma^-_{\mathrm{osc}}|\ldots|\sigma^+_{\mathrm{osc}}\rangle$ which is restricted to the oscillator modes.

The various two–point functions are evaluated in appendix B. Taking advantage of it we find

$$Q(\mathbf{P}_1,x,\bar{x};\mathbf{P}_2,y,\bar{y}) := \prod_{n=1}^{N}\left[1-\omega^{-n}\left(\frac{y}{x}\right)^{\frac{1}{N}}\right]^{P^T_{2R}\Theta^n P_{1R}}\left[1-\omega^{n}\left(\frac{\bar{y}}{\bar{x}}\right)^{\frac{1}{N}}\right]^{P^T_{2L}\Theta^n P_{1L}} . \tag{4.8}$$

for the contribution of oscillator modes to (4.2).

As argued in section 2.2 the correlation function has to satisfy the commutation property

$$Z_{\mathbf{P}_1,\mathbf{P}_2}(x,\bar{x};y,\bar{y}) = Z_{\mathbf{P}_2,\mathbf{P}_1}(y,\bar{y};x,\bar{x}) \quad . \tag{4.9}$$

With the help of the sums

$$\sum_{n=1}^{N}\Theta^n = 0, \quad \sum_{n=1}^{N} n\Theta^n = \frac{N}{1-\Theta^T} \tag{4.10}$$

one shows that

$$Q(\mathbf{P}_1,x,\bar{x};\mathbf{P}_2,y,\bar{y}) \;=\; C(\mathbf{P}_1,\mathbf{P}_2)\,Q(\mathbf{P}_2,y,\bar{y};\mathbf{P}_1,x,\bar{x}) \tag{4.11}$$

where

$$C(\mathbf{P}_1,\mathbf{P}_2) := \prod_{k=1}^{N}\omega^{k\langle\mathbf{P}_1\Theta^k\mathbf{P}_2\rangle} = e^{2\pi i\{p_2^T\frac{1}{1-\Theta}w_1 - p_1^T\frac{1}{1-\Theta}w_2\}} \quad . \tag{4.12}$$

The analogous factor in the context of a *chiral* CFT has first appeared in [28, 29]. $C(\mathbf{P}_1,\mathbf{P}_2)$ exactly cancels the phase stemming from the commutation of the zero mode parts which proves (4.9); as announced in section 3.2.2 there is indeed no obligation to invent a cocycle operator from scratch.

To allow for an arbitrary axionic background $B = B_0 + \Delta$ we rearrange

$$P_{R/L} = (p + \Delta w) \pm (G \mp (B_0 + \Delta))w =: p' \pm (G \mp B)w \quad (p' \in \Lambda_d^*). \quad (4.13)$$

The zero mode part of an exponential type vertex operator now becomes

$$e^{i\{p^T\hat{x}+w^T\hat{q}\}} = e^{i\pi(p'-\Delta w)^T\frac{1}{1-\Theta}w} e^{iw^T\hat{q}} e^{i(p'-\Delta w)^T\hat{x}} \quad . \quad (4.14)$$

Remarkably, the twist invariance and the stability on cosets $[f]$ of the vertex operator $V_{\mathbf{P}}^{\text{inv}}$ are not spoiled[10]. The phase produced upon commutation of two of these zero mode operators is the inverse of

$$C(\mathbf{P}_1, \mathbf{P}_2) = e^{2\pi i\{(p_2'-\Delta w_2)^T\frac{1}{1-\Theta}w_1 - (p_1'-\Delta w_1)^T\frac{1}{1-\Theta}w_2\}} \quad (4.15)$$

which generalizes (4.12). Having dropped the primes on p the four–point function (4.6) turns into

$$Z_{\mathbf{P}_1,\mathbf{P}_2}(x,\bar{x};y,\bar{y}) = g'(\mathbf{P}_1)\, g'(\mathbf{P}_2)\, e^{i\pi\{(p_1-\Delta w_1)^T\frac{1}{1-\Theta}w_1 + (p_2-\Delta w_2)^T\frac{1}{1-\Theta}w_2\}}$$

$$\times\, e^{2\pi i\{(p_2-\Delta w_2)^T\frac{1}{1-\Theta}f_2 + (p_1-\Delta w_1)^T\frac{1}{1-\Theta}(w_2+f_2)\}} \quad (4.16)$$

$$\times\, x^{-\frac{1}{2}P_{1R}^2}\, y^{-\frac{1}{2}P_{2R}^2}\, \bar{x}^{-\frac{1}{2}P_{1L}^2}\, \bar{y}^{-\frac{1}{2}P_{2L}^2}\, Q(\mathbf{P}_1,x,\bar{x};\mathbf{P}_2,y,\bar{y})$$

Applying crossing symmetry and taking the factorization limit $x \to y$, $\bar{x} \to \bar{y}$ allows one to solve for g', because once the OPE (2.35) has been inserted into (4.2) then it must agree with (4.16). Let us first simplify somewhat the non–singular part of the N–fold product in (4.8):

$$\prod_{n=1}^{N-1} \left[1-\omega^{-n}\right]^{P_{2R}^T\Theta^n P_{1R}} \left[1-\omega^n\right]^{P_{2L}^T\Theta^n P_{1L}} =$$

$$= e^{\frac{1}{2}\pi i\{p_1^T w_2 + p_2^T w_1\}} \prod_{n=0}^{N-1} \omega^{-\frac{1}{2}n\langle\mathbf{P}_2,\Theta^n\mathbf{P}_1\rangle} \prod_{n=1}^{N-1} |1-\omega^n|^{P_{2R}^T\Theta^n P_{1R} + P_{2L}^T\Theta^n P_{1L}} \quad .$$

[10] This "minimal" substitution need not be applied if $[\Delta,\Theta]=0$. For more details about this shift of an axionic modulus see section 5.3.

A short calculation then yields

$$g'(\mathbf{P}_1)\, g'(\mathbf{P}_2) \left(\frac{1}{N}\right)^{P_{2R}^T P_{1R} + P_{2L}^T P_{1L}} \prod_{n=1}^{N-1} |1-\omega^n|^{P_{2R}^T \Theta^n P_{1R} + P_{2L}^T \Theta^n P_{1L}} =$$

$$g'(\mathbf{P}_1+\mathbf{P}_2)\, c_{\mathbf{P}_2}(-\mathbf{P}_1)\, e^{-\frac{1}{2}\pi i\{p_1^T w_2 + p_2^T w_1\}}\, e^{-i\pi(p_1-\Delta w_1)^T w_2} \quad . \tag{4.17}$$

Because of (3.31) the phase factors cancel throughout. This enables us readily to extract g':

$$\begin{aligned} g'(\mathbf{P}) &= \left(\frac{1}{N}\right)^{\frac{1}{2}(P_R^2+P_L^2)} \prod_{n=1}^{N-1} |1-\omega^n|^{\frac{1}{2}(P_R^T\Theta^n P_R + P_L^T \Theta^n P_L)} \\ &= \prod_{\mu=1}^{d/2} \delta(k_\mu)^{-\frac{1}{2}(h^\mu+\bar{h}^\mu)} \end{aligned} \tag{4.18}$$

where

$$\begin{aligned} \delta(k_\mu) &= N^2 \prod_{n=1}^{N-1} (2\sin\tfrac{\pi n}{N})^{-2\cos(2\pi n k_\mu)} \\ &= \exp[2\psi(1) - \psi(k_\mu) - \psi(1-k_\mu)] \\ h^\mu &= |\mathcal{P}_R^\mu|^2, \qquad \bar{h}^\mu = |\mathcal{P}_L^\mu|^2 \quad . \end{aligned} \tag{4.19}$$

That the two different expressions offered for the twist-dependent quantity $\delta(k_\mu)$ are equal may be directly demonstrated, if one uses the well-known series representation of $\psi(z) = \frac{d}{dz}\ln\Gamma(z)$ together with a suitable regularization. Amazingly, an alternative proof is contributed by conformal field theory itself: The second form of the coupling strength $\delta(k_\mu)$ emerges also from an untwisted channel factorization of the four-twist field correlation function $\langle \sigma^-_{-f_1}\sigma^+_{f_2}\sigma^-_{-f_3}\sigma^+_{f_4}\rangle$ (see also [15], [17] where the case $B=0$ is treated). Furthermore, we recognize that (3.43) is obviously fulfilled.

Thus we have[11]

$$\begin{aligned} \langle \sigma^-_{-f_1}|V_{\mathbf{P}}^{\mathrm{inv}}(1,1)|\sigma^+_{f_2}\rangle &= \sqrt{N} g'(\mathbf{P})\, e^{i\pi(p-\Delta w)^T \frac{1}{1-\Theta}(2f_2+w)} \\ &\times\ \delta_{\mathbf{0},\,(f_1-f_2-w)\mathrm{mod}(1-\Theta)\Lambda} \quad . \end{aligned} \tag{4.20}$$

[11] $g'(\mathbf{P})$ is nothing else but the finite ratio of the divergent normal ordering factors which arise upon placing $\exp(iP_LX_L + iP_RX_R)_{\mathrm{osc}}$ into the twisted (untwisted) sector (cf. [17]). In fact, $-\ln\delta(k_\mu) = \langle \bar{\mathcal{X}}_L^\mu(1)\mathcal{X}_L^\mu(1)\rangle_t + \langle \mathcal{X}_L^\mu(1)\bar{\mathcal{X}}_L^\mu(1)\rangle_t$
$- \langle \bar{\mathcal{X}}_L^\mu(1)\mathcal{X}_L^\mu(1)\rangle - \langle \mathcal{X}_L^\mu(1)\bar{\mathcal{X}}_L^\mu(1)\rangle \;=\; (L \leftrightarrow R)$.

Alternatively, one may resort to the OPE

$$\sigma^{-}_{-f_1}(z_1,\bar{z}_1)\,\sigma^{+}_{f_2}(z_2,\bar{z}_2) \;=\; \sum_{w\in[f_2]-[f_1]}\;\sum_{p\in\Lambda_d^*}\frac{g'(\mathbf{P})}{\sqrt{N}}\;e^{-i\pi(p-\Delta w)^T\frac{1}{1-\Theta}(2f_2-w)} \times\; z_{12}^{h-2h_\sigma}\,\bar{z}_{12}^{\bar{h}-2h_\sigma}\;V^{\rm inv}_{\mathbf{P}}(z_2,\bar{z}_2)+\dots\;, \tag{4.21}$$

where a factor $1/N$ has been included in order to undo the N–fold degeneracy created by the orbit map $\mathbf{P}\to\Theta\mathbf{P}$. Furthermore the space group selection rule $w\in[f_2]-[f_1]$ was incorporated.

5 Discrete symmetries relate geometrically different orbifold CFTs

5.1 The background moduli

Every model belonging to the class of bosonic orbifold compactifications we have dealt with in this investigation is entirely specified by the background matrix $(g+b)$. Needless to say that several conditions still have to be met in order to arrive at a well–defined modding–out procedure of the underlying torus construction: The twist Θ is subject to (3.16) and (3.17). Temporarily assuming the integer matrix $\tilde{M}$ (which enters the second condition) vanishes, we are left with a pair of homogeneous systems of linear equations for the elements of g and b. The solution spaces for both cases therefore become $\mathbb{R}$–linear vector spaces whose coordinates are denoted as *metrical* and *axionic moduli*, respectively. However, generically we are given an arbitrary integer matrix $\tilde{M}$ in the crystallographic conditions (3.13).

The validity of the main results obtained in section 3.2 and section 4 hinges on there being available an integer representative b_R for $\tilde{M}\neq 0$ (otherwise the cocycle (3.31) for the untwisted sector would fail to work and its twisted sector counterpart (4.14) would cease to act in a well–defined way on states characterized by a conjugacy class).

As it will be outlined in appendix A, there exist however (simple) orbifold constructions which do not admit an integer–valued b_R. Our previous treatment obviously does not carry over to these models. They share several features with heterotic string compactifications which have been equipped with Wilson lines A_i in such a way that $N\, A_i$ are $E_8 \times E_8$ root lattice vectors. To explore these constructions, a deeper understanding of twists leaving fixed subtori is necessary. This topic will therefore be presented separately [27].

After having elucidated the (local) structure of the orbifold background space we raise the question whether there exist global symmetries in our space of consistent models. For reasons to become clear in the sequel we have to restrict ourselves to the continuous space of background moduli (both for the metrical and the axionic degrees of freedom), i.e. we neglect the cases $\tilde{M} \neq 0$ even if there should exist an integer solution b_R of (3.17).

By definition, a *background symmetry* is a map which associates to any reference orbifold a second one (whose moduli are found in terms of (invertible) functions of the reference moduli) such that correlation functions of both models agree. Admittedly, simple redefinitions of the vertex operators which occur in a generic correlation function allow it to display a multitude of values (even if they were to preserve the norm of each vertex operator). Therefore we still have to complement the mapping in the space of background moduli by an *induced* transformation which relates the Hilbert space of the reference model to that of the partner model. This will enable us meaningfully to compare their correlation functions. It follows immediately from the characteristic scaling behaviour of two–point functions that the spectra of the stress–energy tensors $T(z)$, $\bar{T}(\bar{z})$ will agree for models which are linked by a background symmetry.

In our case we are going to demonstrate that both the involutive duality map

$$\tilde{g} \pm \tilde{b} = \frac{1}{g \pm b} \tag{5.1}$$

and the collection of (discrete) axionic shifts

$$g' = g \quad ; \quad b' = b + a \quad (\text{for which } a_{ij} \in \mathbb{Z}\,,\ Q^T a Q = a\,,\ a^T = -a) \tag{5.2}$$

lead to background symmetries after they have been suitably extended to the Hilbert spaces of the respective orbifold constructions.

Alternatively, the duality map (5.1) can also be expressed via

$$\tilde{G} = G, \quad \tilde{B} = -B, \quad \tilde{e} = \frac{1}{G - B} e^* \quad . \tag{5.3}$$

On the other hand, the original lattice will not be modified, if an axionic shift is applied. The stress–energy spectra of the primary fields of the untwisted sector then are recognized to remain invariant under (5.1), (5.2) (use (2.11) together with (2.16)).

As far as string theory is concerned, this property was first addressed in [31], [32] where the string propagation on the compact r–dimensional torus was discussed. In the absence of an axionic background the duality transformation amounts here simply to the exchange of the torus lattice Λ_r with Λ_r^*. The case $r = 1$ (compactification on a circle of radius R) is most easily surveyed: $\tilde{R} = \alpha'/R$ where $\sqrt{\alpha'}$ corresponds to the fundamental Planck length scale.

5.2 Invariance of the string emission process under duality

The extension of a background symmetry to orbifold constructions was proposed in [33] for the orbicircle: Although the spectra of the conformal weights h, $\bar{h}$ of the twisted sector do not depend on the moduli, a background symmetry might act nontrivially there provided that these spectra are degenerate. This situation occurs within the class of Z_N orbifold constructions. Whatever conjugacy class $(\Theta, f + (1-\Theta)\Lambda_d)$ of the first twisted sector is singled out, the associated ground states σ_f^+ display the same conformal weight (4.4) which is solely governed by the eigenvalues of the underlying order N twist. The generalization of the twisted sector duality transformation encountered in [33] turns out to be

$$\tilde{\sigma}_f^+ = \frac{1}{\sqrt{N_1}} \sum_{k \in \mathcal{F}^*} \exp\left(2\pi i f^T \frac{1}{1-\Theta} k\right) \quad \sigma^+_{(G-B)^{-1}k} \quad , \tag{5.4}$$

which is a discrete Fourier–transformation. The finite sum ranges over an arbitrary set of representatives of the quotient space $\mathcal{F}^* = \dfrac{\Lambda_d^*}{(1-\Theta)\Lambda_d^*}$; nonetheless it is well–defined. Hence, the dual partner state $\tilde{\sigma}_f$ of the ground state σ_f from the reference model consists of a linear superposition of *ordinary* ground states $\sigma_{\tilde{f}}$ ($\tilde{f} \in \tilde{\Lambda}_d$) belonging to the partner model. The normalization factor $N_1 = \det(1-\Theta)$ is just the multiplicity of the twisted sector ground states. This relationship has been discovered in [22] by demanding that the four–twist field correlation function be duality invariant[12]. The extension to higher twisted sectors (which permits us to examine duality invariance of the three–point Yukawa couplings) together with quite a comprehensive reference list of articles about background symmetries may also be found in [22].

However, to show the duality invariance of the string emission process with incoming and outgoing twisted sector ground states we also have to consistently derive how duality affects the vertex operators $V_{\mathbf{P}}^{\mathrm{inv}}(z,\bar{z})$. A discussion of what happens in the untwisted sector is already contained in section 2 of [22]. It was found that duality requires the asymmetric replacements

$$\begin{aligned} X_R \mapsto \tilde{X}_R &= \frac{1}{G+B}(G-B)\,X_R \\ X_L \mapsto \tilde{X}_L &= -X_L \end{aligned} \tag{5.5}$$

to be performed inside every functional of the left– and rightmoving coordinates $X_L(z)$, $X_R(\bar{z})$. The fields on the right–hand side of each arrow belong to the dual orbifold model.

These prescriptions are consistent with (5.1). Consequently (cf. (3.39)) the zero–mode operators $\hat{x}$, $\hat{q}$ will be mapped to

$$\begin{aligned} \tilde{\hat{x}} &= \frac{1}{G+B}\hat{q} \\ \tilde{\hat{q}} &= (G-B)\hat{x} \quad . \end{aligned} \tag{5.6}$$

Furthermore we deduce from the mode expansions (2.4), (2.5) that

$$\begin{aligned} \tilde{\hat{p}} &= (G-B)\,\hat{w} \\ \tilde{\hat{w}} &= \frac{1}{G+B}\hat{p} \end{aligned} \tag{5.7}$$

where we profited from the auxiliary formula

$$(G-B)\frac{1}{2G}(G+B) = (G+B)\frac{1}{2G}(G-B) \quad . \tag{5.8}$$

Of course the mode expansion of a coordinate field can no longer depend on the zero modes $\hat{p}$, $\hat{w}$ as soon as we switch to a twisted sector representation.

[12]Although this result has been derived in an analysis of *two*–dimensional target spaces, it applies as well to d–dimensional orbifolds (d even), because the proof given for the invariance under a duality map nowhere refers to a particular value for d.

The fact that they have become superfluous is already encoded in the second class constraints (3.35). Therefore the relations (5.7) are perfectly valid as long as they lead to new constraints for the operators $\hat{p}$, $\hat{w}$ of the dual orbifold model! Having inserted the dual partner fields from (5.6), (5.7) into the previous vector of constraints $\vec{c}$ we end up with another set of prescriptions (after suitable c–number rescalings):

$$\tilde{\vec{c}} = \begin{pmatrix} (1-\Theta)\hat{q} - 2\pi\hat{p} \\ \hat{w} \end{pmatrix} \approx 0 \quad . \tag{5.9}$$

Once more $\hat{p}$, $\hat{w}$ are recognized to be redundant. However, the Dirac quantization of the second pair $\hat{q}$, $\hat{w}$ of zero modes again turns out to be non–trivial because of the presence of $\tilde{\vec{c}}$. This time we infer from (3.36) that

$$\begin{aligned} [\hat{x}, \hat{p}^T] &= i \cdot \mathbf{1} \\ [\hat{q}, \hat{w}^T] &= 0 \\ [\hat{x}, \hat{q}^T] &= \frac{2\pi i}{1-\Theta^T} = -\frac{2\pi i}{1-\Theta} + 2\pi i \cdot \mathbf{1} \quad . \end{aligned} \tag{5.10}$$

On the algebraic level there arise of course some marked differences in comparison to the previous set of commutators (3.37). But we are entirely interested in the group multiplication laws of the operators $\exp(ip^T\hat{x})$ ($p \in \Lambda_d^*$) and $\exp(iw^T\hat{q})$ ($w \in \Lambda_d$) which are decisive for the precise evaluation of correlation functions involving string emission processes. From this point of view the *dual* model displays the same group composition laws as the model we have begun with. Indeed, we need not worry about the difference of $(2\pi i)\cdot\mathbf{1}$ between the values of $[\hat{x}, \hat{q}^T]$ in (3.37) and (5.10). It is immaterial, since it gives rise to only trivial factors $\exp(2\pi i p^T w) = 1$ whenever the BCH formula is applied to permute the above operators of the Weyl–Heisenberg group.

Perhaps we should point out that constraints closely related to (5.9) were also proposed in [23]. Moreover, this second set was tailored exactly to reproduce (3.37). From our approach the physical interpretation (which remained obscure in [23]) is then obvious: The alternative set (5.9) gives the constraints valid for the *dual* model. Above we have also seen that even in the case where the zero mode commutator algebras differ the induced group multiplication laws continue to coincide for groups of finite order. This point has evidently not received due attention in [23].

Finally we establish that the eigenstates of $\hat{p}$ in the twisted sector take the form of the (dual) superpositions (5.4). Explicitly, we have

$$
\begin{aligned}
e^{2\pi i v^T \frac{1}{1-\Theta}\hat{p}}\, |\sigma_k^+\rangle^* &= e^{2\pi i v^T \frac{1}{1-\Theta}k}\, |\sigma_k^+\rangle^* \qquad (v \in \Lambda_d,\ k \in \Lambda_d^*) \\
\text{with} \qquad |\sigma_k^+\rangle^* &:= \frac{1}{\sqrt{N_1}} \sum_{f \in \mathcal{F}} e^{2\pi i k^T \frac{1}{1-\Theta} f} |\sigma_f^+\rangle \\
\mathcal{F} &= \frac{\Lambda_d}{(1-\Theta)\Lambda_d}
\end{aligned}
\tag{5.11}
$$

which follows by using the constraint set (5.9) in conjunction with the translation operator property (3.41) of $\hat{q}$.

We then proceed to construct the dual vertex operator $\tilde{V}_{\mathbf{P}}^{\rm inv}$ associated to (3.42). To this end we substitute the dual quantities (5.5) for the original left– and rightmoving parts of $X^\mu(z,\bar{z})$. Since the twist Θ commutes with the background tensor $(G+B)$ we are able to return to the ordinary fields $X_R(z)$, $X_L(\bar{z})$, if we simultaneously pass to the dual Narain momentum

$$
\left\{ \begin{array}{l} \tilde{P}_R = (G+B)\frac{1}{G-B} P_R \\ \tilde{P}_L = -P_L \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \tilde{p} = (G+B)w \\ \tilde{w} = \frac{1}{G-B} p \end{array} \right\} \quad . \tag{5.12}
$$

In order that the dual vertex operator will again describe the emission of strings carrying the (dual) Narain momentum $\tilde{\mathbf{P}}$ we have to make sure that our construction yields $V_{\tilde{\mathbf{P}}}^{\rm inv}$ in the dual orbifold model apart from some additional c–number phase factor. The form of the string coupling (4.18) guarantees that

$$
g'(\Theta^k \mathbf{P}) = g'(\Theta^k \tilde{\mathbf{P}}) \tag{5.13}
$$

holds even for nonvanishing axionic background as long as $\Theta B = B\Theta$ can be relied upon. All the details concerning the proof of this equality are to be found in appendix C.

When deriving the partner vertex dual to (3.42) one also has to watch whether the appropriate normal ordering comes about. As far as the (fractionally moded) annihilation and creation operators $\hat{\alpha}_{n-k_\mu}, \hat{\hat{\alpha}}_{n+k_\mu}$ and their conjugates are concerned the substitution (5.5) will not lead to a different ordering. But this issue becomes important, if one inspects the zero mode contents (3.40) of the vertex operator. In fact, we have

$$\begin{aligned}\exp(iw^T\tilde{\hat{q}})\ \exp(ip^T\tilde{\hat{x}}) &= \exp(i\tilde{p}^T\hat{x}\,)\ \exp(i\tilde{w}^T\hat{q}) \\ &= \exp(i\tilde{w}^T\hat{q})\ \exp(i\tilde{p}^T\hat{x})\ \exp(-2\pi i\tilde{w}^T\tfrac{1}{1-\Theta}\tilde{p})\end{aligned} \tag{5.14}$$

where we finally returned to a normally ordered expression. In a last step one has to rewrite the local phase factor which appears in (3.40) in terms of the dual quantities $\tilde{p}$, $\tilde{w}$. Then we gain

$$\tilde{V}_{\mathbf{P}}^{\text{inv}} = V_{\tilde{\mathbf{P}}}^{\text{inv}}\ \exp(i\pi p^T w) \qquad , \tag{5.15}$$

as soon as the surplus phases have been combined by employing the basic identity

$$\frac{1}{1-\Theta} + \frac{1}{1-\Theta^T} = \mathbf{1} \tag{5.16}$$

Hence $\tilde{V}_{\mathbf{P}}^{\text{inv}}$ fulfils all conditions to serve as a vertex in the operator formulation of CFT. Next, we need to clarify, if the three–point function (4.20) is invariant under a duality transformation induced by (5.1). This is tantamount to claiming that the identity

$$\begin{aligned}\langle\tilde{\sigma}^-_{-f_1}|e^{i\tilde{p}^T\hat{x}}\,e^{i\tilde{w}^T\hat{q}}|\tilde{\sigma}^+_{f_2}\rangle_{\tilde{g}+\tilde{b}} &= \langle\sigma^-_{-f_1}|e^{iw^T\hat{q}}\,e^{ip^T\hat{x}}|\sigma^+_{f_2}\rangle_{g+b} \\ & \quad (f_1, f_2 \in \Lambda_d)\end{aligned} \tag{5.17}$$

holds among the zero mode contributions to the string emission process; the suffix attached to each matrix element allows to keep track of the respective background values.

To establish (5.17) we first consider the pair of structurally simpler identities

$$\begin{aligned}\langle\tilde{\sigma}^-_{-f_1}|e^{i\tilde{p}^T\hat{x}}|\tilde{\sigma}^+_{f_2}\rangle_{\tilde{g}+\tilde{b}} &= \langle\sigma^-_{-f_1}|e^{iw^T\hat{q}}|\sigma^+_{f_2}\rangle_{g+b} \\ \langle\tilde{\sigma}^-_{-f_1}|e^{i\tilde{w}^T\hat{q}}|\tilde{\sigma}^+_{f_2}\rangle_{\tilde{g}+\tilde{b}} &= \langle\sigma^-_{-f_1}|e^{ip^T\hat{x}}|\sigma^+_{f_2}\rangle_{g+b} \qquad .\end{aligned} \tag{5.18}$$

Using the by now familiar rules $e^{ip^T\hat{x}}|\sigma^+_f\rangle = \exp(2\pi i p^T\frac{1}{1-\Theta}f)|\sigma^+_f\rangle$ and $e^{iw^T\hat{q}}|\sigma^+_f\rangle = |\sigma^+_{f+w}\rangle$ together with (5.4) it is easy to verify the crucial relations collected in (5.18). We mention that one can directly arrive at the second identity, if one suitably trades all twisted sector ground states in the first one for the $\hat{p}$–eigenstates given in (5.11), and if subsequently the notions of background and *dual* background are interchanged (and vice versa).

Now (5.17) is seen to be equivalent to (5.18) if one inserts the decompositions

$$\begin{aligned} \mathbf{1} &= \sum_{t\in\mathcal{F}} |\sigma_t\rangle_{g+b}\langle\sigma_t| \\ \mathbf{1} &= \sum_{t\in\mathcal{F}} |\tilde{\sigma}_t\rangle_{\tilde{g}+\tilde{b}}\langle\tilde{\sigma}_t| \end{aligned} \tag{5.19}$$

of the unit operator; the latter is of course confined to the zero mode state space of the twisted sector.

Thus, we have observed that under a duality map the two ingredients of the cocycle operator, adapted to a twisted sector, are exchanged. They are generating elements of the Weyl–Heisenberg group. In addition, (5.18) informs us that duality acts as a similarity transformation which diagonalizes the cyclic permutation operator $\exp(iw^T\hat{q})$. An alternative way of expressing (5.18) by means of commutative diagrams has been indicated in [22]. There we emphasized an additional property of the above zero mode exponentials which amount to phase–weighted cyclic permutations in the set of twist fields: they also act as generators of a symmetry group $\mathcal{L}$ which — at a prescribed background — leaves every correlation function invariant. From our analysis we learn that the cocycle operators belonging to $V_{\mathbf{P}}^{\rm inv}$ represent this group.

Above, the duality transformation of the vertex operator gave also rise to an additional phase factor $\exp(i\pi p^T w) = \exp(i\pi(h-\bar{h}))$ which takes its values from $\{1,-1\}$. It is local in the sense that it only depends on the characteristic Narain momentum $\mathbf{P}$ of the vertex operator. Hence one might try to get rid of it by a local redefinition of $V_{\mathbf{P}}^{\rm inv}$. This can be rapidly achieved for twists whose order N is odd. Let us introduce

$$U_{\mathbf{P}}^{\rm inv} = V_{\mathbf{P}}^{\rm inv}\ \exp(i\pi p^T \frac{N}{1-\Theta} w) \tag{5.20}$$

as a new vertex operator. With the help of (4.10) we conclude that these vertices differ at most by a sign. Moreover using (5.16) reveals that

$$\tilde{U}_{\mathbf{P}}^{\rm inv} = U_{\tilde{\mathbf{P}}}^{\rm inv} \tag{5.21}$$

provided that $N \equiv 1 \bmod 2$, as announced before. The correlation function (4.20) for string emission is substituted by

$$\langle \sigma^-_{-f_1} | U^{\text{inv}}_{\mathbf{P}}(1,1) | \sigma^+_{f_2} \rangle = \sqrt{N} g'(\mathbf{P}) \left\{ \exp[-2\pi i (p - \Delta w)^T \tfrac{1}{1-\Theta}(f_2 + f_1)] \right\}^{\frac{N-1}{2}} \times \delta_{\mathbf{0},(f_2+w-f_1)\bmod(1-\Theta)\Lambda_d} \quad . \tag{5.22}$$

Thus we have managed to express this correlation function in terms of an N–th order root of unity (before we potentially could have ended up with a $(2N)$–th order root of unity). A counterexample where the sign factor apparent in (5.15) cannot be absorbed by a local redefinition of $V^{\text{inv}}_{\mathbf{P}}$ is delivered by the orbicircle model (having twist order two). Of course additional local phases (as in $U^{\text{inv}}_{\mathbf{P}}$) have no impact on the question of duality invariance, since — being c–numbers — they are not affected by (5.5).

Actually, a multiplication by a new sign distribution as given in (5.20) is completely compatible with the former decomposition of the zero mode exponential (3.40). As far as the c–number factor $\phi = \exp(ip^T \frac{1}{1-\Theta} w)$ in that formula is concerned, there still remains the freedom to change its sign for particular Narain lattice vectors $\mathbf{P}$. Indeed, the Weyl–Heisenberg group commutation laws for $\exp(ip^T \hat{x})$, $\exp(iw^T \hat{q})$ determine only its square ϕ^2 whereas the role of (3.40) merely consists in assigning a particular value to $\exp(ip^T \hat{x} + iw^T \hat{q})$. Since the parameters p, w live in (discrete) lattices there is virtually no possibility to pin down a further sign in front of ϕ. An exception from this indeterminacy occurs when $\phi^2 = 1$ in which case the zero mode exponentials of (3.40) commute. Thus it is natural to choose the BCH–phase to be equal to 1. For N odd, the modification suggested in (5.20) will in fact satisfy this boundary condition.

For a complete discussion of the string emission process it is mandatory to also take the higher twisted sectors into account. The counterparts of the three–point function (4.20) and of the dual twist field prescription (5.4) become more involved. The interested reader should consult appendix D where we treat this generalization.

5.3 Invariance of the string emission process under discrete axionic shifts

An axionic background shift by $a = e^T \alpha e$ is triggered, if we adopt

$$\begin{array}{rclcl} \hat{x}' & = & \hat{x} & \qquad & \hat{q}' = \hat{q} - \alpha \hat{x} \\ \hat{p}' & = & \hat{p} - \alpha \hat{w} & \qquad & \hat{w}' = \hat{w} \end{array} \tag{5.23}$$

as the new zero mode operators in the model characterized by b'. Their substitution into (2.16) and (3.40) indeed causes the initial antisymmetric tensor B to shift by an amount α. Correspondingly the second class constraints (3.35) now read

$$\vec{c}\,' = \begin{pmatrix} (1-\Theta)\hat{x} - 2\pi\hat{w} \\ \hat{p} - \alpha\hat{w} \end{pmatrix} \approx 0 \quad . \tag{5.24}$$

Still, the commutator matrix formed by this vector of constraints is identical to the one known from the reference model. This feature holds in the case of a duality map as well (cf. (5.9)). Using the prescription (3.36) for Dirac commutators we recover the previous results (3.37). Beyond this we come across two additional non–vanishing commutators:

$$\begin{array}{rcl} [\hat{q}, \hat{q}^T] & = & -2\pi i\, \alpha \\ [\hat{q}, \hat{p}^T] & = & -i\, \alpha \quad . \end{array} \tag{5.25}$$

However, the group composition properties of exponentials containing the zero modes $\hat{x}$, $\hat{q}$ are not altered by the latter, because

$$e^{iw_1^T \hat{q}}\, e^{iw_2^T \hat{q}} = e^{iw_2^T \hat{q}}\, e^{iw_1^T \hat{q}} \quad (w_1, w_2 \in \Lambda_d) \tag{5.26}$$

continues to hold (observe that $a_{ij} \in \mathbb{Z}$ ensures the triviality of the BCH phase $\exp(2\pi i w_1^T \alpha w_2)$).

What are the consequences of (5.23) for the first twisted sector's Hilbert space? Although we had to adjust $\hat{q}$ in order to realize the shift (5.2) the right– and leftmoving zero mode parts of the coordinate field $X(z,\bar{z})$ remained the same. We are forced to conclude that even

$$X_R'(z) = X_R(z)\,; \qquad X_L'(\bar{z}) = X_L(\bar{z})\,. \tag{5.27}$$

Therefore we do not have to care about the oscillator exponentials when we are going to determine the associated vertex operator $(V_{\mathbf{P}}^{\text{inv}})'$ of the model whose axionic background is $B' = B + \alpha$. Strings emitted by the new vertex operator still carry the Narain momentum $\mathbf{P}$ (in distinction to what happens in the case of a duality map (see 5.12)). This implies that the required axionic shift in (2.16) is to be achieved by an internal reordering: We simply choose

$$\begin{aligned} p' &= p + \alpha w \\ w' &= w \end{aligned} \tag{5.28}$$

as the new momentum and winding vector. Whence there is no need here to bother about those c–number factors in $V_{\mathbf{P}}^{\text{inv}}$ which are functions of P_R, P_L.

But we still have to cope with the transformation of the product (3.40) of zero mode exponentials under (5.23). A formal application of the BCH formula yields

$$e^{iw^T(\hat{q}-\alpha\hat{x})} = e^{i\pi(\alpha w)^T \frac{1}{1-\Theta} w}\; e^{iw^T\hat{q}}\; e^{i(\alpha w)^T\hat{x}} \quad . \tag{5.29}$$

Admittedly, the set of zero mode operators $\{\hat{x}, \hat{q}, \hat{p}, \hat{w}\}$ with the algebra outlined above does not genuinely exist. Still, the exponentials which contain these modes are well–defined and the algebraic rules can be applied to them in a consistent manner. We concede that again a sign ambiguity is inherent in (5.29): The commutation laws of the Weyl–Heisenberg group permit us only to determine the square of the c–number phase unambiguously.

Assembling all pieces we end up with

$$(V_{\mathbf{P}}^{\text{inv}})' = V_{\mathbf{P}}^{\text{inv}} \tag{5.30}$$

where the right hand side is to be understood as a vertex w.r.t. the new axionic background B'. It is important to have α commute with Θ, since otherwise the new vertex fails to be twist invariant and thus cannot be a physical field for the partner orbifold. The above construction also assures us that the transformed vertex operators commute in the first twisted sector, because their zero mode parts again provide the phase factors compensating (4.12) or (4.15).

The map which yields the partner twist fields in the new orbifold model has already been determined in [22] with the aid of pure twist field correlation functions:

$$
\begin{aligned}
(\sigma_f^+)' &= U_f\,\sigma_f^+ \\
\Rightarrow \quad (\sigma_{-f}^-)' &= \bar{U}_f\,\sigma_{-f}^- \quad .
\end{aligned}
\tag{5.31}
$$

Here the phase factors U_f provide a projective representation of the abelian group Λ_d on the unit circle in $\mathbb{C}$:

$$
U_{f_1+f_2} = U_{f_1}U_{f_2}\exp(-2\pi i f_1^T \frac{1}{1-\Theta}\alpha f_2) \qquad (f_1, f_2 \in \Lambda_d) \quad . \tag{5.32}
$$

In addition, U_f is constrained by

$$
(U_f)^2 = \exp(-2\pi i f^T \frac{1}{1-\Theta}\alpha f) \qquad (f \in \Lambda_d) \quad . \tag{5.33}
$$

The right hand side of (5.32) must evidently be symmetric against exchanging f_1 and f_2. This can be immediately shown provided that $[\alpha, \Theta] = 0$ and that the corresponing shift a in the lattice basis is integer. Thus we came across a further argument why axionic shifts have to commute with the twist. Furthermore, for (5.31) to be well–defined we have to insist on $U_{\mathbf{0}} = U_{(1-\Theta)\lambda}$ $(\lambda \in \Lambda_d)$.

Now we are ready to probe the invariance of the three–point coupling (4.20) under a transformation induced by an axionic background shift (5.2). We expect to have

$$
\langle(\sigma_{-f_1}^-)'|(V_{\mathbf{P}}^{\mathrm{inv}})'|(\sigma_{f_2}^+)'\rangle_{g+b'} = \langle\sigma_{-f_1}^-|V_{\mathbf{P}}^{\mathrm{inv}}|\sigma_{f_2}^+\rangle_{g+b} \tag{5.34}
$$

where $b' = b + a$. As was the case for (5.17) a proof should be based on the auxiliary identities

$$
\begin{aligned}
\langle(\sigma_{-f_1}^-)'|e^{iw^T\hat{q}'}|(\sigma_{f_2}^+)'\rangle_{g+b'} &= \langle\sigma_{-f_1}^-|e^{iw^T\hat{q}}|\sigma_{f_2}^+\rangle_{g+b} \\
\langle(\sigma_{-f_1}^-)'|e^{ip^T\hat{x}'}|(\sigma_{f_2}^+)'\rangle_{g+b'} &= \langle\sigma_{-f_1}^-|e^{ip^T\hat{x}}|\sigma_{f_2}^+\rangle_{g+b}
\end{aligned}
\tag{5.35}
$$

involving matrix elements of the zero mode exponentials. Together with suitable unit operator insertions (5.34) will ensue.

Given a solution of (5.32–5.33) the second relation is rapidly seen to hold since $\hat{x}' = \hat{x}$ and both $e^{ip\hat{x}}$ and U have diagonal matrix representations when acting on the set of ordinary twist fields. As regards the first relation it is recognized to be the addition theorem of (5.33) in disguise, if we are willing to adopt U_w as the BCH phase in the decomposition (5.29). This choice differs at most by a sign from the one we had previously in mind. Likewise there might occur now w–dependent signs in (5.30).

At least for defining automorphisms Θ whose order is an odd number we are able to solve (5.32–5.33) explicitly:

$$U_f = \left\{ \exp(2\pi i f^T \frac{1}{1-\Theta} \alpha f) \right\}^{\frac{N-1}{2}} \quad . \tag{5.36}$$

(For a proof one must rely on $\frac{N-1}{2} \in \mathbb{N}$ and also exploit the fact that $\frac{N}{1-Q}$ is an integer matrix (see (4.10).) Since the case of twists of even order is more involved we have to postpone a complete discussion to a subsequent study [34].

We conclude this section with a glimpse of how an axionic shift (5.2) is realized in the (higher) s–th twisted sector. Again we restrict ourselves to odd order N. Similarly to the way the duality operation is extended to these sectors (cf. appendix D) one first considers Θ^s to be the defining twist which promotes the ingredients $\sigma^{(s)}_{\Theta^\beta f}$ of the physical twist field $\Sigma^{(s)}_f$ to physical fields themselves. Then $B \mapsto B + \alpha$ induces a phase transformation $(\sigma^{(s)}_f)' = U^{(s)}_f \sigma^{(s)}_f$ where $U^{(s)}_f$ this time solves (5.32–5.33) with Θ replaced by Θ^s. We concentrate here on odd N because then the order $N(s)$ of Θ^s will necessarily be again an odd number. Whence we just have to adapt (5.36):

$$U^{(s)}_f = \left\{ \exp(2\pi i f^T \frac{1}{1-\Theta^s} \alpha f) \right\}^{\frac{N(s)-1}{2}} \quad . \tag{5.37}$$

If Θ is reinstalled as the defining twist, care must be taken that (5.37) still leads to a sensible transformation law of the (by now only) physical fields $\Sigma^{(s)}_f$. From the key property $U^{(s)}_f = U^{(s)}_{\Theta f}$ we deduce that $(\Sigma^{(s)}_f)'$ remains a physical field. Thus we have obtained

$$(\Sigma^{(s)}_f)' = U^{(s)}_f \, \Sigma^{(s)}_f \quad . \tag{5.38}$$

We are now ready to establish the invariance of a string emission from the s–th twisted sector under (5.2). Taking into account some minor changes in the quantization of the zero modes which are inherited from the s–th twisted sector analogue of (5.24) and using the algebraic rules available for the phases $U_f^{(s)}$ we indeed find

$$\langle(\Sigma_{-f_1}^{(-s)})'|(V_{\mathbf{P}}^{\mathrm{inv}})'|(\Sigma_{f_2}^{(s)})'\rangle_{g+b'} = \langle\Sigma_{-f_1}^{(-s)}|V_{\mathbf{P}}^{\mathrm{inv}}|\Sigma_{f_2}^{(s)}\rangle_{g+b} \quad . \tag{5.39}$$

6 Discussion

We have obtained the twisted sector string coupling constant $g'(\mathbf{P})$ for the emission of an untwisted string from a twisted one in bosonic Z_N orbifold theories. Furthermore, we applied the Dirac quantization of the bosonic zero modes to calculate the complete coefficients for the OPE of twist fields with anti–twist fields, containing all relative phase factors. Seemingly twist invariant and coset stable cocycle operators (both in the untwisted and twisted sector of the theory), can only be constructed, if the constraint (3.17) holds which is due to Θ's automorphic action on $\Lambda_{\mathcal{N}}$. In this paper we have not covered those models whose axionic background is fractional w.r.t. the lattice basis and fails to commute with Θ (see below).

With complete expressions for the vertex operators at our disposal we then have answered in the affirmative the question of background modular invariance for the twisted sector string emission. This property continues to hold in higher twisted sectors (there, physical twist fields are in general no longer related to a single fixed point, whence somewhat involved expressions were encountered). It was necessary to adapt the constraints which determine the composition laws of zero mode exponentials in the twisted sector each time a modular transformation of the background took place. However, the transformed sets of cocycle operators were found merely to give new representations of the same Weyl–Heisenberg group.

There are some *open problems* which remain to be solved:

- What happens, if a twist Θ leaves some directions in T_d fixed? This case becomes especially interesting in the presence of a fractionally valued axionic background b, which couples the twisted to the fixed coordinates. It clearly resembles those heterotic orbifolds whose space group is embedded via *shifts* into the gauge degrees of freedom. In these models Θ does not rotate the sixteen (chiral) gauge coordinates; moreover non–trivial Wilson lines will communicate a shift of the spatial coordinates to the gauge group part [27].

- We left out the calculation of Yukawa couplings $\langle \sigma^{+}_{f_1} \sigma^{+}_{f_2} \sigma^{--}_{-f_3} \rangle$. They are not directly accessible in the operator approach, since it only applies to correlation functions with at most two twist fields. Perhaps a Fock space description of the twist fields, as it is derived in [30], permits one to evaluate at least the *quantum* correlation part. However this method is quite complicated.
  The path integral formalism seems to be the more promising tool to evaluate Yukawa couplings (see [15], [17], [22], [35]). At the moment, a thorough treatment of these couplings for any background compatible with an automorphic twist action on $\Lambda_{\mathcal{N}}$ is still missing. We will take up this problem in a forthcoming publication [34].

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A The condition on the axionic background B imposed by the twist

Here we solve the constraint

$$R = Q^T b Q - b \tag{A.1}$$

(see (3.18)) for a number of examples and check whether there even exists a solution of type (3.22).

Example 1: Consider a four–dimensional torus $T_4 := \mathbb{R}^4/(2\pi\Lambda_{su(3)} \times \Lambda_{su(3)})$ ($\Lambda_{\mathbf{su}(3)}$ denotes the lattice generated by the simple roots of su(3)). Then a rotation by $\frac{4\pi}{3}$ in the first and by $\frac{2\pi}{3}$ in the second lattice factor is taken to mod out T_4 thus providing a Z_3 orbifold. For the basis

$$e = \begin{pmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & -\sqrt{\frac{3}{2}} \end{pmatrix} \tag{A.2}$$

of $\Lambda_{\mathbf{su}(3)}$ we have

$$Q = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix} \quad , \quad M = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad . \tag{A.3}$$

It is convenient to expand b with respect to the standard basis $\{\rho_{ij} |\, i,j = 1,\ldots,4; i < j\}$ of the set A_4 of real antisymmetric 4×4 matrices:

$$\begin{array}{lcl} (\rho_{ij})_{\alpha\beta} & := & \delta_{i\alpha}\delta_{j\beta} - \delta_{i\beta}\delta_{j\alpha} \\ b & = & \beta_1\rho_{12} + \beta_2\rho_{13} + \ldots + \beta_6\rho_{34}\,; \quad \beta_i \in \mathbb{R} \quad . \end{array} \tag{A.4}$$

Now we find that the homogeneous solution space $\mathcal{S}$ of (A.1) is spanned by $Z_1 = \rho_{12}$, $Z_2 = \rho_{34}$, $Z_3 = \rho_{13} - \rho_{24}$ and $Z_4 = \rho_{14} + \rho_{23} + \rho_{24}$. The following conditions arise from the fact that R is integer (whatever particular choice has been made for R):

$$\begin{array}{lcl} \beta_3 & = & \beta_4 \bmod 1 \\ \beta_2 & = & \beta_3 - \beta_5 \bmod 1 \; . \end{array} \tag{A.5}$$

Obviously (3.22) can always be cast into the form

$$\begin{array}{rcl} b_0 & = & \beta_1 Z_1 + \beta_6 Z_2 + (\beta_4 - \beta_5) Z_3 + \beta_4 Z_4 \\ b_R & = & (\beta_2 - \beta_4 + \beta_5)\,\rho_{13} + (\beta_3 - \beta_4)\,\rho_{14} \end{array} \quad . \tag{A.6}$$

Of course this answer is not unique, since one may add to b_0 (and subtract from b_R) integer linear combinations $\sum_{i=1}^{4} k_i Z_i$ at will.

Example 2: We turn to the case of the six–dimensional Z_7 orbifold where a Coxeter twist acts on the Cartan basis of the su(7) lattice underlying $T_6 := \mathbb{R}^6/(2\pi\Lambda_{\mathrm{su}(7)})$ via

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} . \tag{A.7}$$

The homogenous solution space $\mathcal{S}$ is spanned by (with ρ_{ij} as in (A.4))

$$\begin{array}{rcl} Z_1 & = & \rho_{12} + \rho_{23} + \rho_{34} + \rho_{45} + \rho_{56} \\ Z_2 & = & \rho_{13} + \rho_{24} + \rho_{35} + \rho_{46} - \rho_{16} \\ Z_3 & = & \rho_{14} + \rho_{25} + \rho_{36} - \rho_{15} - \rho_{26} \end{array} \quad . \tag{A.8}$$

The requirement (A.1) for $b = \sum_{i<j} \beta_{ij}\rho_{ij}$ ($\beta_{ij} \in \mathbb{R}$; $1 \le i,j \le 6$) then amounts to

$$\begin{array}{rcl} \beta_{12} & = & \beta_{23} = \beta_{34} = \beta_{45} = \beta_{56} \bmod 1 \\ \beta_{13} & = & \beta_{24} = \beta_{35} = \beta_{46} = -\beta_{16} \bmod 1 \\ \beta_{14} & = & \beta_{25} = \beta_{36} = -\beta_{15} = -\beta_{26} \bmod 1 \end{array} \quad . \tag{A.9}$$

Thus all admissible solutions can be written as

$$b_0 \equiv \beta_{12} Z_1 + \beta_{13} Z_2 + \beta_{14} Z_3 \bmod 1 \quad . \tag{A.10}$$

Counterexample 3: We demonstrate that $\det(\Theta - \mathbf{1}) \neq 0$ is necessary in order that b takes the form (3.22). We inspect the case of a four–dimensional Z_N orbifold with

$$Q = \begin{pmatrix} \tilde{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \tag{A.11}$$

where the blocks $\tilde{Q} \neq \mathbf{1}$, $\mathbf{0}$ and $\mathbf{1}$ are 2×2 matrices. Thus Θ fixes a two–dimensional torus. With the parametrization

$$b = \begin{pmatrix} \beta_1 & \beta_3 \\ -\beta_3^T & \beta_2 \end{pmatrix} \in A_4\,; \quad \beta_1^T = -\beta_1,\ \beta_2^T = -\beta_2 \tag{A.12}$$

we obtain

$$R(b) = \begin{pmatrix} 0 & (\tilde{Q}^T - \mathbf{1})\beta_3 \\ \beta_3^T(\mathbf{1} - \tilde{Q}) & 0 \end{pmatrix} \quad . \tag{A.13}$$

However $R(b)$ determines β_3 unambiguously (i.e., $\beta_3 = 0$ for $R = 0$). Clearly, the decomposition $b = b_0 + b_R$ as in (3.22) can no longer be maintained: b_0 vanishes on the β_3–block while b_R ceases to be integer under particular circumstances. Take for instance $\beta_1 = \beta_2 = 0$, $\beta_3 = (\tilde{Q}^T - \mathbf{1})^{-1}$ and note that $R(b)$ becomes integer. A short calculation reveals that

$$\begin{aligned} \beta_3 &= -\frac{\epsilon\tilde{Q}\epsilon + \mathbf{1}}{\det(\tilde{Q} - \mathbf{1})} \\ \det(\tilde{Q} - \mathbf{1}) &= 2 - \mathrm{tr}\tilde{Q} = 2(1 - \mathrm{Re}\,\omega) \end{aligned} \tag{A.14}$$

where $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and the phases ω, $\bar{\omega}$ stand for the eigenvalues of $\tilde{Q}$. Hence except for $\omega = \exp(\pm\frac{i\pi}{6})$ (Z_6 orbifold as the twisted two–dimensional subspace) one is confronted with $\det(\tilde{Q} - \mathbf{1}) \in \{2, 3, 4\}$. Due to this fact the blocks β_3 are not always integer if the above subspace is a Z_k orbifold ($k \in \{4, 3, 2\}$).

Counterexample 4: Actually there are also orbifold constructions *without* fixed directions which do not possess integer representatives b_R of the axionic background. We modify (A.11), starting from

$$Q = \begin{pmatrix} \tilde{Q} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \qquad . \tag{A.15}$$

Thus Θ acts via $\tilde{Q}$, $-\mathbf{1}$ on the lattices of two–dimensional orbifold subspaces. In slight contrast to (A.14) the off-diagonal block of b may this time be chosen as

$$\beta_3 = \frac{\mathbf{1} - \epsilon\tilde{Q}\epsilon}{2 + \operatorname{tr}\tilde{Q}} \qquad (\tilde{Q} \neq -\mathbf{1}) \tag{A.16}$$

We are led to conclude that if the first subspace should be a Z_4– or a Z_6 orbifold then an integral representative b_R is not available.

B Two–point functions for the twisted sector

In this appendix we calculate all twisted sector two–point functions[13] of the oscillator parts of the bosonic fields $\mathcal{X}^{\mu}_{R/L}$ and $\bar{\mathcal{X}}^{\nu}_{R/L}$. One easily obtains

$$\begin{aligned} \langle \bar{\mathcal{X}}^{\mu}_{R}(x)\mathcal{X}^{\nu}_{R}(y)\rangle_t &= i^2 \sum_{m=0}^{\infty} \sum_{n=0}^{-\infty} \frac{x^{-(m+k_\mu)}\, y^{-(n-k_\nu)}}{(m+k_\mu)(n-k_\nu)} \left[\bar{\alpha}^{\mu}_{m+k_\mu}, \alpha^{\nu}_{n-k_\nu}\right] \\ &= \delta^{\mu\nu} \sum_{m=0}^{\infty} \frac{1}{m+k_\mu} \left(\frac{y}{x}\right)^{m+k_\mu} =: \delta^{\mu\nu} H\left(\frac{y}{x}\right). \end{aligned} \tag{B.1}$$

Instead of $H(w)$ we first treat

$$\partial_w H(w) = w^{k_\mu - 1} \sum_{m=0}^{\infty} w^m = \frac{w^{k_\mu - 1}}{1 - w} \tag{B.2}$$

After the substitution $w = z^N$, one obtains (we set $s_\mu := N k_\mu$)

$$H(w(z)) = -N \int \frac{d\,z \quad z^{s_\mu - 1}}{z^N - 1}. \tag{B.3}$$

In order to solve this integral we make a short excursion to the decomposition of partial fractions. Consider

---

[13] Of course, from the point of view of CFT, these are four–point functions which include two twist fields.

$$R(x) = \frac{a_0 + a_1 x + \cdots + a_m x^m}{b_0 + b_1 x + \cdots + b_k x^k} =: \frac{f(x)}{d(x)}; \quad k > m. \tag{B.4}$$

We may of course normalize $b_k = 1$ whence the linear factor representation of the denominator takes the form

$$d(x) = \prod_{n=1}^{k} (x - \beta_n) \tag{B.5}$$

with $\beta_n \in \mathbb{C}$. If the β_n are mutually different, $R(x)$ decomposes into partial fractions as follows:

$$R(x) = \sum_{n=1}^{N} \frac{f(\beta_n)}{d'(\beta_n)} \frac{1}{x - \beta_n} \quad . \tag{B.6}$$

Therefore

$$\int R(x)\, dx = \sum_{n=1}^{N} \frac{f(\beta_n)}{d'(\beta_n)} \ln(x - \beta_n) + \text{const.} \quad . \tag{B.7}$$

In case of $H(z)$ the zeros of the denominator $d(x)$ are given by $\beta_n = \omega^n$ ($\omega = \exp(2\pi i/N)$), and we obtain

$$H(w(z)) = -\sum_{n=1}^{N} \omega^{n s_\mu} \ln(z - \omega^n) + \text{const.} \quad . \tag{B.8}$$

The integration constant is fixed by the condition $H(0) = 0$. Finally we arrive at

$$\langle \bar{\mathcal{X}}_R^\mu(x) \mathcal{X}_R^\nu(y) \rangle_t = -\delta^{\mu\nu} \sum_{n=1}^{N} \omega^{+n s_\mu} \ln\left[1 - \omega^{-n} \left(\frac{y}{x}\right)^{\frac{1}{N}}\right] \quad , \tag{B.9}$$

and analogously

$$\langle \mathcal{X}_R^\mu(x) \bar{\mathcal{X}}_R^\nu(y) \rangle_t = -\delta^{\mu\nu} \sum_{n=1}^{N} \omega^{-n s_\mu} \ln\left[1 - \omega^{-n} \left(\frac{y}{x}\right)^{\frac{1}{N}}\right] \tag{B.10}$$

$$\langle \bar{\mathcal{X}}_L^\mu(\bar{x}) \mathcal{X}_L^\nu(\bar{y}) \rangle_t = -\delta^{\mu\nu} \sum_{n=1}^{N} \omega^{+n s_\mu} \ln\left[1 - \omega^{n} \left(\frac{\bar{y}}{\bar{x}}\right)^{\frac{1}{N}}\right] \tag{B.11}$$

$$\langle \mathcal{X}_L^\mu(\bar{x})\bar{\mathcal{X}}_L^\nu(\bar{y})\rangle_t \quad = -\delta^{\mu\nu}\sum_{n=1}^{N}\omega^{-ns_\mu}\ln\left[1-\omega^n\left(\frac{\bar{y}}{\bar{x}}\right)^{\frac{1}{N}}\right]. \qquad \text{(B.12)}$$

C Duality invariance of the string coupling constant

As announced in section 5.2 we argue here why the string coupling constant g' sticks to its original value, if a string carrying the *dual* Narain momentum is emitted.

Observe first that for (4.18) to be applicable we must select a coordinate system in which $G = \frac{1}{2}\mathbf{1}_d$, and $\Theta \in SO(d)$ has acquired a 2×2–block diagonal form. Whereas $\bar{h}_\mu (1 \leq \mu \leq \frac{d}{2})$ clearly keeps its original value under the duality map the right–moving quantities h_μ usually assume different values, as (5.12) mixes the components of the right–moving Narain vector P_R. This is to be blamed on the presence of a non–trivial axionic background B. Thanks to the condition

$$\Theta B = B\Theta \qquad \text{(C.1)}$$

(which was supposed to hold from the beginning of section 5) we may show that even the right mover's contribution to $g'(\mathbf{P})$ is an invariant under (5.12). To verify this assertion we look more closely at a 2×2 block

$$\mathcal{C}_{\mu\nu} = \begin{pmatrix} B_{2\mu-1,2\nu-1} & B_{2\mu-1,2\nu} \\ B_{2\mu,2\nu-1} & B_{2\mu,2\nu} \end{pmatrix} \qquad (1 \leq \mu,\nu \leq \frac{d}{2}) \qquad \text{(C.2)}$$

of B. With the notation familiar from (3.6) we then deduce

$$\Theta_\mu \mathcal{C}_{\mu\nu} = \mathcal{C}_{\mu\nu}\Theta_\nu \qquad \text{(C.3)}$$

(neither μ nor ν are summed over here). Adopting a complex basis for the two–dimensional real subspaces which were singled out above we immediately learn that $\mathcal{C}_{\mu\nu}$ has to vanish except for the cases $\Theta_\mu = \Theta_\nu$ or $\Theta_\mu = \Theta_\nu^T$. In other words, the nonvanshing blocks of B relate only those two–dimensional subspaces of the target for which $\delta(k_\mu) = \delta(1-k_\mu)$ possesses the *same* value.

It is therefore adequate to rewrite the twisted string coupling (4.18) as follows:

$$g'(\mathbf{P}) = \prod_{l\in\mathcal{K}} \delta(l)^{-\frac{1}{4}\{P_R^T(l)P_R(l)+P_L^T(l)P_L(l)\}} \qquad . \tag{C.4}$$

Here, $2\pi\mathcal{K}$ comprises all planar rotation angles φ_μ associated to Θ which happen to lie in the interval $(0,\pi]$. Furthermore, the index l attached to the right– and leftmoving momenta P_R, P_L requests to retain only their orthogonal projection onto T_l which is the momentum subspace where Θ exclusively consists of the building blocks Θ_μ with $k_\mu \in \{l, 1-l\}$.

According to the analysis given above for the 2×2 blocks of the axionic background, *both* transformations (5.12) are seen to decompose into $|\mathcal{K}|$ bijective maps on the various subspaces T_l. Now the invariance of $g'(\mathbf{P})$ under $\mathbf{P} \mapsto \tilde{\mathbf{P}}$ is seen immediately, since $(\mathbf{1}+2B)\frac{1}{\mathbf{1}-2B}\mid_{T_l}$ $(1\leq l\leq |\mathcal{K}|)$ are simple rotations in T_l.

D Duality of the process of string emission originating from a higher twisted sector

Let us focus on the s–th twisted sector $(1\leq s<N)$ which imposes global monodromy conditions of the form (Θ^s, f) $(f\in\Lambda_d)$ on the coordinate field $X^\mu(z,\bar{z})$ (here, we apply the space group notation). In what follows we will always stick to the assumption $(\Theta_\mu)^s \neq \mathbf{1}$ $(1\leq\mu\leq\frac{d}{2})$ (see (3.6)). Put differently, Θ^s is not permitted to fix any plane inside the target space.

As has been stressed in [8], [15] the associated twist field $\Sigma_f^{(s)}$ will moreover provide the boundary conditions which result from (Θ^s, f) by conjugation with arbitrary elements of S_d. A short calculation yields the conjugacy class

$$\bigcup_{\beta=0}^{|\mathcal{O}_f|-1} (\Theta^s, \Theta^\beta f + (1-\Theta^s)\Lambda_d) \qquad . \tag{D.1}$$

The set $\mathcal{O}_f$ comprises the *orbit* $\{(\Theta^x f)\,\mathrm{mod}(1-\Theta^s)\Lambda_d\,; 0\leq x\leq N-1\}$ of the fixed point representative f. In distinction to the first twisted sector where $|\mathcal{O}_f|=1$ the translation group part relevant for a higher twisted sector generically consists of a union of cosets in $\mathcal{F}$.

We abbreviate by $\sigma_f^{(s)}$ that portion of $\Sigma_f^{(s)}$ which takes care of the global monodromy subset whose translation vectors are restricted to the coset $\{f + (1-\Theta^s)\Lambda_d\}$. Then the normalized physical twist field $\Sigma_f^{(s)}$ can be represented as

$$\Sigma_f^{(s)} = \frac{1}{\sqrt{|\mathcal{O}_f|}} \sum_{\beta=0}^{|\mathcal{O}_f|-1} \sigma_{\Theta^\beta f}^{(s)} \quad . \tag{D.2}$$

As far as the quantum properties are concerned the Fock spaces obtained upon exciting the various $\sigma_{\Theta^\beta f}^{(s)}$ are indistinguishable. Notice that the coefficients in the superposition (D.2) have one and the same value which is dictated by normalization and twist invariance of $\Sigma_f^{(s)}$.

Now let us modify the vertex operator (3.42) such that it can properly act on the s–th twisted sector ground states $|\Sigma_f^{(s)}\rangle$. The correct quantum monodromy will be assured if we replace the fractional shifts k_μ which determine the mode expansions (3.8), (3.9) by $(sk_\mu)\mathrm{mod}\,1$ throughout. We also recall the contents of the footnote attached to (4.20). It interprets the string coupling constant $g'(\mathbf{P})$ in the twisted sector as the normal ordering factor of the quantum oscillator part of $V_{\mathbf{P}}^{\mathrm{inv}}$ which arises relative to the untwisted vacuum. Evidently, we must include the sector index s as an additional label of the coupling constant which then reads

$$g'(s;\mathbf{P}) = \prod_{\mu=1}^{d/2} \delta((sk_\mu)\mathrm{mod}\,1)^{-\frac{1}{2}(h_\mu+\bar{h}_\mu)} \quad , \tag{D.3}$$

since the two–point functions of the left– and rightmoving coordinate fields now have to be evaluated in the s–th twisted sector (cf. for instance (B.1)).

Likewise the zero mode algebra (3.37) depends on the choice of a specific twisted sector. Here the second class constraint (3.35) has to be replaced by $(1-\Theta^s)\hat{x} - 2\pi\hat{w} \approx 0$. Therefore the nontrivial commutator of the zero modes contained in $X(z,\bar{z})$ changes into

$$[\hat{x}, \hat{q}^T] = -\frac{2\pi i}{1-\Theta^s} \quad . \tag{D.4}$$

Consequently (3.40) will contain Θ^s instead of Θ in its exponentials.

When determining the matrix elements of a single $V_{\mathbf{P}}^{\mathrm{inv}}$ sandwiched between ground states of the s–th twisted sector one may neglect the normal ordered product of the exponentials which contain the oscillator operators; their effect on the full correlation function is already summarized in $g'(s;\mathbf{P})$. Thus it remains to study the action of the zero mode exponentials on the twisted sector ground state $|\Sigma_f^{(s)}\rangle$. In our (higher) sector we arrive at

$$\begin{aligned}&\sum_{k=1}^{N} e^{i(\Theta^k w)^T\hat{q}}\ e^{i(\Theta^k p)^T\hat{x}}\ |\Sigma_f^{(s)}\rangle \\ =\ & \frac{1}{\sqrt{|\mathcal{O}_f|}}\sum_{\beta=0}^{|\mathcal{O}_f|-1}\sum_{k=1}^{N}\exp[2\pi i(\Theta^k p)^T\tfrac{1}{1-\Theta^s}(\Theta^\beta f)]\quad |\sigma^{(s)}_{\Theta^\beta f+\Theta^k w}\rangle \quad .\end{aligned} \tag{D.5}$$

Despite the abundance of phase factors the right hand side in fact turns out to be a superposition of several ground states (D.2). To confirm this claim we first convince ourselves of the twist invariance of (D.5). Admittedly, the left hand side is invariant under Θ's action by construction. However, it is instructive to verify this symmetry property explicitly also for the right hand side: Let us identify the phase–weighted states appearing in (D.5) with $|\beta,k\rangle$ in an obvious way. Θ will then map such a state to $|\beta+1,k+1\rangle$ regardless of the accompanying phase factor. Due to $||\mathcal{O}_f|,k\rangle = |0,k\rangle$ and $|\beta,N+1\rangle = |\beta,1\rangle$ we may restore the original summation ranges.

This consideration proves to be quite useful to settle the above claim. We just found that the twist maps the finite set W of states $|\beta,k\rangle$ bijectively onto itself. Now we single out a particular element and act with Θ repeatedly on it, until we are back to the original state. As discussed above the phase factors along this cycle are identical. Meanwhile, the translation group index of $\sigma^{(s)}$ has traversed a complete orbit. Thereby we have extracted a phase–weighted physical state (D.2) from (D.5). Removing these states $|\beta,k\rangle$ from W the search for a twist invariant contribution can be once more started, because Θ continues to act bijectively on the remaining states. In a finite number of steps W will be exhausted. Of course, it may well occur that the same physical state appears several times (with a priori different phases as prefactors) in (D.5).

Drastic simplifications come about if s is a *prime* number. From the elementary fact that there exists an integer $\dot{s}$ with the property $s\dot{s}\equiv 1 \bmod N$ we deduce that the orbit of every lattice vector f has (the shortest) length 1:

$$
\begin{aligned}
f - \Theta f &= (1-\Theta^s)\frac{(1-\Theta)}{(1-\Theta^s)} f \\
&= (1-\Theta^s)(1+\Theta^s+\ldots+(\Theta^s)^{\dot{s}-1}) f \in (1-\Theta^s)\Lambda_d \quad .
\end{aligned} \tag{D.6}
$$

(Notice that we are entitled to replace Θ by $\Theta^{s\dot{s}}$.) The physical twist fields endow therefore the coordinates X^μ with global monodromy shifts from a single coset ($\Sigma_f^{(s)} = \sigma_f^{(s)}$). Moreover, the right hand side of (D.5) then boils down to a single term (ordinarily, such simplicity is only met in the first twisted sector), namely

$$
\sum_{k=1}^{N} e^{i(\Theta^k w)^T \hat{q}}\, e^{i(\Theta^k p)^T \hat{x}}\, |\Sigma_f^{(s)}\rangle = N \left\{ \exp(2\pi i\, p^T \frac{1}{1-\Theta} f) \right\}^{\dot{s}} |\Sigma_{f+w}^{(s)}\rangle. \tag{D.7}
$$

(The peculiar form of the phase factor is due to the identity $(1-\Theta) \times (1-\Theta^s)^{-1} \equiv \dot{s} \bmod (1-\Theta)$.) The ultimate reason for the simplifications encountered for all prime order Z_N orbifolds is that every higher twisted sector may (equivalently) be looked upon as a *first* twisted sector.

Returning to the general case we must content ourselves with

$$
\langle \Sigma_{-f_1}^{(s)} | V_{\mathbf{P}}^{\mathrm{inv}} | \Sigma_{f_2}^{(s)} \rangle = g'(s;\mathbf{P}) \sqrt{N} \exp(i\pi p^T \frac{1}{1-\Theta^s} w) \left(\frac{1}{\sqrt{|\mathcal{O}_{f_1}|}} \frac{1}{\sqrt{|\mathcal{O}_{f_2}|}} \right)^{-1}
$$

$$
\sum_{\beta=0}^{|\mathcal{O}_{f_2}|-1} \sum_{\gamma=0}^{|\mathcal{O}_{f_1}|-1} e^{2\pi i p^T \frac{1}{1-\Theta^s}(\Theta^\beta f_2)}\, \delta_{\mathbf{0},(w+\Theta^\beta f_2 - \Theta^\gamma f_1) \bmod (1-\Theta^s)\Lambda_d} \tag{D.8}
$$

for the string emission correlation function. Fortunately, we got rid of the sum over zero mode exponentials still present in (D.5): Since the vertex operator $V_{\mathbf{P}}^{\mathrm{inv}}$ is sandwiched between twist invariant physical fields each of its N terms which are generated by repeated action of the twist on — say — the first one yields the same matrix element. Therefore we gain an overall factor N as in (4.20). The two–fold sum in (D.8) guarantees that the three–point coupling is independent of whether representative vectors other than $\mathbf{P}, f_1, f_2$ are taken to describe the emission vertex or the twist fields.

We now enter the discussion of the duality invariance concerning the string emission from the s–th twisted sector. Above, we have already mentioned the slight changes $V_{\mathbf{P}}^{\mathrm{inv}}$ has to undergo if it is placed in a higher twisted sector. Its dual partner field is then determined along the lines of section 5.2. With the help of (5.5) – (5.7) and of (5.10), (5.16) where Θ is always replaced by Θ^s we conclude that the former rule (5.15) is still applicable.

To generalize (5.4) we assume that the unphysical fields $\sigma^{(s)}_{\Theta^\beta f}$ are subject to this transformation law after a substitution $\Theta \mapsto \Theta^s$. This makes sense because in an orbifold construction based on the defining twist Θ^s every $\sigma^{(s)}_{\Theta^\beta f}$ is a valid physical ground state of the first twisted sector. We therefore expect the following map under duality:

$$\tilde{\Sigma}^{(s)}_{\tilde{f}} = \frac{1}{\sqrt{N_s}} \sum_{h \in (\mathcal{F}^*_s)'} \sqrt{\frac{|\mathcal{O}_{\tilde{f}}|}{|\mathcal{O}_f|}} \sum_{u=0}^{|\mathcal{O}_f|-1} \exp[2\pi i f^T \frac{1}{1-\Theta^s}(\Theta^u h)]\, \Sigma^{(s)}_f \quad . \tag{D.9}$$

Here N_s stands for the number $\det(1-\Theta^s)$ of fixed points of the toroidal twist Θ^s; $(\mathcal{F}^*_s)'$ denotes a complete system of orbit representatives for the quotient space $\mathcal{F}^*_s = \frac{\Lambda^*_d}{(1-\Theta^s)\Lambda_{d^*}}$, and we have to identify $\tilde{f} = \frac{1}{G-B}h$ $(\in \tilde{\Lambda}_d)$ while summing over h. The main virtue of (D.9) consists in the fact that the dual twist field is a superposition of *physical* twist fields although we have initially transformed the unphysical ingredients $\sigma^{(s)}_{\Theta^\beta f}$ according to (5.4). In [22], (D.9) was moreover successfully applied to verify the duality symmetry of Yukawa couplings of the form $\langle \sigma^+_{f_1} \sigma^+_{f_2} \Sigma^{(N-2)}_{-f_3} \rangle$.

In this paper we are exclusively concerned with clarifying the issue of duality invariance for the string emission process (this topic was not yet covered in [22]). For (D.8) to be duality invariant it is sufficient to demonstrate

$$\langle \tilde{\Sigma}^{(-s)}_{-\tilde{f}_1} |\, e^{i w^T \tilde{\hat{q}}}\, e^{i p^T \tilde{\hat{x}}}\, | \tilde{\Sigma}^{(s)}_{\tilde{f}_2} \rangle_{\tilde{g}+\tilde{b}} \; = \; \langle \Sigma^{(-s)}_{-f_1} |\, e^{i w^T \hat{q}}\, e^{i p^T \hat{x}}\, | \Sigma^{(s)}_{f_2} \rangle_{g+b} \tag{D.10}$$

which generalizes (5.17) to any higher twisted sector. Recall that we already have confirmed the validity of $g'(s;\tilde{\mathbf{P}}) = g'(s;\mathbf{P})$ thus ensuring duality invariance within the twisted sector Fock space. In analogy to our previous approach (see (5.18), (5.19)) we can prove the above identity involving zero mode exponentials with the help of

$$\begin{aligned} \langle \tilde{\Sigma}^{(-s)}_{-\tilde{f}_1} | e^{i\tilde{p}\tilde{\hat{x}}} | \tilde{\Sigma}^{(s)}_{\tilde{f}_2} \rangle_{\tilde{g}+\tilde{b}} &= \langle \Sigma^{(-s)}_{-f_1} | e^{iw\hat{q}} | \Sigma^{(s)}_{f_2} \rangle_{g+b} \\ \langle \tilde{\Sigma}^{(-s)}_{-\tilde{f}_1} | e^{i\tilde{w}\tilde{\hat{q}}} | \tilde{\Sigma}^{(s)}_{\tilde{f}_2} \rangle_{\tilde{g}+\tilde{b}} &= \langle \Sigma^{(-s)}_{-f_1} | e^{ip\hat{x}} | \Sigma^{(s)}_{f_2} \rangle_{g+b} \end{aligned} \tag{D.11}$$

and the following insertions of restricted unit operators:

$$\begin{aligned} \mathbf{1} &= \sum_{f\in\mathcal{F}'_s} |\Sigma_f^{(s)}\rangle_{g+b}\langle\Sigma_{-f}^{(-s)}| \\ \mathbf{1} &= \sum_{f\in\mathcal{F}'_s} |\tilde{\Sigma}_f^{(s)}\rangle_{\tilde{g}+\tilde{b}}\langle\tilde{\Sigma}_{-f}^{(-s)}| \quad . \end{aligned} \tag{D.12}$$

($\mathcal{F}'_s$ provides a complete system of orbit representatives for the quotient $\mathcal{F}_s = \frac{\Lambda_d}{(1-\Theta^s)\Lambda_d}$.)

The necessity to sum over orbits of lattice vectors in (D.9) complicates the proofs of these auxiliary relations. In order to point out how to proceed let us look closer at the second identity in (D.12) (the other formulae are then to be treated in a similar fashion). Its right hand side turns into

$$\frac{1}{N_s}\sum_{f\in\mathcal{F}'_s}\frac{1}{|\mathcal{O}_f|}\sum_{h,k\in(\mathcal{F}^*_s)'}\sqrt{|\mathcal{O}_h||\mathcal{O}_k|}\sum_{u,v=0}^{|\mathcal{O}_f|-1}\mathcal{E}(f,\Theta^u h-\Theta^v k)\,|\Sigma_{\tilde{f}}^{(s)}\rangle_{\tilde{g}+\tilde{b}}\langle\Sigma_{-\tilde{t}}^{(-s)}| \tag{D.13}$$

when the dual twist fields are expanded according to (D.9). ($\mathcal{E}(f,h)$ is a shorthand form of $\exp[2\pi i f^T(1-\Theta^s)^{-1}h]$; $\tilde{f}=(G-B)^{-1}h$, $\tilde{t}=(G-B)^{-1}k$ with $f\in\Lambda_d$ and $h,k\in\Lambda^*_d$.) The following relations are crucial to finally simplify the five–fold sum in (D.13):

$$\mathcal{E}(f,h) = \mathcal{E}(\Theta^{|\mathcal{O}_f|}f,h) \quad , \tag{D.14}$$

$$\mathcal{E}(f,\Theta^u h) = \mathcal{E}(\Theta^{-u}f,h) \quad . \tag{D.15}$$

Our first aim will be the extension of the summation over h, k to the complete set of coset representatives $\mathcal{F}^*_s$. This step is not at all hampered by the phase factors $\mathcal{E}(f,\Theta^u h-\Theta^v k)$, since e.g. $h\mapsto\Theta h$ can be absorbed by a shift of $\sum_u$ whose original range of summation can even be restored thanks to (D.14). Furthermore, recasting $\mathcal{E}(\ldots)$ according to (D.15) and combining $\sum_{f\in\mathcal{F}'_s}$ with $\sum_{u=0}^{|\mathcal{O}_f|-1}$ allows us to extend the sum over f to the complete set of representatives $\mathcal{F}_s$. We do not have to worry about the u–dependent limits of the remaining sum over $(v-u)$, since we have again (D.14) at our disposal. Meanwhile we have arrived at

$$\frac{1}{N_s}\sum_{f\in\mathcal{F}_s}\frac{1}{|\mathcal{O}_f|}\sum_{v=0}^{|\mathcal{O}_f|-1}\sum_{h,k\in\mathcal{F}_s^*}\frac{1}{\sqrt{|\mathcal{O}_h||\mathcal{O}_k|}}\,\mathcal{E}(f,h-\Theta^v k)\,|\Sigma_{\tilde{f}}^{(s)}\rangle_{\tilde{g}+\tilde{b}}\langle\Sigma_{-\tilde{t}}^{(-s)}|\;. \tag{D.16}$$

The summation $\sum_v$ can be immediately executed if the k–summation is adjusted via $k \mapsto \Theta^{-v}k$. We are permitted to set $v=0$ in (D.16), and to replace $\sum_v$ by a multiplicity factor $|\mathcal{O}_f|$; thereby the normalization factor $\frac{1}{|\mathcal{O}_f|}$ is cancelled. From now on it is fairly easy to continue because $\sum_{f\in\mathcal{F}_s}$ yields already Kronecker's symbol $\delta_{\mathbf{0},(h-k)\mathrm{mod}(1-\Theta^s)\Lambda_d^*}$. We end up with

$$\sum_{h\in\mathcal{F}_s^*}\frac{1}{|\mathcal{O}_h|}|\Sigma_{\tilde{f}}^{(s)}\rangle_{\tilde{g}+\tilde{b}}\langle\Sigma_{-\tilde{f}}^{(-s)}| \tag{D.17}$$

which amounts to the decomposition of the unit operator (acting on the zero mode representation space of the s–th twisted sector) of the *dual* orbifold model. This concludes our check of the second identity in (D.12).

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