SLAC–PUB–5582
June 1991
(A)

BEYOND BASSETTI AND ERSKINE: BEAM-BEAM DEFLECTIONS FOR NON-GAUSSIAN BEAMS.*

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ABSTRACT

Generalizations of the well-known Bassetti and Erskine formula to include tilt angle and non-Gaussian particle distributions, as well as centroid deflections, are derived.

1. Introduction

The electric field of a relativistic upright Gaussian beam was first calculated in closed form by Bassetti and Erskine in Ref. 1. In this note we first generalize their formula to include tilt angle, thereby putting it in a covariant form. We follow the strategy Talman outlines in Ref. 2. Second, we comment on issues arising in the numerical evaluation of this formula. Third, we average the single particle deflection angle that is proportional to the transverse electric field over an offset distribution to derive the centroid's beam-beam deflection angle. Both Gaussian and non-Gaussian beams are considered. A section about possible applications concludes this paper.

2. The Covariant Form

Following Ref. 2, we see that the deflection angle (x_1 = horizontal, x_2 = vertical) a single particle experiences, can be written using a complex Green function

$$\Delta x_2' + i\Delta x_1' = N_1 K \int_{\mathbf{R}^2} d^2\,\tilde{\mathbf{x}}\; G(\mathbf{x}-\tilde{\mathbf{x}})\; \psi_0(\tilde{\mathbf{x}},\sigma) \tag{1}$$

where we define $K = -2r_e/\gamma$ with γ being the normalized energy of the deflected particle and r_e the classical electron radius. N_1 is the number of particles in the field producing bunch described by the distribution function

* Work supported by Department of Energy contract DE–AC03–76SF00515.

Presented at the 7th ICFA Beam Dynamics Workshop, Los Angeles, May 13–16, 1991.

$$\psi_0(\tilde{\mathbf{x}},\sigma) = \frac{1}{2\pi\sqrt{\det\sigma}}\ \exp\left[-\frac{1}{2}\ \sum_{i,j}^{2}(\sigma^{-1})_{ij}\tilde{x}_i\tilde{x}_j\right] \tag{2}$$

that generates the electric field. The Green function is given by

$$G(\mathbf{x}-\tilde{\mathbf{x}}) = \frac{i}{(x_1-\tilde{x}_1)+i(x_2-\tilde{x}_2)}\ . \tag{3}$$

In this report we denote vector valued quantities in bold typeface. It is well known that Eq. 1 can be rewritten in terms of a potential

$$\begin{aligned}
\Delta x_2' + i\Delta x_1' &= -\left(\frac{\partial}{\partial x_2} + i\frac{\partial}{\partial x_1}\right)\ \phi(x_1,x_2) \\
\phi(x_1,x_2) &= \frac{N_1K}{2}\int_0^\infty dq\,\frac{1}{\sqrt{\det(q+2\sigma)}}\ \exp\left[-(q+2\sigma)_{ij}^{-1}x_ix_j\right]
\end{aligned} \tag{4}$$

where we have to deal with a matrix rather than individual sigmas as in Refs. 1,2. The appearing symmetric sigma matrix is explicitely given by

$$(q+2\sigma)_{ij} = \begin{pmatrix} q+2\sigma_{11} & 2\sigma_{12} \\ 2\sigma_{12} & q+2\sigma_{22} \end{pmatrix}\ . \tag{5}$$

In the next step we diagonalize $(q+2\sigma)$ and reduce the required integrals to those appearing in the original derivation. The diagonalization can be done by a simple coordinate rotation given by

$$\bar{x}_1 + i\bar{x}_2 = e^{-i\alpha}\,(x_1+ix_2)\ . \tag{6}$$

The sigma matrix then transforms according to

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix}\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \tag{7}$$

which reduces the potential ϕ to

$$\phi(x_1,x_2) = \frac{N_1K}{2}\int_0^\infty dq\,\frac{1}{\sqrt{(q+2\sigma_x^2)(q+2\sigma_y^2)}}\ \exp\left[-\frac{\bar{x}_1^2}{2\sigma_x^2} - \frac{\bar{y}_1^2}{2\sigma_y^2}\right] \tag{8}$$

which leads to the well known Bassetti and Erskine result

$$\begin{aligned}
\Delta\bar{x}_2' + i\Delta\bar{x}_1' &= \frac{N_1K\sqrt{\pi}}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\left\{w\left[\frac{\bar{x}_1+i\bar{x}_2}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right]\right. \\
&\quad \left. -\exp\left[-\frac{\bar{x}_1^2}{2\sigma_x^2} - \frac{\bar{y}_1^2}{2\sigma_y^2}\right]\ w\left[\frac{\bar{x}_1\frac{\sigma_y}{\sigma_x} + i\bar{x}_2\frac{\sigma_x}{\sigma_y}}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right]\right\}\ .
\end{aligned} \tag{9}$$

Finally, we have to relate all quantities back to those of the original coordinate system. Note that we can write with the help of Eqs. 6 and 7

$$\begin{aligned} \sigma_x^2 - \sigma_y^2 &= e^{-2i\alpha}(\sigma_{11} - \sigma_{22} + 2i\sigma_{12}) \\ \bar{x}_1 \frac{\sigma_y}{\sigma_x} + i\bar{x}_2 \frac{\sigma_x}{\sigma_y} &= e^{-i\alpha} \left[\frac{\sigma_{22} - i\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22} - \sigma_{12}^2}}\, x_1 + i \frac{\sigma_{11} + i\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22} - \sigma_{12}^2}}\, x_2 \right] . \end{aligned} \tag{10}$$

We have to remember that Eq. 9 was derived from the potential given by Eq. 8 by differentiation with respect to the variables $\bar{\mathbf{x}}$. Therefore we have to write

$$\Delta \bar{x}_2' + i\Delta \bar{x}_1' = e^{i\alpha}\left(\Delta x_2' + i\Delta x_1'\right) . \tag{11}$$

Inserting Eqs. 10 and 11 in Eq. 9 leads to the final covariant form of Bassetti and Erskine's formula, which reads

$$\Delta x_2' + i\Delta x_1' = N_1 K F_0(x_1, x_2, \sigma) \tag{12}$$

where F_0 is defined by

$$\begin{aligned} F_0(x_1, x_2, \sigma) = \frac{\sqrt{\pi}}{\sqrt{2(\sigma_{11} - \sigma_{22} + 2i\sigma_{12})}} \Bigg\{ w \left[\frac{x_1 + ix_2}{\sqrt{2(\sigma_{11} - \sigma_{22} + 2i\sigma_{12})}} \right] \\ - \exp\left[-\tfrac{1}{2}\, \Sigma_{i,j}^2 (\sigma^{-1})_{ij} x_i x_j \right]\, w \left[\frac{(\sigma_{22} - i\sigma_{12})x_1 + i(\sigma_{11} + i\sigma_{12})x_2}{\sqrt{\sigma_{11}\sigma_{22} - \sigma_{12}^2}\sqrt{2(\sigma_{11} - \sigma_{22} + 2i\sigma_{12})}} \right] \Bigg\} . \end{aligned} \tag{13}$$

The kick angle given by Eq. 12 and 13 is now valid in any coordinate system and for any tilt angle the beam might have.

3. Numerical Issues

In the evaluation of F_0 various numerical problems can arise. Here a few are mentioned and methods described to circumvent them.

First, for round beams that are characterized by $\sigma_{11} = \sigma_{22}$ and $\sigma_{12} = 0$ there seems to be a singularity, because the root

$$\sqrt{2(\sigma_{11} - \sigma_{22} + 2i\sigma_{12})}$$

vanishes. However, this singularity is compensated by a similar one in the arguments of the complex error functions $w(z)$. Considering that $w(z)$ has an asymptotic expansion $w(z) \rightsquigarrow i/\sqrt{\pi}z$, we see that the singularities cancel and we are led to the result for round beams

$$F_0(x_1, x_2, \sigma_{11} = \sigma_{22}, \sigma_{12} = 0) = i\, \frac{1 - \exp\left[-(x_1^2 + x_2^2)/2\sigma_{11}\right]}{x_1 + ix_2} . \tag{14}$$

Second, for large negative imaginary arguments, the complex error functions grow exponentially. This leads to the evaluation of differences of very large numbers, which can become numerically unstable. The problem can be circumnavigated by rewriting the expression in the curly brackets as

$$w(z_1) - e^g\, w(z_2) = -\overline{[w(\bar{z}_1) - e^g\ w(\bar{z}_2)]}\ . \tag{15}$$

The arguments z_1 and z_2 are those of the complex error functions in Eq. 13 and g is the expression in the exponential. In this way, we flip the sign of the imaginary part in arguments of the complex error functions and make the expression numerically well behaved.

Third, the evaluation of the complex error function is computationally expensive. We adapt Talman's approach and calculate a Pade approximation for $w(z)$ which consists of a tenth order polynomial in the numerator and a eleventh order polynomial in the denominator [3]. Checks show that such a routine is accurate to about five significant digits in the first quadrant of the complex plane. Values in other quadrants can be calculated using well known relations for the complex error functions for negative and complex conjugate arguments [4].

4. Centroid Deflection Angle

In the second section of this paper we were concerned about the deflection angle a single particle experiences. Here we average that over a second Gaussian particle distribution which may be offset with respect to the field generating distribution. We can write this centroid deflection angle in the form

$$\Theta_{x_2} + i\Theta_{x_1} = N_1 K \int_{\mathbf{R}^2} d^2\mathbf{y}\ \psi_2(\mathbf{y} - \mathbf{Y}) \int_{\mathbf{R}^2} d^2\mathbf{x}\ G(\mathbf{y} - \mathbf{x})\ \psi_1(\mathbf{x} - \mathbf{X}) \tag{16}$$

where $\mathbf{X}$ is the centroid position of the first beam and $\mathbf{Y}$ of the second, deflected beam. We can now change the variables, exchange the order of integration, and arrive at

$$\Theta_{x_2} + i\Theta_{x_1} = N_1 K \int_{\mathbf{R}^2} d^2\mathbf{y}\ G(\mathbf{Y} - \mathbf{X} - \mathbf{y}) \int_{\mathbf{R}^2} d^2\mathbf{x}\ \psi_1(\mathbf{x})\ \psi_2(\mathbf{x} - \mathbf{y})\ . \tag{17}$$

Note that this expression has the same structure as Eq. 1, which was used to calculate the single particle kick, except that here the distance between the two bunch centers, $\mathbf{Y} - \mathbf{X}$, appears. The convolution of both the field-producing and deflected distribution functions shows up as the source distribution. This observation, which is valid for arbitrary distribution functions ψ_1 and ψ_2, indicates that it is impossible to obtain information about individual beam-beam centroid deflection data. Only information about the convolution is accessible.

We now evaluate Eq. 17 for two Gaussian distributions, given by

$$\begin{aligned}\psi_1(\mathbf{x},\sigma) &= \frac{1}{2\pi\sqrt{\det\sigma}}\exp\left[-\frac{1}{2}\sum_{i,j}^{2}(\sigma^{-1})_{ij}x_ix_j\right]\\ \psi_2(\mathbf{x}-\mathbf{y},\tilde\sigma) &= \frac{1}{2\pi\sqrt{\det\tilde\sigma}}\exp\left[-\frac{1}{2}\sum_{i,j}^{2}(\tilde\sigma^{-1})_{ij}(x_i-y_i)(x_j-y_j)\right].\end{aligned} \tag{18}$$

The convolution is easily performed with the result

$$\int_{\mathbf{R}^2} d^2\mathbf{x}\,\psi_1(\mathbf{x},\sigma)\psi_2(\mathbf{x}-\mathbf{y},\tilde\sigma) = \frac{1}{2\pi\sqrt{\det\mathbf{\Sigma}}}\exp\left[-\frac{1}{2}\sum_{i,j}^{2}(\Sigma^{-1})_{ij}y_iy_j\right] \tag{19}$$

where we introduce $\mathbf{\Sigma} = \sigma + \tilde\sigma$. Clearly the new "effective" distribution function depends on the sum of the correlation matrices of the individual beams. We can now utilize the discussion in Section 2 to deduce the centroid deflection angle and get

$$\Theta_{x_2} + i\Theta_{x_1} = N_1KF_0(Y_1-X_1, Y_2-X_2, \mathbf{\Sigma})\ . \tag{20}$$

which again depends only on $\mathbf{\Sigma}$.

Note that Eq. 20 also depends on the relative distance between the centroids of both bunches $\mathbf{Y}-\mathbf{X}$. Furthermore, the kick angle for the other bunch is given by Eq. 20 with $\mathbf{X}$ and $\mathbf{Y}$ exchanged and KN_2 substituted for KN_1. Writing the centroid angles for beams 1 and 2 as $\mathbf{X}'$ and $\mathbf{Y}'$, respectively, we can write for their variation during one collision

$$\mathbf{Y}'-\mathbf{X}'|_{\text{after}} = \mathbf{Y}'-\mathbf{X}'|_{\text{before}} + (KN_1+KN_2)\ F_0(\mathbf{Y}-\mathbf{X},\mathbf{\Sigma})\ . \tag{21}$$

Assuming a storage ring with equal tunes for the counter-rotating beams , we can infer that the centroid difference variables $\mathbf{Y}-\mathbf{X}$, $\mathbf{Y}'-\mathbf{X}'$ transform among themselves, and that the dynamics described by such a system is equivalent to that of a single particle in the field of a fixed Gaussian beam with $\mathbf{\Sigma} = \sigma+\tilde\sigma$ and $(NK)_{\text{eff}} = KN_1+KN_2$. We have thus proven that the dynamics of "strong-strong" rigid-bunch models with equal tunes is equivalent to "weak-strong" models with appropriately modified parameters [5].

Next, we generalize this discussion to non-Gaussian beams, which we parametrize by a Gaussian ψ_0 and an arbitrary polynomial P_1

$$\psi_1(\mathbf{x},\sigma_1) = P_1(\mathbf{x})\ \psi_0(\mathbf{x},\sigma_1)\ . \tag{22}$$

An example is the Stratonovich expansion in terms of generalized Hermite polynomials, as used by Hirata in Ref. 6. In order to simplify the algebra, we rewrite Eq. 22 in the form

$$\psi_1(\mathbf{x},\sigma_1) = P_1\left(\frac{\partial}{\partial\mathbf{B}}\right)\frac{1}{2\pi\sqrt{\det\sigma_1}}\exp\left[-\tfrac{1}{2}\ \textstyle\sum_{i,j}^{2}(\sigma^{-1})_{ij}x_ix_j + \sum_i^2 B_ix_i\right]\Big|_{\mathbf{B}=\mathbf{0}}\ . \tag{23}$$

The required convolution with the deflected distribution, parametrized in a similar fashion, yields

$$\int_{\mathbf{R}^2} d^2\mathbf{x}\ \psi_1(\mathbf{x},\sigma_1)\ \psi_2(\mathbf{x}-\mathbf{y},\sigma_2) \ = \ P_1\left(\frac{\partial}{\partial\mathbf{B}}\right)\ P_2\left(\frac{\partial}{\partial\mathbf{C}}\right)$$
$$\times\ \exp\left[\tfrac{1}{2}\ \sigma_1\mathbf{BB}+\tfrac{1}{2}\ \sigma_2\mathbf{CC}\right]\ \ \psi_0(\mathbf{y}-\sigma_1\mathbf{B}+\sigma_2\mathbf{C},\sigma_1+\sigma_2)|_{\mathbf{B}=\mathbf{C}=0}\ . \tag{24}$$

The centroid deflection angle of beam 2 is then trivially calculated by plugging Eq. 24 into Eq. 17, and following the derivation for the covariant version of the Bassetti and Erskine formula. We obtain

$$\Theta_{x_2}+i\Theta_{x_1} \ = \ N_1KP_1\left(\frac{\partial}{\partial\mathbf{B}}\right)\ P_2\left(\frac{\partial}{\partial\mathbf{C}}\right)$$
$$\times\ \exp\left[\tfrac{1}{2}\ \sigma_1\mathbf{BB}+\tfrac{1}{2}\ \sigma_2\mathbf{CC}\right]\ \ F_0(\mathbf{Y}-\mathbf{X}-\sigma_1\mathbf{B}+\sigma_2\mathbf{C},\sigma_1+\sigma_2)|_{\mathbf{B}=\mathbf{C}=0}\ . \tag{25}$$

Now only parametric differentiations of Gaussians and complex error functions (in F_0) are required, which are easily done. Note in particular that multiple derivatives of $w(z)$ can be generated recursively according to [4]

$$w^{(n+2)}(z)=-2[zw^{(n+1)}(z)+(n+1)\ w^{(n)}(z)]\ , \tag{26}$$

which makes the evaluation of Eq. 25 computationally inexpensive, because only $w^{(0)}(z)$ needs to be evaluated directly by a Pade approximation or otherwise.

5. Applications

First, Eq. 25 can be used in tracking codes for the beam-beam interaction to either calculate the coherent centroid deflection angle or the incoherent single particle deflection angle, by setting $\sigma_2 = 0$, $\mathbf{C} = 0$ and $P_2 = 1$. This may serve to make beam-beam codes more self consistent.

Second, it is possible to expand Eq. 25 around the "round beam case" [7]. By this we mean to expand $\Theta_{x_2}+i\Theta_{x_1}$ around $\sigma_{11}=\sigma_{22}$ and $\sigma_{12}=0$ in small parameters such as $\varepsilon_1 = 2\Sigma_{12}/(\Sigma_{11}+\Sigma_{22})$ for the tilt signature, $\varepsilon_2 = (\Sigma_{11}-\Sigma_{22})/(\Sigma_{11}+\Sigma_{22})$, for the ellipticity signature or coefficients of the polynomials to assess non-Gaussian characteristics. By plotting the resulting "signature curves" along a path $(x_1(t), x_2(t))$, it is then possible to discern systematic deviations from the normally used round beam deflection curve given by Eq. 14.

6. Conclusion

In this paper the Bassetti and Erskine formula for the deflection angle from elliptic Gaussian beams was generalized to incorporate tilt angles. Thereby, the formula was transformed into a covariant form. Furthermore, we used this formula to calculate the centroid deflection angles of Gaussian and non-Gaussian beams.

In the analysis it turned out that the centroid deflection angle depends only on the convolution of the field-producing and deflected distribution functions. It is therefore not possible to determine properties of the individual beams from deflection data alone.

As a direct consequence, it was shown that the "strong-strong" rigid-bunch models are equivalent to "weak-strong" models with modified parameters.

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